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RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1914–JUNE, 1915.

Thursday, November 12th, 1914.

ANNUAL GENERAL MEETING.

Prof. A. E. H. LOVE, President, in the Chair.

Present twenty-nine members and five visitors.

Mr. J. P. Clatworthy was elected a member.

Messrs. G. R. Jeffery and A. Lynch were admitted into the Society.

On behalf of the Council the President presented the De Morgan Medal to Prof. Sir J. Larmor.

The Treasurer presented his Report. On the motion of Mr. Dallas, seconded by Mr. Wren, the Report was received.

Lt.-Col. A. Cunningham was appointed Auditor.

The Council and Officers for the ensuing Session were elected. The list is as follows:—President, Prof. Sir J. Larmor; Vice-Presidents, Prof. W. Burnside and Prof. A. E. H. Love; Treasurer, Dr. A. E. Western; Secretaries, Mr. J. H. Grace and Dr. T. J. I'A. Bromwich; other members of the Council, Prof. H. F. Baker, Mr. S. Chapman, Mr. E. Cunningham, Mr. A. L. Dixon, Mr. G. H. Hardy, Prof. E. W. Hobson, Mr. A. E. Jolliffe, Prof. H. M. Macdonald, Major P. A. MacMahon, Mr. H. W. Richmond.

Sir J. Larmor having taken the chair, the retiring President, Prof. Love, delivered an address on "Mathematical Research." It was agreed that this should be printed in the *Proceedings*.

Prof. T. Kubota read a paper "On the Theory of Closed Convex Surfaces."*

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* Printed in this volume.

The following were also communicated :—

Note on an Extension of Sylow's Theorem : Prof. G. A. Miller.

The Conformal Representation of the various Triangles bounded by the Arc of Three Intersecting Circles : Mr. J. Hodgkinson.

*The Dynamical Theory of the Tides in a Zonal Basin : Mr. G. R. Goldsbrough.

*The Modulus of an Analytic Function : Mr. G. H. Hardy.

*(i) On the Modification of a Train of Waves as it Advances into Shallow Water ; (ii) *On a Configuration of 21 Points and 21 Lines which arises from the Complete Quadrilateral and Determines the Group of 168 Plane Collineations : Prof. W. Burnside.

On Integrals and Derivates with respect to a Function : Prof. W. H. Young.

ABSTRACTS.

On the Theory of Closed Convex Surfaces : T. Kubota.

A proof of this theorem is given : "If the sections of a closed convex surface by each set of parallel planes are similar and similarly situated curves, the surface must be an ellipsoid."

Note on an Extension of Sylow's Theorem : Prof. G. A. Miller.

The theorem that the number of cyclic sub-groups of order p^a, p being any odd prime number and $a > 1$, contained in a group which involves at least one non-cyclic group of order p^m, $m > a$, is always divisible by p, was proved by me in these *Proceedings*, Vol. 2, 1904, p. 142. In view of the importance of this theorem it may be desirable to direct attention to a much simpler proof than the one just noted.

For simplicity we shall first assume that the order of the non-cyclic group G is p^m, and observe that every cyclic sub-group of order p^a contained in G is found in a non-cyclic sub-group of order p^{a+1}. Employing a well known method with slight modification,† we let r_a and r_{a+1} represent respectively the total number of sub-groups of order p^a, and the total number of *non-cyclic* sub-groups of order p^{a+1} contained in G. If we count each of these non-cyclic sub-groups as many times as it contains sub-groups of order p^a there results the equation

$$\sum_{x=1}^{x=r} a_x = \sum_{y=1}^{y=r_{a+1}} b_y,$$

---

* Printed in this volume.

† Cf. W. Burnside, *Theory of Groups*, 2nd edition, 1911, p. 129.

where a_x represents the number of non-cyclic sub-groups of order $p^{\alpha+1}$ which contain a particular sub-group of order p^α, and b_y represents the number of sub-groups of order p^α contained in a particular non-cyclic sub-group of order $p^{\alpha+1}$.

Hence it results directly that $r_\alpha \equiv r_{\alpha+1} \pmod{p}$, and as $r_\alpha \equiv 1 \pmod{p}$ according to a well known theorem due to Frobenius, it follows that $r_{\alpha+1} \equiv 1 \pmod{p}$. Hence the number of cyclic sub-groups of order p^α in G is either zero or a multiple of p. The main advantage of this method of proof is that it is only a very slight modification of a well known method usually employed to prove the extension of Sylow's theorem due to Frobenius. The extension of this theorem to the case where G is any group involving non-cyclic Sylow sub-groups of order p^m is evident.

The only point in the above proof that presents any difficulty is the fact that $a_x \equiv 1 \pmod{p}$ for each of the possible values of x. In fact, this case presents no special difficulty unless the sub-group H of order p^α in question is cyclic, and is included in a cyclic sub-group of order $p^{\alpha+1}$. In this special case it is easy to see that all the operators of G which are commutative with every operator of H form a group which involves $kp+1$ non-cyclic sub-groups of order $p^{\alpha+1}$ involving H, since its quotient group with respect to H involves a non-cyclic invariant sub-group of order p^2.

The Conformal Representation of the various Triangles bounded by the Arcs of Three Intersecting Circles: Mr. J. Hodgkinson.

It is shown that three intersecting circles determine thirty two distinct triangles. The triangle the sum of whose angles is least is called the "reduced triangle," the others are styled "associated triangles."

The conformal representation on a z half plane of a triangle in a w plane whose angles are $\lambda\pi$, $\mu\pi$, $\nu\pi$ is known to be given by

$$\tfrac{1}{2}\{w, z\} = \tfrac{1}{4}\left\{\frac{1-\lambda^2}{z^2} + \frac{1-\nu^2}{(z-1)^2} + \frac{\lambda^2-\mu^2+\nu^2-1}{z(z-1)}\right\},$$

where $\{w, z\}$ is the Schwarzian derivative.

The reduced triangle being represented upon a z half plane and the associated triangle upon a Z half plane, the author has considered the relation between Z, z.

It is found (1) that the relation is always transcendental when the three circles have a real common orthogonal circle, *i.e.*, for the reduced triangle $\lambda+\mu+\nu < 1$; (2) that when the three circles have an imaginary common orthogonal circle, *i.e.*, for the reduced triangle $\lambda+\mu+\nu > 1$, the relation is algebraic when, and only when, the reduced triangle is one of Schwarz's XV well-known cases.

The relation between z, Z has been obtained by the author in one case by a method due to Klein. From this the writer has deduced that, if α, β, γ are the elements of one of Schwarz's algebraically integrable cases of the hypergeometric equation, the equation whose elements are $\alpha+n_1$, $\beta+n_2$, $\gamma+n_3$, where n_1, n_2, n_3 are integers, is also algebraically integrable.

The author has also considered under what circumstances the function z, obtained by integrating the equation

$$\tfrac{1}{2}\{w, z\} = \tfrac{1}{4}\left\{\frac{1-\lambda^2}{z^2}+\frac{1-\nu^2}{(z-1)^2}+\frac{\lambda^2-\mu^2+\nu^2-1}{z(z-1)}\right\},$$

has a finite number of branches. It is found that when the triangle whose conformal representation is effected by the integral of the equation is one of Schwarz's XV cases, or any of the associated triangles, this will be the case, but for all the cases in which the reduced triangle has the sum of its angles greater than π, the number of branches is infinite. For the cases when the sum of the angles of the reduced triangle is less than π, the function z always has an infinite number of branches when the triangle represented is an associated triangle, and in the case of a reduced triangle the number of branches of z is finite in special cases, viz., when the triangle can be built up of transformations of a triangle whose angles are $\frac{\pi}{2}$, $\frac{\pi}{m}$, $\frac{\pi}{n}$, where m, n are integers. The author shows by a single case that λ, μ, ν may be commensurable, and such a composition yet be impossible. Finally, a test is given by which the possibility of such a composition may be determined.

Thursday, December 10*th*, 1914.

Prof. Sir J. LARMOR, President, in the Chair.

Present nineteen members and two visitors.

Mr. R. H. Fowler was elected a member.

The Auditor (Lt.-Col. Cunningham) presented his Report.

On the motion of Mr. Neville, seconded by Mr. Wren, the Treasurer's Report was adopted, and the thanks of the Society were voted to the Treasurer and to the Auditor.

The following papers were communicated:—

- *On Simultaneous Equations, Linear or Functional: Mr. E. H. Neville.
- *On Cyclotomic Quinquisection: Prof. W. Burnside.
- *Oscillations near the Isosceles Triangle-Solution of the Three Body Problem: Prof. D. Buchanan.
- *On Lamé's Differential Equation and Ellipsoidal Harmonics: Prof. E. T. Whittaker.

ABSTRACTS.

On Simultaneous Equations, Linear or Functional: Mr. E. H. Neville.

The showman's puzzle of covering a large circle with five equal smaller circles, and the principal mathematical question which it raises having been described, the equations on which the numerical solution depends are written down; they are four simultaneous functional equations. It is then shewn, the explanation being made in the case of three equations, that

If it is known that an approximate simultaneous solution of any number n of independent functional equations

$$F_p(x_1, x_2, \ldots, x_n) = 0 \quad (p = 1, 2, \ldots, n)$$

in the same number of variables is given by

$$x_p = a_{p1} \quad (p = 1, 2, \ldots, n),$$

then rational coefficients ρ_{st} can rapidly be obtained, such that if sets of numbers a_{pm} are calculated in succession from the formulæ

$$a_{p,\,m+1} = \sum_{q=1}^{n} \rho_{qp} F_q(r_{1m}, r_{2m}, \ldots, r_{nm}) \quad (p = 1, 2, \ldots, n),$$

where

$$r_{lm} = \sum_{k=1}^{m} a_{lk} \quad (l = 1, 2, \ldots, n),$$

then the n series $\sum_{m=1}^{\infty} a_{pm}$ are convergent, and if the sum of the series Σa_{pm} is r_p the functional equations are all satisfied for the set of values $r_1, r_2, \ldots, r_p$ of the variables.

A comparison having been made between the method described and the classical method of solving the same problem, in respect of the labour involved in application and the rapidity with which the approximation proceeds, it is pointed out that the method often effects economy even in the two simplest cases, that in which the equations are linear algebraic equations, and that in which there is a single functional equation.

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* Printed in this volume.

In conclusion, the method is illustrated by application to the particular set of equations for whose solution it was devised, and it is found that although the arrangement of the circles which is theoretically the most effective cannot be described simply, yet in practice, even if the covering circles are so small compared with the large circle that this arrangement is only just effective, an arrangement of an extremely simple kind is so nearly effective that its failure must escape detection.

On Cyclotomic Quinquisection : Prof. W. Burnside.

The problem of cyclotomic trisection is known to depend on the solution of the diophantine equation

$$[4p-16-9(A+B)]^2+27(A-B)^2=4p,$$

where p is a prime of the form $6n+1$. This equation has only the solution (A, B) and (B, A), either of which may be used. It is here shewn that the problem of quinquisection depends on the solution of the simultaneous diophantine equations

$$12^2p=[4p-16-25(A+B)]^2+5.15^2(A-B)^2+2.15^2C^2+2.15^2D^2,$$

$$0=[4p-16-25(A+B)(A-B)]+3(C^2+4CD-D^2),$$

where p is a prime of the form $10n+1$. These equations have just four solutions

$$(A, B, C, D),\quad (B, A, -D, C),\quad (A, B, -C, -D),\quad (B, A, D, -C),$$

any one of which may be used.

Thursday, January 14*th*, 1915.

Prof. Sir J. LARMOR, President, and temporarily Prof. A. E. H. LOVE, Vice-President, in the Chair.

Present seventeen members and one visitor.

Dr. Bromwich reported that at the beginning of the Session 1913–1914, there were 306 ordinary members, that during the Session eight members had been elected, four members had died, and five had resigned, so that the number at the beginning of the present Session was 305.

The following communications were made :—

On Halphen's Transformation : Mr. H. E. J. Curzon.
*The Condition that a Quintic Equation should be Soluble by Radicals : Mr. W. E. H. Berwick.
*A Class of Diffraction Problems : Prof. H. M. Macdonald.
A Christmas Problem in Probabilities : Dr. A. Young.
On the Lack of Uniformity in the Earth's Rotation : Sir J. Larmor.

ABSTRACTS.

On Halphen's Transformation : Mr. H. E. J. Curzon.

In the geometrical construction corresponding to Halphen's birational transformation, suppose the initial curve C gives rises to the derived curve Γ. Then to C corresponds only one curve Γ. However, not only does C give rise to Γ but so does C', the reciprocal of C with respect to the directing conic, and it is shown in this paper that C and C' belong to a family of algebraic curves, each of which gives Γ on applying the Halphen transformation, and that while C and C' have a one-one correspondence with Γ, each of the other members of the family has a two-one correspondence. The method adopted depends upon solving a Riccati differential equation, two of whose solutions are known. This two-one correspondence involves relations between the coefficients in the equation to the particular curve chosen out of the family and the constants in the directing conic.

It is shewn further how curves may be constructed such that the Halphen construction gives a $(k, 1)$-correspondence between the initial curve and the derived curve, where k is an arbitrary integer. In the illustration given the derived curve is a conic.

The Condition that a Quintic Equation should be Soluble by Radicals : Mr. W. E. H. Berwick.

In order that an irreducible quintic equation

$$ax^5+bx^4+cx^3+dx^2+ex+f=0$$

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* Printed in this volume.

may be soluble by radicals in a field of rationality R, a non-symmetric rational function of the roots α, β, γ, δ, ϵ which belongs to the metacyclic group $G = 0$ must have a value rational in R. The function adopted by Weber is

$$\psi \equiv \tfrac{1}{4}(\alpha\beta+\beta\gamma+\gamma\delta+\delta\epsilon+\epsilon\alpha-\alpha\gamma-\gamma\epsilon-\epsilon\beta-\beta\delta-\delta\alpha)^2,$$

and he shows that when the quintic is reduced to the form

$$y^5+py+q = 0,$$

ψ is a root of the sextic

$$(\psi-p)^4\,(\psi^2-6p\psi+25p^2)-3125q^4\psi = 0.$$

It is only in the case of a very limited class of quintics, however, that the coefficients p, q of the reduced form are rational in R. As a rule they belong to a soluble field R' of relative degree 6 with regard to R. Even in the simplest case when a, b, c, d, e, f are ordinary rational numbers P' is usually a sextic field, and it is by no means easy to determine whether a sextic equation whose coefficients belong to a sextic field has or has not a rational root in that field.

The object of this paper is to obtain the condition that the quintic should be metacyclic in R without the aid of operations in any wider field. Adopting the function

$$\phi = a^4[(\alpha-\beta)^2\,(\beta-\gamma)^2\,(\gamma-\delta)^2\,(\delta-\epsilon)^2\,(\epsilon-\alpha)^2$$
$$+(\alpha-\gamma)^2\,(\gamma-\epsilon)^2\,(\epsilon-\beta)^2\,(\beta-\delta)^2\,(\delta-\alpha)^2],$$

it is found that

$$\begin{aligned}\phi^6&+10j\phi^5+(35j^2+10k)\,\phi^4+(60j^3+30jk+10l)\,\phi^3\\&+(55j^4+30j^2k+25k^2+50jl)\,\phi^2\\&+(26j^5+10j^3k+44jk^2+59j^2l+14kl)\,\phi\\&+(5j^6+20j^2k^2+20j^3l+20jkl+25l^2) = 0,\end{aligned}$$

where $\qquad j = 5^3J,\quad k = 2^5.5^6K,\quad l = -\,2^{10}.5^9L,$

with the notation of Salmon's *Higher Algebra*, and for the quintic to be soluble by radicals in R this sextic must have a rational root. Simple conditions under which the group of the equation reduces to G_{10} or G_5, the two transitive self-conjugate sub-groups of G_{20}, are also obtained. In fact the group of any irreducible quintic with assigned numerical coefficients is fully determined.

On Integrals and Derivates with respect to a Function: Prof. W. H. Young.

(*This abstract was accidentally omitted in November.*)

From Lebesgue's theorem, that the derivates of the integral of a summable function differ from one another and the function only at a set of content zero, it follows that any summable function has the property that a *lu*- and an *ul*-function can be found which agree with it except at a set of content zero. In a paper on "A New Method in the Theory of Integration," presented to the Society in May, 1910, the author pointed out the possibility of constructing the whole theory of integration by fixing the attention primarily on *lu*- and *ul*-functions having that property. More recently in communications made to the Royal Society, and in a second paper published in these *Proceedings*, the author showed in detail how, in point of fact, such *lu*- and *ul*-functions could be chosen to have the given function between them. The proof was only given for the case of a bounded function, but, if the hypothesis is made that the function is summable, the same argument shows that the property is possessed by any unbounded summable function. This is pointed out in the present paper, and, as in the last mentioned paper, the more general case of integration with respect to a function of bounded variation is considered.

The argument employed, of a very simple character, is closely connected with one of the earlier definitions of integration, given by the author. In this definition the ideas of Darboux are fully developed, and expressed in a form which shows the inevitableness of the modern theory of integration, while at the same time it renders evident the point at which Riemann's considerations are inadequate.

The question naturally presents itself as to whether Lebesgue's important theorems as to the relations between the derivates of a function and their primitive and other fundamental theorems of the new calculus cannot be directly deduced from the definition referred to, without the use of Lebesgue's machinery; and, the further question, closely connected with the former, as to whether Lebesgue's theorem, from which the existence of the associated *lu*- and *ul*-functions can be deduced, is not itself a consequence of the existence of these functions, an existence which, as the author has shown, can be proved independently.

The advantage of such a mode of procedure is great, if one reflects that, in it, the use of transfinite numbers of the Theory of Sets of Points is excluded.

In a slightly different order of ideas, De la Vallée Poussin has obtained Lebesgue's results by means of a theorem, due to himself, elaborated evidently for the purpose in hand, which states the existence of functions majorant and minorant to a Lebesgue integral—namely, such continuous functions as have their derivates respectively all greater and all less than the integrand, and approach the integral as near as we please. De la Vallée Poussin's proof avoids transfinite numbers, but employs reasoning in the Theory of Sets of Points analogous in character to that adopted by Lebesgue in his proofs of his theorems.

In the present communication it is shown that this theorem of De la Vallée Poussin is an immediate deduction from the considerations exposed above, and that equally whether the independent variable be the continuous one contemplated by De la Vallée Poussin, or be taken to be any monotone increasing function. By availing himself therefore of De la Vallée Poussin's elegant exposition, the author is able to obtain a systematic solution of the problem he had proposed to himself, and the theorems obtained are at the same time proved in the most general form yet given to them by him.

Closely connected with the subject of integration is that of derivation, and the paper contains a number of results with respect to the derivates of semi-continuous functions which appear not to have been remarked, but which appear to be not without importance, in view of the fundamental part played by semi-continuous functions in the author's mode of treatment. Certain of them are moreover required in the course of the argument.

Thursday, February 11*th*, 1915.

Prof. A. E. H. LOVE, Vice-President, in the Chair.

Present also seventeen members.

The following papers were communicated :—

(i) The Zeros of the Riemann Zeta Function, (ii) On an Assertion of Tchebychef : Messrs. G. H. Hardy and J. E. Littlewood.

Apolar Generation of the Quartic Curve : Mr. W. P. Milne.

*On the Steady Rotation of a Solid of Revolution in a Viscous Fluid : Mr. G. B. Jeffery.

Relations amongst Bernoulli's and Euler's Numbers : Mr. S. T. Shovelton.

The last was an informal communication.

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* Printed in this volume.

ABSTRACTS.

The Zeros of the Riemann Zeta-Function: Messrs. G. H. Hardy and J. E. Littlewood.

It was conjectured by Riemann that the complex zeros of $\zeta(s) = \zeta(\sigma + ti)$ have all the real part $\frac{1}{2}$, but no proof of the correctness of this hypothesis has ever been published. The existence of *an infinity* of zeros on the line $\sigma = \frac{1}{2}$ was proved recently by Hardy (*Comptes Rendus*, April 6th, 1914), who based his proof on a definite integral of Cahen and Mellin, and a lemma concerning the elliptic theta-functions. Important simplifications in and extensions of this method have been effected by Landau, in a paper as yet unpublished. Landau has shown in particular that, for all sufficiently large values of T, there is a zero on the line whose imaginary part lies between T and T^{e}. From this it follows that the number $N_0(T)$ of such roots, whose imaginary part is positive and does not exceed T, is at least of order $\log\log T$.

If the Riemann hypothesis is true, then

$$N_0(T) \sim \frac{T \log T}{2\pi}.$$

Failing a direct proof of the hypothesis, it is a problem of the greatest interest to show that $N_0(T)$ is of as high an order as possible. In this paper we prove that, if ϵ is any positive number, there is a zero on the line whose imaginary part lies between T and $T + T^{\frac{1}{4}+\epsilon}$, for all sufficiently large values of T. The order of $N_0(T)$ is therefore at least

$$T^{\frac{3}{4}-\epsilon}.$$

We have entertained hopes of showing that the order is at least $T^{1-\epsilon}$; but the difficulties of the analysis have so far defied us. It hardly seems likely that any natural extension of our method can carry us further than this.

Our method is now quite free from any reference either to the Cahen-Mellin integral or the theory of elliptic functions. Writing

$$X(t) = f(\tfrac{1}{4} + ti) = f(s) = \pi^{-s} e^{-\frac{1}{2}(s-\frac{1}{4})\pi i}\, \Gamma(s)\, \zeta(2s) = -\frac{2e^{\frac{1}{2}\pi t}\, \Xi(2t)}{\frac{1}{4} + 4t^2},$$

we prove that if $H = T^{\frac{1}{4}+\epsilon}$, where ϵ is a sufficiently small positive number, then

$$\int_T^{T+H} X(t)\, dt = O(T^{\delta})$$

for all positive values of δ, and that, if $\delta < \epsilon$, this equation is inconsistent with the hypothesis that $X(t)$ is of the same sign throughout the interval $(T, T+H)$.

On an Assertion of Tchebychef: Messrs. G. H. Hardy and J. E. Littlewood.

A letter of Tchebychef (*Œuvres*, Vol. 1, p. 697) contains the assertion that the function

$$F(y) = -\Sigma(-1)^{\frac{1}{2}(p-1)} e^{-py} = e^{-3y} - e^{-5y} + e^{-7y} + e^{-11y} - \ldots,$$

where y is positive, p a prime, and the sign is positive or negative according as p is of the form $4m+3$ or of the form $4m+1$, tends to infinity as $y \to 0$. No proof of the truth of this assertion has been published: "da bis heute nicht entschieden ist, ob dies der Fall ist oder nicht, so ist zu vermuten, dass Tchebychef's unpublizierte Begründung seiner Behauptung einen Irrtum enthalten hat" (Landau, *Rendiconti di Palermo*, Vol. 24, p. 156).

We have proved that Tchebychef's assertion is correct if all the complex zeros of the function

$$L(s) = 1^{-s} - 3^{-s} + 5^{-s} - \ldots,$$

have their real part equal to $\frac{1}{2}$. This hypothesis stands on the same footing as the Riemann hypothesis concerning the zeros of $\zeta(s)$. It is highly probable that, if this hypothesis is false, then Tchebychef's assertion is also false; but of this we have no rigorous proof.

It follows from a well known formula of Cahen and Mellin that

$$(1) \qquad f(y) = -\sum_{p,\, m} (-1)^{\frac{1}{2}m(p-1)} \log p\, e^{-p^m y} = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \Gamma(s)\, y^{-s} \frac{L'(s)}{L(s)}\, ds,$$

if $y > 0$ and $\kappa > 1$. An application of Cauchy's theorem leads to the result

$$(2) \qquad f(y) = A + o(1) + \Sigma\, \Gamma(\rho) y^{-\rho},$$

where A is a constant, ρ a complex zero of $L(s)$, and $y \to 0$ by positive values.

Assuming that every ρ has the real part $\frac{1}{2}$, it is not difficult to show that

$$(3) \qquad \Sigma\, |\Gamma(\rho)\, y^{-\rho}| < \tfrac{1}{50} y^{-\frac{1}{2}}.$$

Further, writing

(4)

$$f(y) = -\sum_{p}(-1)^{\frac{1}{2}(p-1)}\log p\, e^{-py} - \sum_{p}\log p\, e^{-p^2 y} - \sum_{p,\ m>3}(-1)^{\frac{1}{2}m(p-1)}\log p\, e^{-p^m y}$$

$$= f_1(y)+f_2(y)+f_3(y),$$

say, we have

(5) $$f_2(y) \sim -\tfrac{1}{2}\pi^{\frac{1}{2}}y^{-\frac{1}{2}}, \quad f_3(y) = O(y^{-\frac{1}{3}}).$$

From (3), (4), and (5), it follows that

(6) $$f_1(y) = \log 3\, e^{-3y} - \log 5\, e^{-5y} + \log 7\, e^{-7y} + \log 11\, e^{-11y} - \ldots > Ky^{-\frac{1}{2}},$$

where K is a positive constant. And, from (6), it is easily deduced that

$$F(y) = e^{-3y}-e^{-5y}+e^{-7y}+e^{-11y}-\ldots > \frac{K}{\sqrt{y}\log(1/y)}.$$

We have thus proved in one direction a good deal more than was asserted by Tchebychef.

On the Steady Rotation of a Solid of Revolution in a Viscous Fluid: Mr. G. B. Jeffery.

The problem discussed in this paper is the motion of a viscous fluid generated by the slow rotation of a solid of revolution about its axis. The characteristic of this motion is that the velocity is everywhere perpendicular to the axis. If z, ϖ, ϕ are cylindrical coordinates, and if there is a solution of Laplace's equation of the form $f(z\varpi)\sin\phi$, it is shown that there is a solution of the present problem given by $v = f(z, \varpi)$, where v is the velocity of the fluid. There are solutions of Laplace's equation of the required type in terms of spherical harmonics, Bessel functions, and oblate and problate spheroidal harmonics. From these we obtain the solutions for the fluid motion generated by the rotation of

(i) Concentric spheres. This solution is well known.

(ii) Confocal problate ellipsoids of revolution.

(iii) Oblate ellipsoids of revolution whose meridian sections are confocal.

(iv) A circular disc. Two solutions of this problem are obtained: one as a particular case of (iii), and one in terms of Bessel functions.

All these solutions, with the exception of the second solution of (iv), are in finite terms, harmonics of the first degree only being necessary. The method is then applied to the investigation of the mutual influence of two non-concentric spheres rotating about their common diameter with different angular velocities. The extent of this influence may be gathered from the following examples :

(i) If two equal spheres A, B are surrounded by a viscous fluid of infinite extent, and A is caused to rotate about the common diameter with a prescribed spin while B is left free to turn under the stresses on its surface ; and if the two spheres are so close together that the distance between their surfaces is but one fiftieth of the distance between their centres ; then B will rotate with about one eighth of the spin of A.

(ii) If a sphere in the presence of an infinite fixed plane rotates about an axis perpendicular to the plane, and if the shortest distance between the surfaces is one fiftieth of the radius of the sphere, then the couple which resists the rotation of the sphere is greater than what it would be in the absence of the plane only by about 12 per cent.

Thursday, March 11*th*, 1915.

Prof. Sir J. LARMOR, President, in the Chair.

Present eighteen members and two visitors.

Messrs. T. Kubota and C. E. Weatherburn were elected members.

The President announced the deaths of Mr. G. W. Hill, who was an honorary member of the Society, and of Prof. H. W. Lloyd Tanner.

The following papers were communicated :—

*On some Theorems in the Theory of Series of Orthogonal Functions : Prof. E. W. Hobson.

Investigations in the Theory of the Partition of Numbers by a New Method of Partial Fractions : Major P. A. MacMahon.

Reciprocal and Parallelogram Linkages : Col. R. L. Hippisley.

*A Pseudo-Sphere whose Equation is Expressible in Terms of Elliptic Functions : Dr. J. R. Wilton.

Circles and Spheres associated with a Triangle, Orthocentric Tetrahedron, etc. : Mr. T. C. Lewis.

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* Printed in this volume.

ABSTRACT.

Circles and Spheres associated with a Triangle, Orthocentric Tetrahedron, etc.: Mr. T. C. Lewis.

1. Let ABC be a triangle, P the orthocentre.

Let A, B, C, P denote the four orthogonal circles having their centres at these points.

Let I, E_1, E_2, E_3 denote circles concentric with the inscribed and escribed circles of the triangle, but with radii increased in the ratio $\sqrt{2}:1$.

The angle of intersection of two circles, spheres, etc., is taken to be the angle between the radii drawn from a point of intersection to the two centres.

The cosine of the angle of intersection of two circles, etc., say A and E_1, will be denoted by AE_1 for short. Hence, if $AB = CD$, the angles of intersection are equal in the two cases; if $AB = -CD$, the angles are supplementary.

The following results are obtained.

Any group of three circles similar to A, I, E_1 are co-axal, *i.e.*, have a common chord.

The following seven circles, viz., the four circles I, E_1, E_2, E_3, and the circles A, B, C, have a common radical centre, viz., P.

Also $$AI = AE_2 = AE_3 = -AE_1,$$

that is, the circles I, E_2, E_3 intersect A at the same angle θ_1, whilst E_1 and A intersect at an angle $\pi-\theta_1$. If the similar intersections with the circles B and C be θ_2 and θ_3, then

$$\cos^2\theta_1+\cos^2\theta_2+\cos^2\theta_3 = 1.$$

The line AP is the locus of centres of circles which make equal angles of intersection with the circles I, E_2, E_3.

Moreover, $$IE_1 = -E_2E_3, \;\ldots,$$

i.e., the angle of intersection of any two of the four circles I, E_1, E_2, E_3 is supplementary to that of the other two.

Also $$IE_1+IE_2+IE_3 = 1 = -\Sigma EE',$$

i.e., the sum of the cosines of the angles of intersection of I with the other three circles is unity, and the sum of the cosines of the angles of intersection of all pairs of circles that may be chosen from the four circles is zero.

In fact, if ϕ_1 be the angle of intersection of I and E_1, etc.,

$$\cos\phi_1 = \cos^2\theta_2+\cos^2\theta_3-\cos^2\theta_1 = 1-2\cos^2\theta_1,$$

whence also it appears that θ_1 and $\frac{1}{2}\phi_1$ are complementary, *i.e.*, at a common point of A, I, E_1 the tangent to A bisects the angle between the radii of I and E_1.

2. If $ABCD$ be an orthocentric tetrahedron, and P the orthocentre, let A, B, C, D, P be used to denote the five mutually orthogonal spheres with centres at these points.

Let I, E_1, E_2, E_3, E_4, F_{hk} denote eight spheres concentric with the spheres which touch, on one side or the other, the faces of the tetrahedron, but with radii increased in the ratio $\sqrt{3} : 1$. The suffixes indicate the faces which any sphere touches externally. There are three spheres F which touch two faces externally, and they are either (i) F_{12}, F_{13}, F_{14} *or* (ii) F_{23}, F_{34}, F_{42}, where the vertex A is taken to be that one whose corresponding number, 1, occurs in all or in none of the suffixes.

The following results are obtained.

Any group of three spheres such as A, I, E_1 have a common section.

The following twelve spheres, viz., the eight spheres I, E, F, and the four spheres A, B, C, D, have a common radical centre, viz., P.

Also the spheres I, E_2, E_3, E_4, and in (ii) F_{23}, F_{34}, F_{42} intersect A at the same angle θ_1, whilst E_1 intersects it at an angle $\pi-\theta_1$; while I, E_2, E_3, E_4 in (i) intersect A at the same angle θ_1, and the remaining four spheres intersect it at an angle $\pi-\theta_1$. If the similar intersections with the spheres B, C, D be θ_2, θ_3, θ_4, then

$$\cos^2\theta_1+\cos^2\theta_2+\cos^2\theta_3+\cos^2\theta_4 = 1.$$

The line AP is the locus of centres of spheres which make equal angles of intersection with the spheres I, E_2, E_3, E_4.

If ϕ_1 be the angle of intersection of I and E_1, etc., then θ_1 and $\frac{1}{2}\phi_1$ are complementary.

Also $$IE_1 = E_2F_{12} = E_3F_{13} = E_4F_{14} \text{ in (i)},$$

and $$IE_1 = -E_2F_{34} = -E_3F_{42} = -E_4F_{23} \text{ in (ii)}.$$

Thus in (i) the angles of intersection of the pairs I, E_1; E_2, F_{12}; E_3, F_{13}; E_4, F_{14} are equal to one another, while in (ii) the angles of intersection of the pairs E_2, F_{34}; E_3, F_{42}; E_4, F_{23} are equal but complementary to the angle of intersection of I and E_1.

The suffixes 2, 3, 4 are by supposition not admitted or omitted in the same way as the suffix 1; and we obtain

$$IE_2 = E_1F_{12} = -E_3F_{14} = -E_4F_{13} \text{ in (i)},$$

and $$IE_2 = -E_1F_{34} = E_3F_{23} = E_4F_{42} \text{ in (ii)}.$$

Also $\Sigma IE = 2$, and $\Sigma EE' = 0$ $[E_1E_2+E_3E_4 = 0, \ldots]$,

i.e., the sum of the cosines of the angles of intersection of I with the spheres E is 2, and the sum of the cosines of the angles of intersection of pairs of spheres E is zero.

So
$$E_1E_k = IF_{1k} \text{ in (i)},$$
$$E_hE_k = IF_{hk} \text{ in (ii)},$$

where h and k are other than 1. And
$$F_{1h}F_{1k} = E_hE_k \text{ in (i)},$$
$$F_{hk}F_{hl} = E_kE_l \text{ in (ii)}.$$

There are very many other similar geometrical relations.

3. In the same manner corresponding relations are established in space of n dimensions, the radii of the n spheres under consideration being $\sqrt{n}$ times those of the inscribed and escribed n spheres.

We obtain
$$\Sigma IE = n-1,$$
$$\Sigma EE' = \tfrac{1}{2}n(n-3), \ \ldots, \ \ldots .$$

The relations between the radii of the n-spheres $I, E, F, \ldots$ may also be easily written down.

When n is odd there are always two sets of cases to be considered, as in three dimensions.

Thursday, April 22nd, 1915.

Prof. Sir J. LARMOR, President, and afterwards Prof. A. E. H. LOVE, Vice-President, in the Chair.

Present fifteen members.

Mr. C. Walmsley was elected a member.

The President announced the death of Prof. W. G. Adams, who had been a member of the Society for nearly thirty years.

The following papers were communicated:—

*The Influence of the Oceanic Waters on the Law of Variation of Latitudes: Sir J. Larmor.

Note on Dirichlet's Divisor Problem: Mr. G. H. Hardy.

*Division of the Lemniscate into Seven Equal Parts: Mr. G. B. Mathews.

Note on a New Form of Closed Linkage: Col. R. L. Hippisley.

Col. Hippisley exhibited a model in connexion with his paper.

ABSTRACTS.

The Influence of the Oceanic Waters on the Law of Variation of Latitude: Sir Joseph Larmor.

The prolongation of the periodic time of the small free orbital motion of the Pole over the Earth's surface from 304 to 428 days has been recognized to be due to the centrifugal strain of the Earth's rotation changing in step with the changing axis of rotation. This regular circular precession is found to be strongly disturbed by irregular surface displacements of terrestrial masses. But among these disturbances the adaptation of the ocean surface to the changing axis ought not to be included, for being synchronous with the precessional motion it must affect its period, and so fundamentally alter it instead of merely disturbing it. It is roughly estimated that if the Earth were elastically unyielding, the effect of the existing ocean would be to lengthen the period of free precession from 304 to about 332 days. The remainder of the actual increase to about 428 days would be, as now, ascribed to elastic centrifugal strain of the solid Earth, and the necessary slight revision of current estimates of its yielding is made on this basis. The question is broached, what would be the course of history of a planet so nearly spherical that the incumbent ocean would destroy secular stability for all possible axes of rotation?

*Division of the Lemniscate into Seven Equal Parts: Mr. G. B. Mathews.

The division of the lemniscate into seven equal arcs depends upon an equation of degree 24 with ordinary integral coefficients. It is shown in the paper how, by a chain of three auxiliary pure quadratics, this equation can be broken up into eight cubics, one of which corresponds to all the real solutions of the problem.

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* Printed in this volume.

Note on a New Form of Closed Linkage: Col. R. L. Hippisley.

The linkage consisted of a series of distinct and dissimilar 3-bar linkages with unequal links.

The formulæ given on p. 205, *Proc. London Math. Soc.*, Ser. 2, Vol. 13, and two others, not there stated, but equally verifiable, give rise to the following vector equations

$$\left.\begin{array}{ll}\alpha_{11}+\beta_5+\delta_1=D_1, & \alpha_9+\beta_{11}+\delta_1=D_1\\ \alpha_1-\gamma_3-\delta_1=D_2, & \alpha_8-\gamma_{11}-\delta_1=D_2\\ \alpha_5+\gamma_1+\epsilon_1=D_3, & \alpha_1+\gamma_4+\epsilon_1=D_3\\ \beta_1-\delta_3+\epsilon_1=D_4, & \beta_8-\delta_{11}+\epsilon_1=D_4\\ \beta_{11}+\gamma_5-\epsilon_1=D_5, & \beta_9+\gamma_{11}-\epsilon_1=D_5\end{array}\right\}, \qquad \text{(A)}$$

where $\alpha_s, \beta_5, \dots$ represent $r_1 e^{\iota\alpha}, r_2 e^{\iota\beta_s}, \dots,$ and

$$\begin{aligned}D_1&=\tfrac{1}{3}(\ d_1+2d_2+\ d_3+2d_4+\ d_5),\\ D_2&=\tfrac{1}{3}(\ 2d_1+\ d_2-\ d_3-2d_4-\ d_5),\\ D_3&=\tfrac{1}{3}(\ 2d_1+\ d_2+2d_3+\ d_4-\ d_5),\\ D_4&=\tfrac{1}{3}(\ d_1+2d_2+\ d_3-\ d_4-2d_5),\\ D_5&=\tfrac{1}{3}(-d_1+\ d_2+2d_3+\ d_4+2d_5).\end{aligned}$$

The equations (A) represent linkages which can be constructed upon bases lying on $O_1 O_4$ (see Fig. 3 of article quoted). The terms on the left of each equation can be permuted in six ways, all indicating different linkages. There are therefore 60 typical vector equations and 11 variations of their subscripts, or in all 660 equations. After a little acquaintance with them it is not difficult to pick out a series like the following

$$\begin{aligned}+\alpha_1+\beta_7+\delta_{11}&=\ \ D_1,\\ -\delta_{11}+\alpha_{11}-\gamma_9&=\ \ D_2,\\ +\gamma_9+\alpha_{10}+\epsilon_5&=\ \ D_3,\\ -\epsilon_5-\alpha_3-\gamma_6&=-D_3,\\ +\gamma_6-\alpha_2+\delta_8&=-D_2,\\ -\delta_8-\beta_{11}-\alpha_1&=-D_1,\end{aligned}$$

in which each begins with the vector that its predecessor ended with, but with sign reversed; the last one ending with the negative of the vector that leads the first one; and in which the sum of the constants vanishes. This compound linkage will close and will have one degree of freedom.

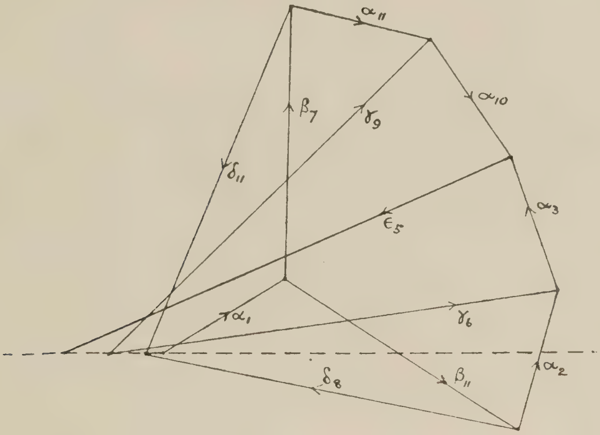

FIG. 1.

It will be observed that there is a kind of symmetry in the arrangement of this linkage (Fig. 1). But quite unsymmetrical ones can be put together, such as

$$\begin{aligned}
+\alpha_1 + \beta_7 + \delta_{11} &= +D_1,\\
-\delta_{11} + \alpha_{11} - \gamma_9 &= +D_2,\\
+\gamma_9 + \alpha_{10} + \epsilon_5 &= +D_3,\\
-\epsilon_5 + \gamma_{10} + \beta_{10} &= +D_5,\\
-\beta_{10} + \delta_8 - \epsilon_9 &= -D_4,\\
+\epsilon_9 - \beta_9 - \gamma_4 &= -D_5,\\
+\gamma_4 - \alpha_2 + \delta_3 &= -D_2,\\
-\delta_3 + \beta_1 + \epsilon_1 &= +D_4,\\
-\epsilon_1 - \gamma_4 - \alpha_1 &= -D_3,
\end{aligned}$$

which will close although the sum of the constants does not vanish because the first and the last radial links are always equal and parallel and can be connected by a link at their extremities, the series being, in fact, completed by the linkage

$$+a_1 - D_1 - a_1 = -D_1,$$

which is a parallelogram (Fig. 2).

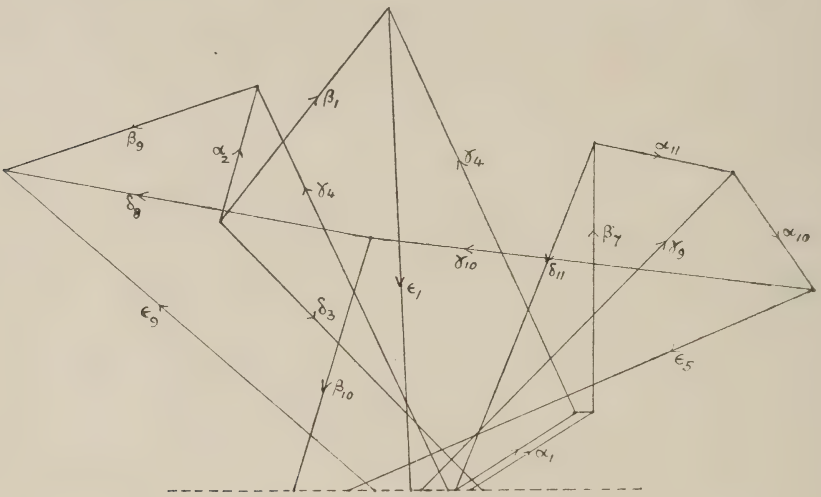

FIG. 2.

A large number of additional formulæ are obtainable from these linkages relating to the amplitudes of the elliptic arguments $u+2rK/n$ and $u+4rK/n$.

Thursday, May 13*th*, 1915.

Prof. Sir J. LARMOR, President, in the Chair.

Present fifteen members.

The following papers were communicated:—

*The Diffraction of Waves by a Wedge: Dr. Bromwich.

An Invariant Modular Equation of the Fifth Order: Mr. W. E. H. Berwick.

*A Direct Method in the Multiplication Theory of the Lemniscate Functions and other Elliptic Functions: Mr. G. B. Mathews.

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* Printed in this volume.

Dr. Bromwich made an informal communication on "The Bright Spot in the Shadow of a Circular Disc," and Mr. S. T. Shovelton on "A New Method for Calculating Bernoulli's Numbers to any Order."

ABSTRACTS.

An Invariant Modular Equation of the Fifth Order: Mr. W. E. H. Berwick.

This paper contains the expanded form of the six-six algebraic equation between the modular functions

$$J \equiv j(\omega) \equiv 2^8 \frac{(1-k^2+k^4)^3}{k^4(1-k^2)^2} \equiv \frac{1}{a^2}+744+\ldots,$$

and $$I \equiv j(5\omega).$$

A Direct Method in the Multiplication Theory of the Lemniscate Function: Mr. G. B. Mathews.

In this paper the calculation of $\psi_m(u)$, where m is a real or complex odd integer, is made to depend upon two ordinary differential equations of the second order, the independent variable being $\wp(u;\ 4,\ 0)$. The explicit forms of several coefficients are given, and a number of particular values of ψ_m are tabulated. It is pointed out that ψ_m satisfies an ordinary differential equation of the fourth order, and reference is made to previous work by Gauss and Eisenstein.

Thursday, June 10th, 1915.

Prof. Sir J. LARMOR, President, in the Chair.

Present eleven members.

Messrs. G. R. Goldsbrough and H. Jeffreys were elected members.

The President announced the death of Prof. Morgan W. Crofton, a former member of the Society, member of the Council and Vice-President.

The following papers were communicated :—

On Periodic Irrotational Waves at the Surface of Deep Water: Prof. W. Burnside.

A Theorem on Series of Orthogonal Functions : Prof. M. Kuniyeda.

The Effect on the Tides of the Variation in the Depth of the Sea : Mr. G. R. Goldsbrough.

Oscillations near an Isosceles Triangle Solution of the Problem of Three Bodies, as the Finite Masses become Unequal : Prof. D. Buchanan.

LIBRARY

Presents.

Between August, 1914, and July, 1915, the following presents were made to the Library, from their respective Authors or Publishers:—

Burgess, H. T.—"Proof of Fermat's celebrated Unproved Theorem."
Carmichael, R. D.—"Theory of Numbers."
Carslaw, H. S.—"Discovery of Logarithms by Napier of Merchiston."
Dickson, L. E.—"Elementary Theory of Equations."
Hill, M. J. M.—"The Theory of Proportion."
Hilton, H.—"Homogeneous Linear Substitutions."
Marsh, H. W.—"Constructive Text Book."
Moritz, R. E.—"Text Book on Spherical Trigonometry."
Neuman, C.—"Über die Dirichletche Theorie der Fourierschen Reihen."
Norland, N. E.—"Sur les Séries de Facultés."
Prasad, Chandi.—"First Course of Coordinate Geometry."
Schaller, T. G.—"Beweis der Richtigkeit des grossen Fermatschen Satzes."
Smolar, G.—"Beweis des grossen Fermatsches Satzes über die Gleichung $a^m + b^m = c^m$."
Stieltjes, T. J.—"Œuvres Complètes."

Calcutta: Indian Association for the Cultivation of Science, Bulletin, nos. 10–12, 1914.
Calcutta: Indian Engineering, vols. 56, 57, 1914.
Cape Town: South African Journal of Science, vol. 10, nos. 9–14, 1914; vol. 11, nos. 2–9, 1914–15.
London: Educational Times, vol. 67, nos. 639–644, 1914; vol. 68, nos. 645–650, 1915.
London: Educational Times, Math. Questions and Solutions, new series, vol. 25, 1914; vol. 26, 1914.
London: Mathematical Gazette, vol. 7, nos. 110–114, 1914; vol. 8, nos. 115–117, 1915.
London: Nautical Almanac for 1917.
Madras: Indian Mathematical Society, Journal, vol. 5, no. 6, 1913; vol. 6, 1914; vol. 7, nos. 1–3, 1915.
Paris: L'Enseignement Math., ann. 16, nos. 3–5, 1914; ann. 17, nos. 1, 2, 1915.
Sendai: Tôhoku Imperial University, Science Reports, vol. 3, nos. 4–6, 1914; vol. 4, no. 1, 1915.
Sendai: Tôhoku Mathematical Journal, vol. 5, nos. 3, 4; vol. 6, nos. 1–4; vol. 7, nos. 1, 2.
Tokyo: Sūgaku-Butsurigaku, vol. 7, nos. 15–22, 1914; vol. 8, nos. 1–4, 1915.

Exchanges.

Between August, 1914, and July, 1915, the following exchanges were received:—

American Journal of Mathematics, vol. 36, nos. 1-4, 1914; vol. 37, nos. 1, 2, 1915.
American Mathematical Society, Transactions, vol. 15, nos. 3, 4, 1914; vol. 16, nos. 1, 2, 1915.

American Mathematical Society, Bulletin, vol. 20, nos. 9, 10, 1913–14; vol. 21, nos. 1–9, 1914–15.
American Philosophical Society, Proceedings, vol. 53, nos. 213–215, 1914.
Amsterdam: Nieuw Archiev, deel 11, stuk 1, 1914.
Amsterdam: Wiskundige Opgaven, deel 11, stuk 6, 7, 1914.
Belgique: Académie Royale des Sciences, Bulletin, 1914, nos. 2–4.
Berlin: Journal für die Mathematik, bd. 144, 1914.
Bordeaux: Société des Sciences, Procès-Verbaux, 1912–13.
Calcutta: Mathematical Society, Journal, vol. 2, nos. 1, 2.
Cambridge Philosophical Society, Proceedings, vol. 17, pt. 6, 1914; vol. 18, pts. 1–3, 1914.
Cambridge Philosophical Society, Transactions, vol. 22, pts. 5–7, 1914.
Cambridge, Mass.: Annals of Mathematics, vol. 16, nos. 1, 2, 4, 1914.
Catania: Accademia Gioenia, Bollettino, ser. 2, fasc. 31, 1914.
Dublin: Index to Serial Publications of Royal Irish Academy.
France: Société Mathématique, Bulletin, tome 41, fasc. 4, 1913; tome 42, fasc. 1, 2, 1914.
France: Société Mathématique, Comptes Rendus des Séances, 1913–14.
Holland: Archives Néerlandaises, tome 3, ser. A, 1914.
København: Kgl. Danske Vidensk. Selsk., Forhandlinger, Oversigt, 1914, nos. 3–6; 1915, nos. 1–6.
Leipzig: Beiblätter zu den Annalen der Physik, bd. 38, hefte 12–14, 1914.
Leipzig: K. Sächsische Gesell., Math. Klasse, Berichte, 1913, nos. 4, 5; 1914, no. 1.
Livorno: Periodico di Matematica, anno 29, fasc. 2–6, 1914; anno 30, fasc. 1–4, 1914–15.
Livorno: Periodico di Matematica, Supplemento, anno 17, fasc. 3–9, 1914; anno 18, fasc. 1–7, 1914–15.
London: Nature, vol. 93, no. 2339; vols. 94, 95, 96.
London: Royal Society, Proceedings, Series A, vol. 90, nos. 619–622; vol. 91, nos. 623–629, 1914–15.
London: Royal Society, Transactions, Series A, vol. 214, 1914.
London: Physical Society, Proceedings, vol. 26, pts. 3–5, 1914; vol. 27, pts. 1–3, 1914–15.
Napoli: Accademia delle Scienze, Rendiconti, vol. 19, fasc. 6–12, 1913; vol. 20, fasc. 1–6, 1914.
Napoli: Accademia delle Scienze, Atti, vol. 15, 1914.
Palermo: Rendiconti del Circolo Matematico, tome 37, 1914.
Paris: Bulletin des Sciences Mathématiques, Juin-Déc., 1914; Jan.–Juillet, 1915.
Paris: Journal de l'Ecole Polyt., cah. 17, 1913.
Porto: Academi Polytechnica de Porto, vol. 9, nos. 2–4, 1914; vol. 10, no. 1, 1915.
Roma: Reale Accademia dei Lincei, Rendiconti, vol. 23, sem. 10–12, 1914.
Torino: R. Accademia delle Scienze, Atti, vol. 49, disp. 8–15, 1914.
Toulouse: Faculté des Sciences, Annales, sér. 3, tome 4, 1912.
Washington: Naval Observations for Year 1914.
Zurich: Vierteljahrsschrift, hefte 1–4, 1914; hefte 1, 2, 1915.

OBITUARY NOTICES

MORGAN WILLIAM CROFTON.*

THE late Morgan William Crofton, F.R.S., though a man of reflective and retiring disposition, and therefore not well known to the public, yet had an unusually remarkable and interesting career.

He was born in Dublin in 1826, eldest son of Rev. W. Crofton, Rector of Skreen, Sligo, a living which must have been held just previously by the father of Sir George Gabriel Stokes, the great mathematician and physicist, who was born at Skreen in 1819.†

Taking his degree in Trinity College, Dublin, with the highest mathematical honours in 1848, the late Dr. Johnstone Stoney being second on the list, he was precluded from standing for Fellowship in those days on account of his having joined the Roman Catholic Church.

He held from 1849 to 1852 the Professorship of Natural Philosophy at the Queen's College, Galway, then newly founded, and later he worked in some of the Jesuit educational institutions in France; he was succeeded at Galway by Dr. G. J. Stoney, who remained there until his appointment as Secretary of the Queen's University in Ireland in 1857.

Coming to Woolwich, Crofton made the acquaintance of the late J. J. Sylvester, F.R.S., who was then Professor of Mathematics at the Royal Military Academy; on his recommendation Mr. Crofton was appointed an Instructor on the Mathematical Staff. Prof. Sylvester, as was his manner, stimulated into activity the exceptional mathematical powers of his colleague, and communicated some of his original work to the Royal Society ("Local Probability," *Phil. Trans.*, 1868, "Law of Error," *Phil. Trans.* 1870), with the result that at his first nomination, in 1868, he was elected a Fellow. He was a frequent contributor of short elegant papers to the early volumes of *Proc. Lond. Math. Soc.* When, in 1870, Prof. Sylvester vacated his office, the other members of the Mathematical Staff at Woolwich are said to have written to the Governor (Sir Lintorn Simmons) to say that if Mr. Crofton, who was the junior member, applied for the Professorship none of them would do so. He was appointed, and held the position till he retired under the age limit in 1884.

Many distinguished officers of the Royal Artillery and Royal Engineers passed under Prof. Crofton at the Royal Military Academy; among them may be mentioned Lord Kitchener, Sir Leslie Rundle, Sir Reginald Wingate, Sir Matthew Nathan, Major P. A. MacMahon, F.R.S., and also the late Prince Imperial of France, with whom Prof. Crofton's religious and French associations formed a special tie.

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* The writer is indebted for valuable material to the Rev. James White, who was a colleague of Prof. Crofton, at Woolwich, from 1870 to 1878.

† See Obituary Notice by Lord Rayleigh, *Roy. Soc. Proc.*, 1903, or Stokes' *Collected Papers*, Vol. v.

While at Woolwich he wrote two brief manuals for the cadets, which exhibit the high level of the teaching to which they were supplementary. One of them, on Applied Mechanics, belongs to a period when the only adequate representative of its class in English was the treatise of Macquorn Rankine, for whose work Crofton had a high admiration. His own methods of demonstration were terse and lucid in their direct geometrical presentation, just what is wanted for the training of practical officers, who require to have command of the ideas of the subject but may be distracted by analytical processes. Especially elegant and original was his treatment of the theory of structures—for example, that of jointed frames, which was just then being worked up into the subject of graphical mechanics. In fact, these short works have lost none of their value from the lapse of time.

The subject of most of his original memoirs was that beautiful combination of geometry with the integral calculus to which has been given (perhaps by himself) the name of Local Probability. In these investigations the elegant geometrical touch which he had acquired in the Dublin school found full scope. The whole subject of Probabilities fascinated him, as it had fascinated the somewhat kindred mind of Pascal; and he always expressed high admiration for Laplace's great treatise. His own lucid and elegant exposition of the subject of Probability in the ninth edition of the "Encyclopædia Britannica" might well be recovered from oblivion, and should, in fact, survive as a classic on the subject.

When the Royal University of Ireland came into being in 1882, Crofton's services were obtained as one of the Fellows on the Catholic side. For some ten years following, the present writer had the advantage as an examiner colleague of sharing in his friendship and intellectual interests, and observing the subtle play of his mathematical tastes. Considerations of health led to the abandonment of this work, and the last twenty years of his life were spent in retirement.

Prof. Crofton had a large acquaintance with classical and general literature. This, in addition to a most amiable and kind-hearted disposition, endeared him greatly to those who had the privilege of his acquaintance.

He had two sons by his first marriage, one a Jesuit priest engaged in educational work, who has been active in the astronomical observatories of the Order; the other an officer of the Royal Engineers, who died of enteric in India before the age of twenty-one. His second wife survives him.

J. L.

GEORGE WILLIAM HILL.

GEORGE WILLIAM HILL was born in New York City on March 13, 1838, of English-Huguenot stock. Having shown at school some signs of that mathematical ability which was afterwards so conspicuous in him, at the age of seventeen he was sent to Rutger's College, New Brunswick, N.J. His professor there was Dr. Theodore Strong, a friend of Nathaniel Bowditch, the translator of Laplace's *Mécanique Céleste;* while still an undergraduate Hill read in his professor's library this and other works by the great French mathematicians—Laplace, Lagrange, Poisson, Legendre, and others. Strong was keenly interested in applied mathematics of the older school, and his influence helped to turn the thoughts of his pupil Hill in the same direction. Before he was twenty-one Hill began original research, and his first published paper, which contains an attempt to extend Laplace's investigation on the figure of the earth, already showed that mastery of analytical resources and facility in exposition which distinguished his later work.

In 1861, soon after leaving college he accepted a position in the *Nautical Almanac* Office, then situated at Cambridge, Mass., and subsequently at Washington. Here, under the successive directorships of Runkle and Newcomb, he laboured for thirty years on the great problems of dynamical astronomy. In his earlier years, indeed, he published a few papers on mathematics unconnected with astronomy, but as time went on he became exclusively absorbed in the latter subject; here also he gradually tended to leave the side more directly connected with observations and their computation, and to devote himself to the distinctively theoretical branches of gravitational astronomy.

After retiring from the *Nautical Almanac* Office in 1892, he was able to give his full time to those favourite studies which had formerly been prosecuted largely in the hours of leisure from official duties. These twenty-two years, during which he was "out of harness," were spent for the most part in much seclusion at his quiet home at West Nyack, where he died on April 16, 1914, at the age of seventy-six. During this interval, however, he gave considerable time and service to the American Mathematical Society, of which he was President in 1894–96; and he served as lecturer on celestial mechanics in Columbia University during the academic years 1898 to 1901.

Hill never married, and he entered very little into society of any kind.

His mode of living was of great simplicity, and he avoided any undertakings in ordinary spheres of life which might interfere with his astronomical investigations. His labours were broken only by long walks around his home, by occasional visits to his more intimate friends, and by infrequent vacation tours to distant parts of the world. Those who knew him well describe him as being, in the privacy of his own circle, a delightful companion and singularly attractive in personality. He had a strong liking for geographical exploration, and made two trips into Canada for this purpose: one into the Hudson Bay region and one into the North-west. It was during the latter journey that he worked out his famous solution of the problem involving an infinite determinant, a solution (in Poincaré's words) *aussi originale que hardie*. The value of the work was immediately recognized in this country through the notice of J. C. Adams; and it presently appeared that Adams had himself in lectures at Cambridge treated the similar problem presented by the progression of the lunar node substantially in the same manner, though with less elaboration of analytical development. Hill visited this country in 1892, and received the honorary doctorate of Cambridge. This was one of many honours bestowed upon him, in Europe as well as in his own country. In this country he received the Copley medal and foreign membership of the Royal Society, the Gold Medal and associateship of the Royal Astronomical Society, and in 1907 he was elected to honorary membership of our own Society.

Over the long period from his entry in the *Nautical Almanac* Office until his death his output of original work remained at a high level both of quality and quantity. His papers appeared mainly in the *Nautical Almanac* Office publications and in American journals; those written up to about 1905 have been republished in four volumes of his *Collected Mathematical Works* by the Carnegie Institution of Washington, and his subsequent papers will suffice to make a fifth volume. Among his early writings were a few which appeared in *The Analyst*, which are of some general interest to mathematicians; they consist of occasional work on such topics as the theory of numbers, the theory of equations, finite differences, and the theory of gravitation. But the works by virtue of which his name will endure are on subjects which must always remain little known to most, even among mathematicians, and the number in a single generation who will study his works is never likely to be anything but small. Nevertheless, in his chief work, the theory of the moon's motion, he was led to a new development in mathematics, the infinite determinant already mentioned, which is doubtless destined to prove as useful in other spheres as in that for which it was invented, and which is of considerable interest to the pure mathematician.

In this new lunar theory, by the introduction of several ideas of considerable novelty, he laid down the lines of a new method of calculating the moon's motion, which bids fair to hold the field against the works of all his predecessors in the attack on this great problem. The treatment differed profoundly in many respects from that of Delaunay, Hansen, and the earlier dynamical astronomers. Instead of polar coordinates referred to fixed axes, rectangular coordinates referred to moving axes were introduced, leading to differential equations in a simple algebraic form suitable for solution by infinite series, and the convergence was increased by the use of a new parameter. A new orbit of reference, called the variational curve, obviated the difficulties met with in using the usual reference orbit (the Keplerian ellipse), on account of the troublesome motion of the perigee of the latter; this is because the variational curve contains a most important part of the solar perturbation, called the "variation." Apart from its value in the lunar theory, moreover, the variational curve was the starting point of a general theory of periodic orbits in the problem of three bodies, which has been developed by Poincaré, Darwin, Brown, Moulton, and others. As regards the lunar theory itself, one of the best tributes to it is the enormous labour, extending over years, devoted to its development and application by Prof. E. W. Brown; subsequent observations have shown, however, that our refractory satellite has even now not yet yielded up all her secrets to the mathematicians.

Extensive as was Hill's work on the moon, both in the development of his own theory and in the discussion of Delaunay's researches, his labours were not confined to this branch of gravitational astronomy. The orbits of Venus, the minor planets, and especially of Jupiter and Saturn, received much of his attention, and in his later years he interested himself greatly in the investigations in periodic and other orbits by Poincaré and Gyldén. His last paper was published in January 1914 in the *Astronomical Journal*, of which he was an associate editor.

At the *Nautical Almanac* Office, Hill, while he occupied only a humble post, lived contentedly in the liberty it afforded him, and never sought to better his position materially; all he desired was peace and freedom to prosecute his work. Newcomb did his best to secure official recognition for his illustrious assistant, but Hill seemed to take little interest in the matter. In Newcomb's words, "here was perhaps the greatest living master in the highest and most difficult branch of astronomy, winning world-wide recognition for his country in the science, and receiving the salary of a department clerk." Newcomb showed the respect in which he held Hill by leaving to him the most difficult section of his scheme for the revision of all the tables of the planets on a uniform basis. Hill's share, the theories of Jupiter and Saturn, occupied him nearly fifteen years, and the investigation takes up the whole of one volume of his *Collected Works*. Its aim was to form theories of these planets such as would be "practically serviceable for a space of three hundred years on each side of the central epoch, taken near the centre of gravity of all the times of observation; theories whose errors in this interval would result, not from neglected terms in the developments, but from the unavoidable imperfections in the values of the arbitrary constant and the masses, adopted from the indications of observation" (Hill). All the original calculations he made himself, and though Newcomb pressed him to accept assistance, he consented only so far as duplication was concerned. In this and all his work he manifested not only the greatest clarity of mind and high mathematical genius, but also the sincerest altruism in the cause of science, labouring in the pursuit of knowledge in its most abstruse branches without a thought of the personal reward which he might fairly have claimed, but to which the simplicity of his life and the innate modesty of his character rendered him almost wholly indifferent. S. C.

WILLIAM GRYLLS ADAMS.

DR. WILLIAM GRYLLS ADAMS was born at Laneast, in Cornwall, in 1836, entered as a student at St. John's College, Cambridge, in 1855, graduated as twelfth wrangler in the Mathematical Tripos of January 1859, and was elected a Fellow of the College in 1864. In his year at Cambridge Prof. William Jack was fourth wrangler and Prof. R. B. Clifton eighth wrangler, and they were first and second Smith's Prizemen respectively. In 1865 he became Professor of Natural Philosophy and Astronomy at King's College, London, in succession to Clerk Maxwell, who at that time retired to his Scottish home near Dalbeattie, and occupied himself with the preparation of the classical treatises that were practically completed when he accepted the invitation to return to Cambridge as the first Cavendish Professor in 1872. Adams's position gave him a prominent place in scientific affairs in London during the evolution of the modern electrical age of engineering. He was a member of the London Mathematical Society since 1866. He assisted in founding the Physical Society in 1875, and was President in 1879; he was President of the Institution of Electrical Engineers in 1884; he served on the Council of the Royal Society from 1882 to 1884 and from 1896 to 1898; he presided over the Mathematical and Physical Section of the British Association in 1880. He was for many years a member of the Kew Observatory Committee and a Visitor of Greenwich Observatory. His mathematical studies were subsidiary to physical investigation; thus in 1875 he delivered the Bakerian lecture to the Royal Society on the equipotential surfaces and lines of electric flow between point electrodes in rectangular regions in two and three dimensions. He showed that the effect produced on the electric resistance of steel by magnetism varies as the square of the magnetization (as was to be expected in the absence of unilateral influence): and he showed that the influence of light on the resistance of selenium is a true radiation effect. Of later years much of his attention was devoted to the subject of terrestrial magnetism, especially following out the influence of earth currents and the Aurora Borealis, and the comparison of simultaneous records at distant observatories, in relation to the earlier ideas of Balfour Stewart. On the decease of his brother, John Couch Adams, the famous astronomer, he prepared his *Collected Papers* for press in two quarto volumes, concerning himself especially with the preparation of the second volume, which consisted of his brother's long continued and unpublished calculation of the changes in the Gaussian constants of terrestrial magnetic distribution in the half-century succeeding Gauss's determinations. In 1906 he retired from his professorship, and soon after settled at Broadstone, Dorset, where, after some years of failing health, he died on April 10, 1915, leaving a widow, three sons, and one daughter.

J. L.

HENRY WILLIAM LLOYD TANNER.

HENRY WILLIAM LLOYD TANNER was born at Burham in Kent on February 17, 1851, his father being Prof. Henry Tanner, F.C.S., a well known authority on agriculture and agricultural chemistry. The son went to Bristol Grammar School, and afterwards to the Royal School of Mines. In October 1868 he matriculated at Oxford as Natural Science Scholar of Jesus College. At Oxford he obtained First Classes in Mathematical Moderations (1870), Mathematical Finals (1872), and Natural Science (1873). He also found time for much music, wide reading of literature, and some debating at the Union. He was Librarian of the Union in Easter Term 1873, just between the times at which Mr. Asquith was Treasurer and President.

During the next ten years he held a number of educational posts for comparatively short periods, the most important being the Professorship of Mathematics and Physical Sciences at the Royal Agricultural College, Cirencester. In 1883 he was chosen first Professor of Mathematics and Astronomy in the newly founded University College of South Wales and Monmouthshire at Cardiff. In this College, and for it and for the young University of Wales into which it and its companion colleges were consolidated some years later, the great work of his life was done. That work is spoken of by all who knew it in terms of most unqualified praise. He organized well, taught devotedly, and endeared himself to colleagues and to generation after generation of students. Given a difficult administrative problem, Tanner was the man to face it; given a wayward cutter of lectures, Tanner was the last to lose and the first to reclaim him; given a social effort in student life or a college entertainment to be got up, his was the youthful merry soul which boyish life made its ally.

He was elected a member of our Society on November 11, 1875, and served on the Council in the years 1879–81. Most of his best work was presented to us. He became F.R.S. in 1899, and was also F.R.A.S., A.R.C.M., and D.Sc. of Oxford. His strong belief in the value of education found scope for action, not only in the Welsh University and his College, but on the Glamorganshire Governing Body for Intermediate Schools. Music, to which he was devoted, he served well as Vice-President of the Cardiff Musical Society. He was invaluable as Local Secretary when the British Association met at Cardiff in 1891. In 1901, during the illness and after the death of Prof. J. Viriamu Jones, he for a time directed his College most ably as Acting Principal.

He was a wise protector of others, but far too unsparing of himself. How much of his practical success resulted from the loving guardianship of his devoted wife (*née* Saunders), none can say; but when she died in 1902, after a long and painful illness, he soon became a failing man. He struggled on till 1909; but memory as well as bodily strength were going, and he then retired with the dignity of Emeritus Professor. He died peaceably on March 6, 1915. Two sons and one daughter survive him.

He produced mathematical work of high quality quite early. During the years 1875–80 there poured from his pen a steady stream of original papers on differential equations which can be spoken of without exaggeration as brilliant. Our volumes VII–XI contain most of these; and the rest appeared in the *Quarterly Journal* and the *Messenger of Mathematics*. Readers of the sixth volume of Forsyth's *Theory of Differential Equations* will recall many references to and results taken from them. He examined many classes of partial differential equations for integrals of lower orders expressible in finite form. In dealing with equations of the second order which admit of intermediate integrals he went far beyond what had been done previously, and in dealing with those of the third order he did pioneer service.

Then came a busy time when his writings were few and short, though always in perfect form. He had no toleration for hasty work, but must highly finish everything he undertook. Thus large ideas had to wait till they could be worthily grappled with.

When he did return to big matters it seems strange that it was not to the theory of differential equations which he had earlier enriched. The fascinations of the theory of numbers took possession of him. Cyclotomy and group-theory in connexion therewith formed a part of this theory in which all that was possible had by no means been done. Our volumes XVIII–XXVII contain a number of masterly papers by him on these subjects, which are now classical.

His early series of papers, and probably also his latest, stand among the greater treasures of our *Proceedings*.

E. B. E.

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PAPERS

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ON THE RATIONAL SOLUTIONS OF THE EQUATION $X^3+Y^3+Z^3=0$ IN QUADRATIC FIELDS

By W. BURNSIDE.

[Read March 12th, 1914.]

IN a memoir with the title of "Die diophantische Gleichung $X^3+Y^3+Z^3=0$" (*Sitzungsberichte der Heidelberger Akademie*, December 1913), Prof. Fueter discusses the relation between the class-number in a quadratic field and the possibility of satisfying the equation $X^3+Y^3+Z^3=0$ by rational numbers in the field. The paper, which contains a number of very interesting results, suggested to me the possibility of dealing with the solution of the equation in quadratic fields by quite elementary methods. I sent the results I arrived at to Prof. Fueter, and he suggested to me to make them public. In the short note that follows I have done this.

1. I assume, as known, the fact that the only solution of the equation

$$X^3+Y^3+Z^3=0, \tag{i}$$

in rational numbers is given by

$$X=0, \quad Y=-Z=\text{any rational number.}$$

Such a solution, whether Y and Z are rational or not, I call a zero-solution. Let x be the root of an irreducible quadratic equation with rational coefficients. Any rational function of x with rational coefficients is expressible in the form $ax+b$, where a and b are rational numbers. If (i) has a non-zero solution in which X, Y, Z are rational functions of x, there must be six rational numbers a_1, b_1, a_2, b_2, a_3, b_3, such that

$$(a_1x+b_1)^3+(a_2x+b_2)^3+(a_3x+b_3)^3 \equiv 0.$$

Neither $(a_2x+b_2)/(a_1x+b_1)$ nor $(a_3x+b_3)/(a_1x+b_1)$ can be rational numbers; for if they were (i) would admit a non-zero rational solution.

Put $$\frac{a_2x+b_2}{a_1x+b_1} = y.$$

Then y satisfies an irreducible quadratic equation with rational coefficients, and $(a_3x+b_3)/(a_1x+b_1)$ is a rational function of y, say $\alpha y+\beta$. Also

$$1+y^3+(\alpha y+\beta)^3 \equiv 0.$$

Consider now the cubic equation with rational coefficients

$$1+z^3+(\alpha z+\beta)^3 = 0.$$

It admits as roots the two roots of the irreducible quadratic equation which y satisfies, and it must therefore have a rational root z_0 so that

$$1+z_0^3+(\alpha z_0+\beta)^3 \equiv 0.$$

Hence 1, z_0 and $\alpha z_0+\beta$, being a rational solution of (i), must be a zero solution. We may take, without loss of generality,

$$\alpha z_0+\beta = 0, \quad z_0 = -1,$$

as the alternative supposition is merely equivalent to a change of notation. It follows that $\alpha = \beta =$ a rational number, say k; while y satisfies the equation

$$1+y^3+k^3(1+y)^3 = 0,$$

or $$y^2+\frac{2k^3-1}{k^3+1}y+1 = 0.$$

This gives $$y = \frac{1-2k^3+\sqrt{-3(1+4k^3)}}{2(1+k^3)},$$

$$k(y+1) = \frac{3k+k\sqrt{-3(1+4k^3)}}{2(1+k^3)},$$

and, clearing the denominators, the solution of (i) takes the form

$$[2(1+k^3)]^3+[1-2k^3+\sqrt{-3(1+4k^3)}]^3+[3k+k\sqrt{-3(1+4k^3)}]^3 \equiv 0.$$

If each term of this identity is multiplied by $\left[\dfrac{3-\sqrt{-3(1+4k^3)}}{2(1+k^3)}\right]^3$ it takes the more symmetrical form

$$(6k)^3+[3+\sqrt{-3(1+4k^3)}]^3+[3-\sqrt{-3(1+4k^3)}]^3\equiv 0. \qquad \text{(ii)}$$

2. It evidently follows from what precedes that every non-zero solution of (i), which is rational in a quadratic field, is obtained by multiplying the identity (ii) by $[\alpha+\beta\sqrt{-3(1+4k^3)}]^3$, and taking for α, β, k any rational numbers, the values 0 and -1 of k being excluded.

In fact, if (ii) is a rational solution of (i), it must be a zero solution, and therefore either $k=0$, or

$$3\pm\sqrt{-3(1+4k^3)}=0,$$

giving

$$k^3+1=0.$$

3. If two solutions of (i), in which

$$\frac{X}{X'}=\frac{Y}{Y'}=\frac{Z}{Z'}$$

are called equivalent, and two solutions, one of which is not, in this sense, a multiple of the other are called distinct, it may be shewn that in every quadratic field in which (i) admits non-zero solutions, it admits an infinite number of distinct solutions.

First, it is clear that the two forms of (ii) with k and k' for parameters cannot be equivalent. If they were, then either

$$\frac{k}{k'}=\frac{3\pm\sqrt{-3(1+4k^3)}}{3\pm\sqrt{-3(1+4k'^3)}}=\frac{3\mp\sqrt{-3(1+4k^3)}}{3\mp\sqrt{-3(1-4k'^3)}},$$

or

$$\frac{k}{3\pm\sqrt{-3(1+4k'^3)}}=\frac{3\pm\sqrt{-3(1+4k^3)}}{k'}=\frac{3\mp\sqrt{-3(1+4k^3)}}{3\mp\sqrt{-3(1+4k'^3)}},$$

where the ± in numerator and denominator are independent.

The first supposition gives $k=k'$, and the second

$$3\pm\sqrt{-3(1+4k'^3)}=\frac{k(6+k')}{6+k},$$

neither of which is admissible.

Secondly, an infinite number of values of the parameter k correspond to the same quadratic field. In fact, the field determined by

$$x^2=n,$$

where n is a positive or negative integer with no square factor, is the same as the field determined by $\sqrt{-3(1+4k^3)}$, if there is a rational number l such that

$$nl^2+12k^3+3=0. \tag{iii}$$

But, if this equation (n being regarded as a given positive or negative integer with no square factor, and the value $n=1$ being excluded) can be satisfied by rational values of k and l at all, it can be satisfied by an infinite number of such values. In fact, if k, l are rational numbers satisfying (iii), it is easy to verify that so also are k' and l' where

$$k'=\frac{k^4-2k}{4k^3+1}, \qquad l'=\frac{3(2k^6+10k^3-1)}{nl(4k^3+1)},$$

and from these an infinite number of such values can be constructed.

The equation (i) has therefore non-zero solutions in the quadratic field defined by $\sqrt{n}$ if (iii) admits rational solutions in k and l; and when this condition is satisfied there is an infinite number of distinct solutions.

ON THE REPRESENTATION OF THE SYMMETRICAL NUCLEUS OF A LINEAR INTEGRAL EQUATION

By E. W. HOBSON.

[Received February 10th, 1914.—Read February 12th, 1914.]

IF $K(s, t)$ denote a symmetrical function of s, t defined in the fundamental square $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, then it may be regarded as the nucleus (Kern) of the linear integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt.$$

If $\{\phi_n(s)\}$, $\{\lambda_n\}$ denote the characteristic functions and numbers of the equation corresponding to the nucleus $K(s, t)$, the following fundamental theorems relating to the representation of $K(s, t)$ are well known for the case in which $K(s, t)$ is a continuous function:—

(*a*) If the series $\sum\limits_{n=1} \dfrac{\phi_n(s)\, \phi_n(t)}{\lambda_n}$ converges uniformly in the fundamental square, then $K(s, t)$ is its sum-function.

(*b*) If all the characteristic numbers $\{\lambda_n\}$ with the possible exception of a finite number of them are of one and the same sign, then $\sum\limits_{n=1} \dfrac{\phi_n(s)\, \phi_n(t)}{\lambda_n}$ certainly converges uniformly to the sum $K(s, t)$. This theorem was first established by Mercer.

It can easily be shewn that the uniform convergence of the series in the whole square necessarily involves its absolute convergence at each point.

The present communication contains the results obtained in the course of an endeavour to ascertain how far the nucleus $K(s, t)$ is represented by the series when that nucleus is not necessarily continuous, but may have finite or infinite discontinuities in the fundamental square. In many of the most important applications of the theory of linear integral equations the case in which $K(s, t)$ is discontinuous on the line $s = t$ arises; it is therefore desirable on this account, as well as on the general ground of the desirability of restricting the generality of the theorems as little as possible, that the relation of the series to a discontinuous nucleus should be elucidated. Throughout the investigation it is assumed that all the integrals employed exist in accordance with Lebesgue's definition of an integral. The Riesz-Fischer theory relating to series of normal functions and to integral equations of the first kind is extensively used.

It is shown that a nucleus $K(s, t)$ such that $\int_a^b \{K(s\ t)\}^2 dt$ is a limited function of s exists, such as to have prescribed characteristic functions and numbers $\{\phi_n(s)\}$, $\{\lambda_n\}$, provided the series $\sum\limits_{n=1} \left\{\frac{\phi_n(s)}{\lambda_n}\right\}^2$ converges for each value of s to a value which is a limited function of s in (a, b). It is further shewn that the function $K(s, t)$ determined in accordance with the prescribed conditions is unique, except for equivalent functions not differing essentially from it. Several theorems are deduced from this result, and a simple proof of Mercer's theorem is obtained by these means.

By means of an extension of a well known theorem due to Hilbert, to the case of a discontinuous nucleus, an extension of Mercer's theorem is obtained which applies to all nuclei which are not infinitely discontinuous on the line $s = t$. Various other special theorems are obtained relating to the convergence of the series.

It is shown that, in case the repeated function of $K(s, t)$ is continuous, or at least equivalent to a continuous function, the nucleus $K(s, t)$ may be divided into the sum of two functions $K^{(1)}(s, t)$, $K^{(2)}(s, t)$ that are orthogonal to one another and are such that $K^{(1)}(s, t)$ has for its sole characteristic numbers those characteristic numbers of $K(s, t)$ that are positive, and that $K^{(2)}(s, t)$ has for its sole characteristic numbers those characteristic numbers of $K(s, t)$ that are negative. The characteristic function corresponding to a characteristic number of $K^{(1)}(s, t)$ or of $K^{(2)}(s, t)$ is the same as the characteristic number of $K(s, t)$ corresponding to the same characteristic number. This result may possibly help to decide whether, or under what conditions, Mercer's theorem can be extended to the case of a continuous nucleus that has an infinite number of characteristic numbers of each sign. The question is shewn to depend upon whether $K^{(1)}(s, t)$ and $K^{(2)}(s, t)$ are infinitely discontinuous, or not, upon the line $s = t$.

1. If $K(s, t)$ be a summable and symmetrical function defined for the points of the square $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, it may be regarded as the

nucleus of the linear integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt,$$

with symmetrical nucleus.

The method of Fredholm is directly applicable to the equation in case $K(s, t)$ is a limited function. In case $K(s, t)$ is not limited, but when all the repeated nuclei from and after the one of some fixed order are limited, Fredholm's method is capable of extension so as to provide the solution of the integral equation.

If $K_n(s, t)$, $K_{n+1}(s, t)$, ..., are all limited, and $\phi_r(s)$ is a characteristic function for the nucleus $K_n(s, t)$, with λ_r^n as the corresponding characteristic number, then $\phi_r(s)$ is also,* in general, a characteristic function for the nucleus $K(s, t)$, the corresponding characteristic number being λ_r, a real n-th root of λ_r^n; and, conversely, every characteristic function for $K(s, t)$ with the corresponding characteristic number can be obtained in this manner. The case in which n is even, and two or more characteristic functions correspond to one and the same characteristic number, is exceptional.

Thus, let
$$\frac{\phi_r^{(1)}(s)\, \phi_r^{(1)}(t) + \phi_r^{(2)}(s)\, \phi_r^{(2)}(t)}{\lambda^n - \lambda_r^n}$$
be that part of the resolvant for the nucleus $K_n(s, t)$ which becomes infinite when $\lambda^n = \lambda_r^n$. In this case the characteristic functions $\phi_r^{(1)}(s)$, $\phi_r^{(2)}(s)$ are not uniquely determinate. For we may replace $\phi_r^{(1)}(s)$, $\phi_r^{(2)}(s)$ by $\cos\alpha . \phi_r^{(1)}(s) + \sin\alpha . \phi_r^{(2)}(s)$, $\sin\alpha . \phi_r^{(1)}(s) - \cos\alpha . \phi_r^{(2)}(s)$ respectively, where α is arbitrary, without altering the expression $\phi_r^{(1)}(s)\phi_r^{(1)}(t) + \phi_r^{(2)}(s)\phi_r^{(2)}(t)$. The corresponding part of the resolvant of $K(s, t)$ will be† of one of the forms

$$\frac{\phi_r^{(1)}(s)\, \phi_r^{(1)}(t) + \phi_r^{(2)}(s)\, \phi_r^{(2)}(t)}{\lambda - \lambda_r},$$

$$\frac{\phi_r^{(1)}(s)\, \phi_r^{(1)}(t) + \phi_r^{(2)}(s)\, \phi_r^{(2)}(t)}{\lambda + \lambda_r},$$

$$\frac{[\cos\alpha . \phi_r^{(1)}(s) + \sin\alpha . \phi_r^{(2)}(s)][\cos\alpha . \phi_r^{(1)}(t) + \sin\alpha . \phi_r^{(2)}(t)]}{\lambda - \lambda_r}$$

$$+ \frac{[\sin\alpha . \phi_r^{(1)}(s) - \cos\alpha . \phi_r^{(1)}(s)][\sin\alpha . \phi_r^{(1)}(t) - \cos\alpha . \phi_r^{(2)}(t)]}{\lambda + \lambda_r},$$

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* See my paper "On the Linear Integral Equation," *Proc. London Math. Soc.*, Ser. 2, Vol. 13, p. 307.

† *Ibid.*, p. 332, where it is shewn that there is one canonical sub-group for $\bar{K}(s, t, \lambda)$ corresponding to each such sub-group for $\bar{K}_n(s, t, \lambda^n)$, and *vice versa*.

where a has some particular value. It is now clear that the characteristic functions for $K_n(s, t)$, corresponding to $\lambda^n = \lambda_r^n$, may be so chosen that they are also characteristic functions for the nucleus $K(s, t)$ corresponding to $\lambda = \lambda_r$, to $\lambda = -\lambda_r$, or to the two characteristic numbers $\lambda = \lambda_r$, $\lambda = -\lambda_r$.

We may thus in all cases regard the characteristic functions for $K(s, t)$ as identical with those for $K_n(s, t)$.

Unless $K(s, t)$ is in some way restricted it will not necessarily be the case that the characteristic functions $\{\phi_r(s)\}$ are continuous functions.

It is, however, a sufficient condition that they may all be continuous, that $K(s, t)$ should be a limited function with its discontinuities regularly distributed, in accordance with the extended meaning of that term given in the paper referred to above. It is there shewn that, if $\phi(t)$ is any summable function, then for such a function $K(s, t)$, $\int_a^b K(s, t)\,\phi(t)\,dt$ is a continuous function of s; thus the relation

$$\phi_r(s) = \lambda_r \int_a^b K(s, t)\,\phi_r(t)\,dt$$

can be satisfied only by a continuous function $\phi_r(s)$.

If $K(s, t)$ be unlimited, but $K_n(s, t)$, $K_{n+1}(s, t)$, ..., be limited functions, it is sufficient to ensure the continuity of the characteristic functions that $K_n(s, t)$ should have its discontinuities regularly distributed, in the sense referred to above.

When $K(s, t)$ is any symmetrical summable function that differs from zero at points of a set of which the plane measure is greater than zero, the proof due to Kneser* that $K(s, t)$ has at least one characteristic number is applicable, whether the function be continuous or not, provided it be limited. In case the function is unlimited, but the repeated nuclei $K_n(s, t)$, $K_{n+1}(s, t)$, ..., are limited, the proof establishes that $K_n(s, t)$ has at least one characteristic number, and this will also be a characteristic number for $K(s, t)$.

Two symmetrical nuclei $K(s, t)$, $K'(s, t)$ of the linear integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\,\phi(t)\,dt$$

will be said to be *equivalent* to one another when they differ from one another only at points of some set such that its section by any straight line parallel to either the s- or t-axis is a set of which the linear measure is zero. It is clear that equivalent nuclei have the same characteristic functions $\{\phi_n(s)\}$ and the same characteristic constants $\{\lambda_n\}$. Also they have the same repeated nuclei.

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* See the *Rendiconti di Palermo*, Vol. XXII, p. 235.

2. Let it be assumed that the symmetrical nucleus $K(s, t)$, although not necessarily continuous or limited, is such that $\int_a^b \{K(s, t)\}^2 ds$ is a limited function of t, and that the repeated nucleus $\int_a^b K(s, t')\, K(t, t')\, dt'$ is either equal to, or equivalent to, the continuous function $K_2(s, t)$. The characteristic functions $\{\phi_n(s)\}$ are then all continuous.

It will then be shewn that $\sum\limits_{n=1} \frac{\phi_n(s)\, \phi(t)}{\lambda_n^2}$ converges uniformly to the value of $K_2(s, t)$; the convergence being absolute.

Since

$$\int_a^b \left\{ K(s, t) - \sum_{n=1}^{n=m} \frac{\phi_n(s)\, \phi_n(t)}{\lambda_n} \right\}^2 ds = \int_a^b \{K(s, t)\}^2 ds - \sum_{n=1}^{n=m} \frac{\{\phi_n(t)\}^2}{\lambda_n^2},$$

it follows that the series $\sum\limits_{n=1} \frac{\{\phi_n(t)\}^2}{\lambda_n^2}$ is convergent for every value of t, and that its sum is $\leqslant \int_a^b \{K(s, t)\}^2 ds$, which is less, for every value of t, than some fixed positive number M. We have

$$\left\{ \sum_{n=m}^{m+r} \left| \frac{\phi_n(s)\, \phi_n(t)}{\lambda_n^2} \right| \right\}^2 \leqslant \sum_{n=m}^{m+r} \frac{\{\phi_n(s)\}^2}{\lambda_n^2} \sum_{n=m}^{m+r} \frac{\{\phi_n(t)\}^2}{\lambda_n^2} < M \sum_{n=m}^{m+r} \frac{\{\phi_n(t)\}^2}{\lambda_n^2}.$$

For a fixed value of t, let m be so chosen that

$$\sum_{n=m}^{m+r} \frac{\{\phi_n(t)\}^2}{\lambda_n^2} < \frac{\epsilon^2}{M},$$

for every value of r, where ϵ is an arbitrarily chosen positive number; we have then

$$\left| \sum_{n=m}^{m+r} \frac{\phi_n(s)\, \phi_n(t)}{\lambda_n^2} \right| \leqslant \sum_{n=m}^{m+r} \left| \frac{\phi_n(s)\, \phi_n(t)}{\lambda_n^2} \right| < \epsilon$$

for every value of r. Since ϵ is arbitrary, it follows that, for a fixed value of t, the series $\sum\limits_{n=1} \frac{\phi_n(s)\, \phi_n(t)}{\lambda_n^2}$ converges absolutely and uniformly, and has therefore a sum that is continuous with respect to s. From the symmetry of the variables s and t, we see that for all values of s and t the series converges to a function that is continuous with respect to the variables s and t separately, at every point (s, t).

Considering the series $\sum\limits_{n=1} \dfrac{\{\phi_n(t)\}^2}{\lambda_n^4}$, we have

$$\sum_{n=m}^{m+r} \frac{\{\phi_n(t)\}^2}{\lambda_n^4} < \frac{1}{\lambda_m^2} \sum_{n=m}^{m+r} \frac{\{\phi_n(t)\}^2}{\lambda_n^2} < \frac{M}{\lambda_m^2};$$

hence, if m be sufficiently large, $\sum\limits_{n=m}^{m+r} \dfrac{\{\phi_n(t)\}^2}{\lambda_n^4}$ is less than an arbitrarily chosen positive number, for all values of r and of t. Thus the series $\sum\limits_{n=1} \dfrac{\{\phi_n(t)\}^2}{\lambda_n^4}$ is uniformly convergent. From this we deduce at once that the series $\sum\limits_{n=1} \dfrac{\phi_n(s)\,\phi_n(t)}{\lambda_n^4}$ is absolutely and uniformly convergent, since

$$|\phi_n(s)\,\phi_n(t)| \leqslant \tfrac{1}{2}\{\phi_n(s)\}^2 + \tfrac{1}{2}\{\phi_n(t)\}^2.$$

It follows, from a known theorem, that the series converges to $K_4(s, t)$; and we have

$$K_4(t, t) = \sum_{n=1}^{\infty} \frac{\{\phi_n(t)\}^2}{\lambda_n^4}.$$

Since

$$\int_a^b \left\{ K_2(s, t) - \sum_{n=1}^{m} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n^2} \right\}^2 ds = K_4(t, t) - \sum_{n=1}^{m} \frac{\{\phi_n(t)\}^2}{\lambda_n^4},$$

we now see that

$$\lim_{m \sim \infty} \int_a^b \left\{ K_2(s, t) - \sum_{n=1}^{m} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n^2} \right\}^2 ds = 0.$$

From this, we infer, by employing the Riesz-Fischer theory,* that for any fixed value of t, there exists a sequence of values of m, such that $\sum\limits_{n=1}^{m} \dfrac{\phi_n(s)\,\phi_n(t)}{\lambda_n^2}$ converges to $K_2(s, t)$ as m passes through the values of the sequence, the convergence taking place for all values of s with the possible exception of those belonging to some set of linear measure zero. Since it has been shewn that the series $\sum\limits_{n=1} \dfrac{\phi_n(s)\,\phi_n(t)}{\lambda_n^2}$ is everywhere convergent, it now follows that, for each fixed value of t, it converges to $K_2(s, t)$, for all values of s except possibly those belonging to some set of linear measure zero. The points of convergence to $K_2(s, t)$ are, for each value of t, everywhere dense on the line $a \leqslant s \leqslant b$, and the sum of the series is continuous with regard to s; therefore the series converges to the continuous function $K_2(s, t)$ for every value of s; t having any fixed value.

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* See, for example, the account of this theory given by W. H. Young and G. Chisholm Young in the *Quarterly Journal of Mathematics*, Vol. XLIV, p. 47.

It has now been shewn that $\sum\limits_{n=1} \dfrac{\phi_n(s)\,\phi_n(t)}{\lambda_n^2}$ converges for all values of s and t to the sum $K_3(s, t)$.

Since $\sum\limits_{n=1} \dfrac{\{\phi_n(s)\}^2}{\lambda_n^2}$, in which all the terms are positive, converges to the continuous function $K_2(s, s)$, its convergence must be uniform.

Since

$$\left|\sum_{n=m}^{m+r} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n^2}\right| \leqslant \sum_{n=m}^{m+r} \frac{|\phi_n(s)\,\phi_n(t)|}{\lambda_n^2} < \tfrac{1}{2}\sum_{n=m}^{m+r} \frac{\{\phi_n(s)\}^2}{\lambda_n^2} + \tfrac{1}{2}\sum_{n=m}^{m+r} \frac{\{\phi_n(t)\}^2}{\lambda_n^2},$$

the integer m can be so chosen that, for all values of r,

$$\sum_{n=m}^{m+r} \frac{|\phi_n(s)\,\phi_n(t)|}{\lambda_n^2} \quad \text{and} \quad \left|\sum_{n=m}^{m+r} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n^2}\right|,$$

are less than an arbitrarily chosen number, for all values of (s, t). It follows that the convergence of $\sum\limits_{n=1} \dfrac{\phi_n(s)\,\phi_n(t)}{\lambda_n^2}$ to $K_2(s, t)$ is uniform and absolute.

The following theorem has been established:—

I. *If* $K(s, t)$ *is any symmetrical nucleus, whether continuous or not, and for which* $\int_a^b \{K(s, t)\}^2\,ds$ *is less than some fixed positive number independent of* t, *and if* $K(s, t)$ *is such that its repeated nucleus is equal to, or equivalent to, a continuous function* $K_2(s, t)$, *then* $\sum\limits_{n=1} \dfrac{\phi_n(s)\,\phi_n(t)}{\lambda_n^2}$ *converges uniformly and absolutely to the value of* $K_2(s, t)$.

The case of this theorem in which $K(s, t)$ is continuous has been established by Kowalewski.*

3. It is known that, if $\phi_1(t)$, $\phi_2(t)$, ..., be a sequence of orthogonal functions defined for the interval (a, b), and normalized so that

$$\int_a^b \{\phi_n(t)\}^2 dt = 1,$$

and if $c_1, c_2, \ldots, c_n, \ldots$, be a sequence of real numbers, the necessary and sufficient condition that a function $F(t)$, of which the square is summable in (a, b), should exist and be such as to satisfy the conditions

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* See his treatise *Einführung in die Determinantentheorie*.

$$\int_a^b F(t)\,\phi_n(t)\,dt = c_n, \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

is that the series $c_1^2 + c_2^2 + \ldots + c_n^2 + \ldots$ should be convergent.

The function $F(t)$ is unique, disregarding the distinction between functions that differ from one another at points of a set of measure zero, only in case the set of orthogonal functions is *closed; i.e.* when no other function orthogonal to them all exists, such that its square is summable, that it differs from zero at points of a set of measure greater than zero, and such that it is not a linear function of a finite number of the given orthogonal functions. Even when the set of orthogonal functions is not closed the function $F(t)$ can be so determined that

$$\int_a^b \{F(t)\}^2\,dt$$

is equal to the sum of the series $c_1^2 + c_2^2 + \ldots$.

If $\chi(t)$ is any independent function, of which the square is summable, and such as to be orthogonal to all the functions $\{\phi_n(t)\}$, then we have

$$\int_a^b F(t)\,\chi(t)\,dt = 0.$$

The truth of this last statement follows from a consideration of the fact that

$$\int_a^b \left[F(t) - \chi(t) \int_a^b F(t)\,\chi(t)\,dt - \sum_{n=1}^{m} c_n \phi_n(t)\right]^2 dt$$

has a value that is essentially positive. It is equivalent to

$$\int_a^b \{F(t)\}^2\,dt - \left\{\int_a^b F(t)\,\chi(t)\,dt\right\}^2 - \sum_{n=1}^{m} c_n^2,$$

it being assumed that $\int_a^b \{\chi(t)\}^2\,dt = 1.$

As this expression is positive for every value of m, we see that

$$\int_a^b F(t)\,\chi(t)\,dt = 0.$$

Let us now suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ is any sequence of real numbers arranged in order of increasing absolute magnitude, and such that $|\lambda_n|$ increases indefinitely as n does so.

Let it be assumed that, for a fixed number s in the interval (a, b), the series $\sum\limits_{n=1} \dfrac{\{\phi_n(s)\}^2}{\lambda_n^2}$ is convergent.

Then the equations

$$\frac{\phi_n(s)}{\lambda_n} = \int_a^b F(t)\,\phi_n(t)\,dt \quad (n = 1, 2, 3, \ldots)$$

are all satisfied by a function $F(t)$, such that

$$\int_a^b \{F(t)\}^2\,dt = \sum_{n=1}^{\infty} \frac{\{\phi_n(s)\}^2}{\lambda_n^2}.$$

Conversely, the convergence of the series is necessary for the existence of the function $F(t)$, such that its square is summable and that it satisfies the equations. The function $F(t)$ is unique in the sense that it can be replaced only by another function which differs from it at points of a set of which the measure is zero.

When we consider different values of s for which the series $\sum\limits_{n=1} \dfrac{\{\phi_n(s)\}^2}{\lambda_n^2}$ is convergent, we may denote $F(t)$ by $K(s, t)$.

Let it now be assumed that $\sum\limits_{n=1} \dfrac{\{\phi_n(s)\}^2}{\lambda_n^2}$ is convergent for all values of s in the interval (a, b). Then $K(s, t)$ is defined, except that it may be replaced by a function which differs from it at points of a set of which the section by each line parallel to the t-axis has linear measure zero.

The function $K(s, t)$, so defined, is such that

$$\phi_n(s) = \lambda_n \int_a^b K(s, t)\,\phi_n(t)\,dt,$$

for all values of n. Also, if $\chi(t)$ is any independent function such that $\chi(s)$ is orthogonal to all the functions $\{\phi_n(s)\}$, we have

$$\int_a^b K(s, t)\,\chi(t)\,dt = 0,$$

and thus $\chi(t)$ is not a characteristic function.

We have now

$$\int_a^b \left\{ K(s, t) - \sum_{n=1}^{n=m} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n} \right\}^2 dt = \int_a^b \{K(s, t)\}^2\,dt - \sum_{n=1}^{n=m} \frac{\{\phi_n(s)\}^2}{\lambda_n^2};$$

and therefore for each fixed value of s we have

$$\lim_{m \sim \infty} \int_a^b \left\{ K(s, t) - \sum_{n=1}^{n=m} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n} \right\}^2 dt = 0.$$

From this it follows, by using the Riesz-Fischer theory, that for each value of s a sequence of values of m can be determined so that

$$\sum_{n=1}^{n=m} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$$

converges to $K(s, t)$, except for a set of values of t of measure zero, as the values of the sequence are assigned to m. It thus appears that the function $K(s, t)$ is symmetrical, and that it may be regarded as the nucleus of an integral equation of Fredholm's type, of which the functions $\{\phi_n(s)\}$ are the characteristic functions and $\{\lambda_n\}$ the characteristic constants.

The following theorem has now been established :—

II. *If* $\{\phi_n(s)\}$ *denotes a sequence of prepared orthogonal functions for the interval* (a, b), *and* $\{\lambda_n\}$ *denotes a sequence of real numbers in ascending order of absolute magnitude, the necessary and sufficient condition that a symmetrical nucleus* $K(s, t)$ *should exist, for which* $\int_a^b \{K(s, t)\}^2 dt$ *is finite for each value of* s, *and for which* $\{\phi_n(s)\}$ *are the characteristic functions, and* $\{\lambda_n\}$ *the characteristic numbers, is that the series* $\sum\limits_{n=1} \frac{\{\phi_n(s)\}^2}{\lambda_n^2}$ *should be convergent for all values of* s *in* (a, b).

It should be observed that the plane set of values of (s, t) for which no sequence of values of m exists such that $\sum\limits_{n=1}^{m} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ converges to $K(s, t)$ is such that its sections by lines parallel to the axes are all sets of linear measure zero. It follows that the plane set has its plane measure zero.

In case $\sum\limits_{n=1} \frac{\{\phi_n(s)\}^2}{\lambda_n^2}$ is divergent for a set of values of s of linear measure zero, the method given fails to determine a function $K(s, t)$ for which $\{\phi_n(s)\}$ and $\{\lambda_n\}$ are the characteristic functions and numbers. In this case the given functions $\{\phi_n(s)\}$ may be replaced by a modified set of orthogonal functions for which $K(s, t)$ may be determined so as to have the given characteristic numbers $\{\lambda_n\}$. Let $\bar{\phi}_n(s) = \phi_n(s)$, for all values of s for which $\sum\limits_{n=1} \frac{\{\phi_n(s)\}^2}{\lambda_n^2}$ is convergent, and let $\bar{\phi}_n(s) = 0$ for all other values of s. The function $K(s, t)$ will then be determined, as above, for those values and s and t which do not belong to the exceptional set ; when s or t belongs to the exceptional set, we assign to $K(s, t)$ the value zero.

The equations

$$\bar{\phi}_n(s) = \lambda_n \int_a^b K(s,\, t)\, \bar{\phi}_n(t)\, dt$$

are then satisfied for all values of s in $(a,\, b)$, and for all values of n.

4. Let it now be assumed that the given functions $\{\phi_n(s)\}$ and the given numbers $\{\lambda_n\}$ are such that $\sum\limits_{n=1} \dfrac{\{\phi_n(s)\}^2}{\lambda_n^2}$ is not only convergent for each value of s, but also that its sum is a limited function of s, in the interval $(a,\, b)$.

In accordance with what has been established in § 3, a symmetrical function $K(s,\, t)$ exists, such that

$$\phi_n(s) = \lambda_n \int_a^b K(s,\, t)\, \phi_n(t)\, dt, \quad \text{for} \quad n = 1,\, 2,\, 3,\, \ldots,$$

and such that

$$\int_a^b \{K(s,\, t)\}^2\, dt = \sum_{n=1}^{\infty} \frac{\{\phi_n(s)\}^2}{\lambda_n^2}.$$

It will now be proved that there can exist no other function $H(s,\, t)$ not equivalent to $K(s,\, t)$, such that $\int_a^b \{H(s,\, t)\}^2\, dt$ is finite for each value of s, and limited for all values of s, which shall have the functions $\{\phi_n(s)\}$ for its sole characteristic functions, and $\{\lambda_n\}$ for the corresponding characteristic numbers.

For assume $H(s,\, t)$ to be such that

$$\int_a^b H(s,\, t)\, \phi_n(t)\, dt = \frac{\phi_n(s)}{\lambda_n}, \quad \text{for} \quad n = 1,\, 2,\, 3,\, \ldots.$$

We need consider only the case in which $\{\phi_n(s)\}$ is not a closed system, for, if it were closed, $H(s,\, t)$ and $K(s,\, t)$ must be equivalent.

It is known * that any normal orthogonal system is enumerable; we

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* It was shewn by E. Schmidt (*Comptes Rendus de l'Académie des Sciences*, Dec. 10, 1906, p. 955), that any orthogonal set of continuous functions $\{\phi(s)\}$ must be enumerable. The proof will here be adapted by means of a slight alteration to the case of functions that are not necessarily continuous. Let $F(s)$ be any function whose square is summable in $(a,\, b)$, and let $\phi_{p_1}(s),\, \phi_{p_2}(s),\, \ldots,\, \phi_{p_r}(s)$ be any finite set of the functions $\{\phi(s)\}$; we have then

$$\int_a^b \left\{ F(s) - \sum_1^r \phi_{p_r}(s) \int_a^b F(s)\, \phi_{p_r}(s)\, ds \right\}^2 ds = \int_a^b \{F(s)\}^2\, ds - \sum_1^r \left\{ \int_a^b F(s)\, \phi_{p_r}(s)\, ds \right\}^2.$$

It follows from this equality, since the expression on the right-hand side is essentially positive, that, if k be any fixed positive number, there can exist only a finite number of the functions may therefore assume that $\{\phi_n(s)\}$ is a part of some* closed system of normal orthogonal functions which is obtained by adjoining an orthogonal system $\{\chi_n(s)\}$, where $n = 1, 2, 3, \ldots$, so that $\{\phi_n(s)\}$, $\{\chi_n(s)\}$ taken together form a closed orthogonal system.

Let us suppose that

$$\int_a^b H(s, t)\, \chi_n(t)\, dt = \psi_n(s),$$

where $\psi_n(s)$ may be zero for some values of n, but not for all values of n, in which latter case $H(s, t)$ would be necessarily equivalent to $K(s, t)$.

We have now

$$\int_a^b \{H(s, t)\}^2\, dt = \sum_{n=1}^{\infty} \{\psi_n(s)\}^2 + \sum_{n=1}^{\infty} \frac{\{\phi_n(s)\}^2}{\lambda_n^2},$$

the convergence of the series being necessary for the existence of $H(s, t)$.

$\phi(s)$, such that

$$\left| \int_a^b F(s)\, \phi(s)\, ds \right| > k.$$

Assigning to k a sequence of values diminishing towards the limit zero, we see that the set of those functions $\phi(s)$, for which

$$\left| \int_a^b F(s)\, \phi(s)\, ds \right| > 0$$

is enumerable. We have then only to consider those functions $\phi(s)$ for which

$$\int_a^b F(s)\, \phi(s)\, ds = 0.$$

Taking for simplicity $a = -\pi$, $b = \pi$, which involves no loss of generality, let $F(s)$ have successively the values 1, $\cos s$, $\sin s$, $\cos 2s$, $\sin 2s$, ...; we have then an enumerable set of enumerable sets in each of which one at least of the integrals is in absolute value > 0; the whole composite set is enumerable. There is no function $\chi(s)$ such that $\int_{-\pi}^{\pi} \{\chi(s)\}^2\, ds$ exists as a positive number, and such that

$$\int_{-\pi}^{\pi} \chi(s) \begin{matrix}\cos \\ \sin\end{matrix}\, ns\, ds = 0,$$

for every value of n. For, in accordance with a known theorem, first established by Fatou, for such a function $\chi(s)$ we have

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \{\chi(s)\}^2\, ds = \sum_{n=0}^{\infty} \left\{ \int_{-\pi}^{\pi} \chi(s) \cos ns\, ds \right\}^2 + \sum_{n=1}^{\infty} \left\{ \int_{-\pi}^{\pi} \chi(s) \sin ns\, ds \right\}^2,$$

and thus it is impossible that all the integrals on the right-hand side can vanish. It has thus been shewn that the set $\{\phi(s)\}$ must be enumerable, whether it be closed or not.

* It may be noticed that a set of orthogonal functions may form part of more than one closed set of orthogonal functions. For example, in the interval $(-\pi, \pi)$ consider the orthogonal set $\sin x$, $\sin 2x$, ..., $\sin nx$, ... This forms part of the closed set 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, ...; but we might also adjoin to the set $\sin x$, $\sin 2x$, ..., $\sin nx$, ... the orthogonal functions $P_2(\pi x)$, $P_4(\pi x)$, ..., $P_2\ (\pi x)$, ...; where $P_{2n}(x)$ denotes the Legendre's function of order $2n$; the extended set is then still an orthogonal set.

Let us consider $H(s, t) - K(s, t)$ as a nucleus; we have

$$\int_a^b \{H(s, t) - K(s, t)\} \phi_n(t)\, dt = 0,$$

and thus the functions $\{\phi_n(s)\}$ are not characteristic functions for this nucleus. Also

$$\int_a^b \{H(s, t) - K(s, t)\} \chi_n(t)\, dt = \psi_n(s);$$

and therefore it appears from these two sets of equations that

$$\int_a^b \{H(s, t) - K(s, t)\}^2\, dt = \sum_{n=1}^{\infty} \{\psi_n(s)\}^2.$$

It is now seen that the repeated nucleus of $H(s, t) - K(s, t)$, viz.,

$$\int_a^b \{H(s, t') - K(s, t')\} \{H(t, t') - K(t, t')\}\, dt',$$

is numerically less than

$$\left[\sum_{n=1}^{\infty} \{\psi_n(s)\}^2 \sum_{n=1}^{\infty} \{\psi_n(t)\}^2\right]^{\frac{1}{2}},$$

and this is a limited function since it has been assumed that

$$\int_a^b \{H(s, t)\}^2\, dt,$$

and consequently $\sum_{n=1}^{\infty} \{\psi_n(s)\}^2$, is a limited function of s. It now follows, in accordance with what has been pointed out in § 1, that the nucleus $H(s, t) - K(s, t)$ has at least one characteristic function $\mu(s)$, and we have seen that this cannot belong to the set $\{\phi_n(s)\}$. We have therefore

$$\int_a^b \{H(s, t) - K(s, t)\} \mu(t)\, dt = \frac{\mu(s)}{\nu},$$

where ν is the characteristic number corresponding to $\mu(s)$.

From this we see that

$$\frac{1}{\nu} \int_a^b \mu(s)\, \phi_n(s)\, ds = \int_a^b \mu(t)\, dt \int_a^b [H(s, t) - K(s, t)]\, \phi_n(s)\, ds = 0;$$

hence $\mu(s)$ is orthogonal to all the functions $\{\phi_n(s)\}$, and hence we have

$$\int_a^b K(s, t)\, \mu(t)\, dt = 0.$$

It follows that $$\int_a^b H(s, t)\,\mu(t)\,dt = \frac{\mu(s)}{\nu};$$

and therefore $\mu(s)$ is a characteristic function for the nucleus $H(s, t)$. This is contrary to the supposition that $H(s, t)$ has $\{\phi_n(s)\}$ for its sole characteristic functions. Hence $H(s, t)$ must be equivalent to $K(s, t)$.

The following theorem has now been established:—

III. *If* $\{\phi_n(s)\}$ *be a system of normal orthogonal functions, and* $\{\lambda_n\}$ *a set of numbers arranged in order of increasing absolute magnitude, the necessary and sufficient condition that a symmetrical nucleus* $K(s, t)$, *such that* $\int_a^b \{K(s, t)\}^2\,dt$ *is a limited function of* s, *may exist and be such that* $\{\phi_n(s)\}$, $\{\lambda_n\}$ *are its sole characteristic functions and numbers, is that* $\sum_{n=1} \left\{\frac{\phi_n(s)}{\lambda_n}\right\}^2$ *should converge for each value of* s *to a value which is a limited function of* s. *This function* $K(s, t)$ *is the only nucleus, apart from equivalent ones, which satisfies the above conditions.*

The set of values of (s, t) for which no sequence of values of m exists such that $\sum_{n=1}^{n=m} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ converges to $K(s, t)$, is such that its sections by lines parallel to the axes are all of linear measure zero, and therefore the set has plane measure zero.

We have now established the following theorem:—

IV. *If* $K(s, t)$ *is any symmetrical nucleus, not necessarily continuous or limited, but such that* $\int_a^b \{K(s, t)\}^2\,dt$ *exists as a limited function of* s, *for all values of* s *in* (a, b), *then for each set of values of* (s, t) *with the possible exception of those belonging to a set of which all the sections by lines parallel to the axes have linear measure zero, there exists a sequence of values of* m *such that* $\sum_{n=1}^{n=m} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ *converges to* $K(s, t)$ *as* m *passes through the values in the sequence.*

If, at any point (s, t) not belonging to the exceptional set of points, the series $\sum_{n=1} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ is convergent, it must converge to the value $K(s, t)$; we have then the following theorem:—

V. *If the series* $\sum_{n=1} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ *corresponding to a nucleus* $K(s, t)$, *for* *which* $\int_a^b \{K(s,t)\}^2 dt$ *is a limited function of s, be such that it is convergent for every point* (s, t) *which does not belong to some set of plane measure zero, then the series converges to* $K(s, t)$ *with a similar exception. If the series is everywhere convergent, it converges to* $K(s, t)$ *except possibly at points of a set of which the sections by lines parallel to the axes have linear measure zero.*

The following theorem is immediately deducible:—

VI. *If* $K(s, t)$ *be such that the functions* $\phi_n(s)$ *are all continuous, and such that* $\int_a^b \{K(s,t)\}^2 dt$ *is a limited function of s, for all values of s in* (a, b), *and if it be continuous in a closed connex set G, consisting of a domain and its boundary, and if the series* $\sum\limits_{n=1} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ *is uniformly convergent in G, then it converges throughout G to the sum* $K(s, t)$.

For the series converges in G to a function that is continuous in G. Moreover, G has its measure greater than zero, and thus, in the arbitrary small neighbourhood of any point of G, there are points which do not belong to the exceptional set. Hence $K(s, t)$ cannot differ at any point of G from that function, continuous in G, which represents the sum of the series.

For example, if $K(s, t)$ is continuous except when $s = t$, where it may have any finite or infinite discontinuities consistent with the condition that $\int_a^b \{K(s,t)\}^2 dt$ should exist for each value of s as a limited function of s, and if $\sum\limits_{n=1} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ is uniformly convergent for all values of s and t, such that $|s-t| \geqslant \epsilon$, when ϵ is an arbitrary positive number, then the series $\sum\limits_{n=1} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ converges to $K(s, t)$ at all points such that $s \neq t$. This applies, for example, to the case $K(s, t) = \frac{f(s,t)}{|s-t|^\alpha}$, where $f(s, t)$ is a continuous function, and $\alpha < \frac{1}{2}$.

5. Let all the numbers $\lambda_1, \lambda_2, \ldots,$ be positive, and let $K(s, t)$ be continuous; then, in accordance with a known theorem due to Hilbert,[*]

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[*] See E. Schmidt's paper in the *Math. Annalen*, Vol. 63, p. 453.

we have

$$\int_a^b\int_a^b K(s, t) f(s) f(t)\, ds\, dt = \Sigma \frac{1}{\lambda_n}\left\{\int_a^b f(s)\,\phi_n(s)\, ds\right\}^2,$$

where $f(s)$ is continuous in (a, b). If $f(s)$ is chosen so as to vanish except in the interval $(s_1, s_1+\epsilon)$, and to be positive within that interval, the number ϵ being so chosen that $K(s, t)$ has the same sign as $K(s_1, s_1)$ in the square $s_1 \leqslant s \leqslant s_1+\epsilon$, $s_1 \leqslant t \leqslant s_1+\epsilon$, unless $K(s_1, s_1) = 0$. It is clear that that sign must be positive, since the right-hand side of the above equation is positive; hence $K(s_1, s_1) \geqslant 0$, and, as s_1 is any point, we have $K(s, s) \geqslant 0$. If we now consider the nucleus

$$K(s, t) - \sum_{n=1}^{n=m} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$$

which is symmetrical and continuous, and has only the positive characteristic numbers $\lambda_{n+1}, \lambda_{n+2}, \ldots$, we see that

$$K(s, s) - \sum_{n=1}^{n=m} \frac{\{\phi_n(s)\}^2}{\lambda_n} \geqslant 0.$$

Thus the series $\sum\limits_{n=1} \dfrac{\{\phi_n(s)\}^2}{\lambda_n}$ is convergent for every value of s, and its sum is limited for all values of s.

It follows from theorem III that there exists a nucleus $k(s, t)$ such that

$$\int_a^b \{k(s, t)\}^2\, dt = \sum_{n=1} \frac{\{\phi_n(s)\}^2}{\lambda_n},$$

of which the characteristic numbers are $\lambda_1^{\frac{1}{2}}, \lambda_2^{\frac{1}{2}}, \ldots$, and of which $\phi_1(s), \phi_2(s), \ldots$ are the characteristic functions. This nucleus is unique, except for equivalent ones.

The repeated nucleus

$$k_2(s, t) = \int_a^b k(s, t')\, k(t, t')\, dt'$$

has for its sole characteristic values $\lambda_1, \lambda_2, \ldots$, and for its sole characteristic functions $\phi_1(s), \phi_2(s), \ldots$ (see § 1).

Hence, since $\int_a^b \{k_2(s, t)\}^2\, dt$ is a limited function of s, it must, in accordance with what has been shewn in theorem III, be equivalent to $K(s, t)$, and therefore by the theorem I of § 2, we see that $\sum\limits_{n=1} \dfrac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ converges uniformly to $K(s, t)$.

If a finite number of the numbers $\{\lambda_n\}$ be negative, we may apply the above result to the nucleus

$$K(s,\, t) - \sum_{n=1}^{n=r} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n},$$

where r is so large that all the negative characteristic numbers are included in the first r. This nucleus has its characteristic constants $\lambda_{r+1}, \lambda_{r+2}, \ldots$ all positive, and the above result may be applied to it. It has thus been shewn that—

VII. *If the symmetrical nucleus* $K(s, t)$ *be continuous, and all its characteristic constants, with the possible exception of a finite number of them, are of one sign, then* $\sum\limits_{n=1} \dfrac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ *converges uniformly to* $K(s, t)$ *as sum.*

This is Mercer's theorem.*

6. The theorem of Hilbert quoted in § 5 will now be extended to the case of discontinuous functions of which the squares are summable.

We shall assume that $K(s, t)$ is limited only by the condition that $\int_a^b \{K(s, t)\}^2 dt$ exists for each value of s, and is a limited function of s in the interval (a, b). Let $p(t)$ be a function whose square is summable in the interval (a, b). Then, if $g(s) = \int_a^b K(s, t)\,p(t)\,dt$, we see that $g(s)$ is a limited summable function.

Consider the function

$$F(s) = g(s) - \sum_{n=1}^{\infty} \phi_n(s) \int_a^b g(t)\,\phi_n(t)\,dt.$$

The series in the second term on the right-hand side is equivalent to

$$\sum_{n=1}^{\infty} \lambda_n \int_a^b K(s, t)\,\phi_n(t)\,dt \int_a^b \int_a^b K(t, t')\,p(t')\,\phi_n(t)\,dt\,dt',$$

or to

$$\sum_{n=1}^{\infty} \int_a^b K(s, t)\,\phi_n(t)\,dt \int_a^b \phi_n(t')\,p(t')\,dt',$$

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* See *Phil. Trans.*, Vol. 209A. Another proof has been given by Kneser, in the *Rend. di Palermo*, Vol. XXXVII, p. 195.

which, in accordance with a theorem[*] due to E. Schmidt, is absolutely and uniformly convergent.

We have now

$$\int_a^b F(s)\,\phi_n(s)\,ds = \int_a^b g(s)\,\phi_n(s)\,ds - \int_a^b g(t)\,\phi_n(t)\,dt = 0,$$

and hence, in accordance with what has been shewn in § 3, we see that

$$\int_a^b K(s, t)\,F(t)\,dt = 0.$$

We have now

$$\int_a^b \{F(s)\}^2 ds = \int_a^b F(s)\,g(s)\,ds = \int_a^b\int_a^b F(s)\,K(s, t)\,p(t)\,ds\,dt = 0.$$

It follows that $F(s) = 0$ for every value of s, with the possible exception of those belonging to a set of measure zero.

Therefore the uniformly convergent series

$$\sum_{n=1} \phi_n(s) \int_a^b g(t)\,\phi_n(t)\,dt$$

converges to $g(s)$ for all values of s with the possible exception.

The series is equivalent to

$$\sum_{n=1} \frac{\phi_n(s)}{\lambda_n} \int_a^b \phi_n(t)\,p(t)\,dt.$$

Multiplying by $q(s)\,ds$ and integrating, we have

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_a^b \phi_n(s)\,q(s)\,ds \int_a^b \phi_n(t)\,p(t)\,dt = \int_a^b\int_a^b K(s, t)\,p(t)\,q(s)\,ds\,dt,$$

where $q(s)$ is a function whose square is summable. The legitimacy of the term-by-term integration is unaffected by the possible existence of the exceptional set of points of measure zero.

We have now shewn that

$$\int_a^b\int_a^b K(s, t)\,p(t)\,q(s)\,ds\,dt = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_a^b \phi_n(s)\,q(s)\,ds \int_a^b \phi_n(s)\,p(s)\,ds,$$

where $p(s)$, $q(s)$ are any functions whose squares are summable in (a, b), and where $K(s, t)$ is subject only to the condition that $\int_a^b \{K(s, t)\}^2 dt$ exists for each value of s and is limited as a function of s.

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[*] *Math. Annalen*, Vol. LXIII, p. 440.

Let $p(s)$ and $q(s)$ be identical; the theorem then becomes

$$\int_a^b \int_a^b K(s, t)\, p(s)\, p(t)\, ds\, dt = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left\{ \int_a^b \phi_n(s)\, p(s)\, ds \right\}^2.$$

7. Let it now be assumed that $K(s, t)$ is such that all the characteristic numbers $\{\lambda_n\}$ are positive. Let $p(s)$ be defined by $p(s) = 1$, when $s_1 - \epsilon \leqslant s \leqslant s_1 + \epsilon$, and $p(s) = 0$, for all other values of s, where ϵ is a fixed positive number, and s_1 a fixed value of s.

We see then that $$\iint K(s, t)\, ds\, dt,$$

taken over a square with its centre at the point (s_1, s_1) and its sides of length 2ϵ parallel to the axes, is essentially positive.

Let us now assume that all the characteristic functions $\{\phi_n(s)\}$ are continuous, and let us apply the result just obtained to the nucleus

$$K(s, t) - \sum_{n=1}^{n=m} \frac{\phi_n(s)\, \phi_n(t)}{\lambda_n},$$

which is such that $$\int_a^b \left\{ K(s, t) - \sum_{n=1}^{n=m} \frac{\phi_n(s)\, \phi_n(t)}{\lambda_n} \right\}^2 dt,$$

or $$\int_a^b \{K(s, t)\}^2 dt - \sum_{n=1}^{m} \left\{ \frac{\phi_n(s)}{\lambda_n} \right\}^2,$$

is a limited function of s, the value of m being fixed.

We see that

$$\frac{1}{4\epsilon^2} \iint K(s, t)\, ds\, dt - \frac{1}{4\epsilon^2} \iint \sum_{n=1}^{n=m} \frac{\phi_n(s)\, \phi_n(t)}{\lambda_n}\, ds\, dt$$

is essentially positive for each value of m and each value of ϵ; the integrals being taken over the square with centre at (s_1, s_1).

Now $\sum_{n=1}^{n=m} \frac{\phi_n(s)\, \phi_n(t)}{\lambda_n}$ being a continuous function of (s, t), we see that

$$\frac{1}{4\epsilon^2} \iint \sum_{n=1}^{n=m} \frac{\phi_n(s)\, \phi_n(t)}{\lambda_n}\, ds\, dt = \sum_{n=1}^{n=m} \frac{\{\phi_n(s_1)\}^2}{\lambda_n} + \eta_m,$$

where η_m converges to zero as ϵ does so, and thus can be made arbitrarily small by taking ϵ small enough, for each fixed value of m.

Thus
$$\frac{1}{4\epsilon^2}\iint K(s,t)\,ds\,dt - \sum_{n=1}^{n=m}\frac{\{\phi_n(s_1)\}^2}{\lambda_n} - \eta_m$$
is essentially positive.

If $K(s, t)$ be such that it is not infinitely discontinuous at the point (s_1, s_1), we can choose ϵ so small that $|K(s, t)|$ is limited in the square $s_1-\epsilon \leqslant s \leqslant s_1+\epsilon$, $s_1-\epsilon \leqslant t \leqslant s_1+\epsilon$.

In that case
$$\frac{1}{4\epsilon^2}\iint K(s,t)\,ds\,dt \leqslant N,$$
where N denotes the upper limit of $|K(s, t)|$ in the square so chosen. We now see that the series $\sum\limits_{n=1}\frac{\{\phi_n(s_1)\}^2}{\lambda_n}$ cannot diverge, for if it diverges we can choose m so large that
$$\sum_{n=1}^{n=m}\frac{\{\phi_n(s_1)\}^2}{\lambda_n} > N,$$
and since η_m may for this value of m be taken arbitrarily small by proper choice of ϵ, the expression
$$\frac{1}{4\epsilon^2}\iint K(s,t)\,ds\,dt - \sum_{n=1}^{n=m}\frac{\{\phi_n(s)\}^2}{\lambda_n} - \eta_m$$
would then be negative, which is impossible.

Therefore the series $\sum\limits_{n=1}\frac{\{\phi_n(s_1)\}^2}{\lambda_n}$ is convergent, provided $K(s, t)$ is not infinitely discontinuous at the point (s_1, s_1).

In case $\sum\limits_{n=1}\frac{\{\phi_n(s_1)\}^2}{\lambda_n}$ is divergent, $K(s, t)$ must be infinitely discontinuous at (s_1, s_1).

In case $K(s, t)$ is such as to have no infinite discontinuity on the straight line $s=t$, the series $\sum\frac{\{\phi_n(s)\}^2}{\lambda_n}$ is convergent for every value of s in (a, b). It then follows that the series $\sum\limits_{n=1}\frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ is everywhere absolutely convergent.

Employing the theorem V established in § 4, we now have the following theorem:—

VIII. *If $K(s, t)$ be any symmetrical nucleus for which all the characteristic numbers are positive and the characteristic functions all continuous, such that $\int_a^b \{K(s,t)\}^2\,dt$ is a limited function of s, and also such* *as not to be infinitely discontinuous at any point on the straight line* $s=t$, *the series* $\Sigma \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ *converges to* $K(s, t)$ *everywhere with the possible exception of points of a set of which the sections by lines parallel to the axes have linear measure zero.*

This is an extension of Mercer's theorem to the case of discontinuous nuclei with positive characteristic numbers. It is easily seen that it can be extended to the case in which a finite set of the characteristic numbers are negative.

8. Let $K(s, t)$ be any symmetrical nucleus for which $\{\phi_n(s)\}$ and $\{\lambda_n\}$ are the characteristic functions and constants, and let it be assumed that the series $\sum\limits_{n=1} \frac{1}{|\lambda_n|}$ is convergent.

Let $\epsilon_1, \epsilon_2, \ldots,$ denote a sequence of decreasing positive numbers such that $\epsilon_1+\epsilon_2+\ldots$ converges to a positive number ζ arbitrarily chosen. Let $m_1, m_2, m_3, \ldots,$ be a sequence of increasing integers so chosen that

$$\sum_{m_1}^{\infty} \frac{1}{|\lambda_n|} < \epsilon_1^2, \quad \sum_{m_2}^{\infty} \frac{1}{|\lambda_n|} < \epsilon_2^2, \quad \ldots .$$

We have then

$$\int_a^b \sum_{m_1}^{m_2-1} \frac{\{\phi_n(s)\}^2}{|\lambda_n|} ds < \epsilon_1^2,$$

$$\int_a^b \sum_{m_2}^{m_3-1} \frac{\{\phi_n(s)\}^2}{|\lambda_n|} ds < \epsilon_2^2, \quad \ldots .$$

It follows that

$$\sum_{m_1}^{m_2-1} \frac{\{\phi_n(s)\}^2}{|\lambda_n|} < \epsilon_1,$$

in the points of a set of linear measure $> b-a-\epsilon_1$; that

$$\sum_{m_2}^{m_3-1} \frac{\{\phi_n(s)\}^2}{|\lambda_n|} < \epsilon_2,$$

in the points of a set of linear measure $> b-a-\epsilon_2$, and so on.

It follows that in the points of a set of linear measure $> b-a-\zeta$,

$$\sum_{m_1}^{m_1+r} \frac{\{\phi_n(s)\}^2}{|\lambda_n|} < \zeta,$$

for all values of r.

Now choose a set of values of ζ, say $\zeta_1, \zeta_2, \ldots$ converging to zero, and such that $\zeta_1+\zeta_2+\ldots$ converges to a positive number δ arbitrarily small.

We see then that in a set of measure $> b-a-\delta$, the series $\sum\limits_{n=1}^{\infty} \frac{\{\phi_n(s)\}^2}{|\lambda_n|}$ converges uniformly. Since δ is arbitrarily small, we see that $\Sigma \frac{\{\phi_n(s)\}^2}{\lambda_n}$ is convergent for all values of s that do not belong to some set of linear measure zero.

Since
$$\left| \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n} \right| \leqslant \tfrac{1}{2} \frac{\{\phi_n(s)\}^2}{|\lambda_n|} + \tfrac{1}{2} \frac{\{\phi_n(t)\}^2}{|\lambda_n|},$$

we see that the series $\Sigma \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ is absolutely convergent, provided neither s nor t belong to the exceptional linear set. We thus see that $\sum\limits_{n=1} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ is convergent, except possibly at points of a set of which the plane measure is zero.

It has thus been shewn that :—

IX. *If $K(s, t)$ be any symmetrical nucleus not necessarily continuous or limited, such that the series of which the terms are the reciprocals of its characteristic numbers is absolutely convergent, then $\sum\limits_{n=1} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ is absolutely convergent, except possibly at points of a set of which the plane measure is zero.*

Employing theorem V, we now obtain the following theorem :—

X. *If $K(s, t)$ be any symmetrical nucleus, limited or unlimited, such that its characteristic functions are continuous, and such that $\int_a^b \{K(s, t)\}^2 dt$ is a limited function of s, then, if the series of which the terms are the reciprocals of the characteristic numbers is absolutely convergent, the series $\sum\limits_{n=1} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ converges to $K(s, t)$ with the possible exception of points (s, t) of a set of which the plane measure is zero.*

9. Let $K(s, t)$ be such that, provided $|s-t|$ is $\geqslant \epsilon$, where ϵ is an arbitrarily chosen positive number, $\left| \sum\limits_{n=1}^{n=m} \phi_n(s)\,\phi_n(t) \right|$ is less than some fixed positive number, for all values of m, and for all the values of s and t which satisfy the prescribed condition. Also let us assume that all the characteristic numbers $\{\lambda_n\}$ corresponding to $K(s, t)$ are positive.

It can then be shewn, by employing Abel's lemma, that the series $\sum\limits_{n=1} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ converges uniformly for all the values of s and t, such that

$$|s-t| \geqslant \epsilon.$$

For

$$\sum_{n=m+1}^{n=m+p} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n} = \frac{1}{\lambda_{m+p+1}} \sum_{m+1}^{m+p} \phi_n(s)\,\phi_n(t)$$

$$+\sum_{q=1}^{q=p}\left[\left(\frac{1}{\lambda_{m+q}}-\frac{1}{\lambda_{m+p+1}}\right)\sum_{m+1}^{m+q} \phi_n(s)\,\phi_n(t)\right];$$

and since

$$\frac{1}{\lambda_{m+q}}-\frac{1}{\lambda_{m+p+1}} \geqslant 0,$$

we have $$\left|\sum_{n=m+1}^{n=m+p} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}\right| < \frac{P}{\lambda_{m+p+1}} + P\left(\frac{1}{\lambda_{m+1}}-\frac{1}{\lambda_{m+p+1}}\right) < \frac{P}{\lambda_{m+1}},$$

where P is a fixed positive number, dependent only on ϵ. Since $\frac{1}{\lambda_{m+1}}$ is arbitrarily small, by making m large enough, the uniform convergence of the series is established.

If we assume that $K(s, t)$ is continuous except on the line $s = t$, where it is infinitely discontinuous, and that $\int_a^b \{K(s, t)\}^2 dt$ is a limited function of s, the functions $\{\phi_n(s)\}$ are all continuous, and the theorem VI of § 4 is applicable.

We obtain then the following theorem :—

XI. *If $K(s, t)$ is continuous except on the line $s = t$, where it may be infinitely discontinuous, but such that $\int_a^b \{K(s, t)\}^2 dt$ is a limited function of s, and if $\left|\sum\limits_{n=1}^{n=m} \phi_n(s)\,\phi_n(t)\right|$ is less than some fixed finite number, independent of m, for all values of s and t such that $|s-t| \geqslant \epsilon$, where ϵ is an arbitrarily chosen positive number, then the series $\sum\limits_{n=1} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ converges uniformly to $K(s, t)$ for all values of s and t such that*

$$|s-t| \geqslant \epsilon.$$

This theorem is applicable to the case of the Sturm-Liouville functions and to other orthogonal functions which arise in connexion with linear differential equations.

10. Let $K(s, t)$ be such that $\int_a^b \{K(s, t)\}^2 dt$ is a limited function of s, and that the repeated function of $K(s, t)$ is equivalent to a continuous function $K_2(s, t)$.

Then, as has been shewn in theorem I, $\sum\limits_{n=1} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ converges uniformly to $K_2(s, t)$.

Therefore $\Sigma \left\{ \frac{\phi_n(s)}{\lambda_n} \right\}^2$ converges to a continuous function of s.

Let us consider separately those values of λ which are positive and those which are negative; let the former be denoted by

$$\lambda_1^{(P)},\ \lambda_2^{(P)},\ \lambda_3^{(P)},\ \ldots,$$

and the latter by $\qquad \lambda_1^{(N)},\ \lambda_2^{(N)},\ \lambda_3^{(N)},\ \ldots,$

each series being arranged in order of increasing absolute magnitude. It is then clear that the two series

$$\sum_{n=1} \left\{ \frac{\phi_n^{(P)}(s)}{\lambda_n^{(P)}} \right\}^2, \qquad \sum_{n=1} \left\{ \frac{\phi_n^{(N)}(s)}{\lambda_n^{(N)}} \right\}^2$$

are each convergent and have limited sum functions.

Let $K^{(1)}(s, t)$ be that nucleus, constructed as in § 3, which has $\{\phi_n^{(P)}(s)\}$, $\{\lambda_n^{(P)}\}$ for its sole characteristic functions and numbers, and let $K^{(2)}(s, t)$ be the nucleus, similarly constructed, which has $\{\phi_n^{(N)}(s\}$, $\{\lambda_n^{(N)}\}$ for its sole characteristic functions and numbers. We have then

$$\int_a^b \{K^{(1)}(s, t)\}^2 dt = \sum_{n=1} \left\{ \frac{\phi_n^{(P)}(s)}{\lambda_n^{(P)}} \right\}^2, \quad \int_a^b \{K^{(2)}(s, t)\}^2 dt = \sum_{n=1} \left\{ \frac{\phi_n^{(N)}(s)}{\lambda_n^{(N)}} \right\}^2;$$

and therefore

$$\int_a^b \{K^{(1)}(s, t)\}^2 dt + \int_a^b \{K^{(2)}(s, t)\}^2 dt = \int_a^b \{K(s, t)\}^2 dt.$$

Since $\phi_n^{(N)}(t)$ is orthogonal to all the characteristic functions of $K^{(1)}(s, t)$, we have

$$\int_a^b K^{(1)}(s, t)\,\phi_n^{(N)}(t)\,dt = 0;$$

and similarly, we have $\int_a^b K^{(2)}(s, t)\,\phi_n^{(P)}(t)\,dt = 0.$

It follows that

$$\int_a^b \{K^{(1)}(s,\ t)+K^{(2)}(s,\ t)\}\ \phi_n^{(P)}(t)\,dt = \lambda_n^{(P)}\phi_n^{(P)}(s),$$

and

$$\int_a^b \{K^{(1)}(s,\ t)+K^{(2)}(s,\ t)\}\ \phi_n^{(N)}(t)\,dt = \lambda_n^{(N)}\phi_n^{(N)}(s);$$

Hence the nucleus $K^{(1)}(s,\ t)+K^{(2)}(s,\ t)$ has the same characteristic functions and constants as $K(s,\ t)$. It will be shewn that it has no other characteristic functions than these.

Since $K^{(1)}(s, t')$ is orthogonal to $\phi_1^{(N)}(t'),\ \phi_2^{(N)}(t'),\ \ldots,$ we have

$$\int_a^b K^{(1)}(s,\ t')\,K^{(2)}(t,\ t')\,dt' = 0,$$

hence
$$\int_a^b [K^{(1)}(s,\ t')+K^{(2)}(s,\ t')][K^{(1)}(t',\ t)+K^{(2)}(t',\ t)]\,dt'$$
$$= \int_a^b K^{(1)}(s,\ t')\,K^{(1)}(t,\ t')\,dt' + \int_a^b K^{(2)}(s,\ t')\,K^{(2)}(t',\ t)\,dt',$$

and thus the repeated function of $K^{(1)}(s,\ t)+K^{(2)}(s,\ t)$ is the sum of the repeated functions of $K^{(1)}(s,\ t)$ and $K^{(2)}(s,\ t)$.

Also
$$\int_a^b \{K(s,\ t)\}^2 dt = \int_a^b \{K^{(1)}(s,\ t)+K^{(2)}(s,\ t)\}^2 dt,$$

since
$$\int_a^b K^{(1)}(s,\ t)\,K^{(2)}(s,\ t)\,dt = 0.$$

Since
$$\int_a^b \{K^{(1)}(s,\ t)+K^{(2)}(s,\ t)\}^2 dt = \sum_{n=1} \left\{\frac{\phi_n^{(P)}(s)}{\lambda_n^{(P)}}\right\}^2 + \sum_{n=1} \left\{\frac{\phi_n^{(N)}(s)}{\lambda_n^{(N)}}\right\}^2,$$

it now follows that $K^{(1)}(s,\ t)+K^{(2)}(s,\ t)$ has no characteristic functions and constants other than those of $K(s,\ t)$.

The following result has been established :—

XII. *If* $K(s,\ t)$ *be such that its repeated function is equivalent to a continuous function, then* $K(s,\ t)$ *may be expressed as the sum of two functions* $K^{(1)}(s,\ t)$, $K^{(2)}(s,\ t)$, *orthogonal to one another, and such that* $K^{(1)}(s,\ t)$ *has for its sole characteristic numbers those numbers corresponding to* $K(s,\ t)$ *which are positive; and such that* $K^{(2)}(s,\ t)$ *has for its sole characteristic numbers those numbers corresponding to* $K(s,\ t)$ *which are negative. The characteristic function corresponding to a characteristic number of* $K^{(1)}(s,\ t)$ *or of* $K^{(2)}(s,\ t)$ *is the same as the characteristic function of* $K(s,\ t)$ *corresponding to the same characteristic number.*

This theorem is applicable in particular to a continuous nucleus $K(s, t)$, which may have an infinite number of characteristic numbers of each sign.

If it were certain that $K^{(1)}(s, t)$, $K^{(2)}(s, t)$ were not infinitely discontinuous on the straight line $s = t$, theorem VIII would be applicable to these nuclei, and the uniform convergence of $\Sigma \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ to $K(s, t)$ could be deduced. As, however, this possibility cannot be excluded, at all events *a priori*, no further general result has been obtained for the case of a continuous nucleus with an infinite number of characteristic numbers of each sign beyond that obtained in § 3; that, for each point (s, t), with the possible exception of points belonging to a set of measure zero, a sequence of values of m can be so determined that $\sum_{n=1}^{n=m} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ converges to $K(s, t)$ as m goes through the values in the sequence.

THE DYNAMICAL THEORY OF THE TIDES IN A POLAR BASIN

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1. *Introductory.*

THE problem here attacked is that of finding the tidal oscillations on a rotating globe in an ocean which is bounded by parallels of latitude. In this paper the case examined is that of a polar basin, so that only one boundary is to be considered. A similar analysis, however, can be applied to the double boundary problem—the zonal sea. In a further paper I intend to give the results of the latter case.

The only complete solution of a tidal problem hitherto found is that of Hough [*] for the case of an ocean wholly covering a rotating globe. His results are in the form of series of spherical surface harmonics, and, in consequence, it is possible to include the effects of the self-attraction of the water particles. Poincaré [†] and Love [‡] have suggested the use of the method of Ritz for the determination of tides in restricted basins; and Poincaré [§] has also mentioned the possible application of Fredholm's integral equation method to these problems. Previously Kelvin [‖] had indicated a direct method of attacking the simpler problem of the tides in a zonal sea. Upon his method the following analysis is based. The form in which his results were left was such that it was almost impossible to make a numerical discussion of them; so that several alterations, suggested by Hough's work, have been made in the following paper. Since series of powers of cosines of the latitude are used, it is not possible to include the self-attraction. But the effects of the latter appear to be small, except in the vicinity of a coincidence of free and forced oscillations. The final results show that such an event cannot occur in basins of the dimensions considered; and hence the results without the self-attraction will represent well the complete expressions.

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[*] *Phil. Trans.*, A, t. 189, p. 201, and t. 191, p. 139.

[†] *Leçons de Méc. Cél.*, t. III, p. 297.

[‡] *Proceedings of Fifth Inter. Congress of Math.* (1912), Vol. II.

[§] *Göttinger Vorträge* (1910); *Leçons*, t. III, p. 233.

[‖] *Phil. Mag.*, 1875.

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2. *The Equations of Motion.*

Consider an ocean of uniform depth covering a rotating globe of negligible ellipticity, and bounded, if at all, by parallels of latitude. We shall principally discuss the case where the ocean includes one of the poles.

Let θ = co-latitude of a point on the surface;
ϕ = the longitude, reckoned eastward;
u = the southward component of velocity;
v = the eastward component of velocity;
ω = the angular velocity of the earth's rotation;
a = the mean radius of the earth;
h = the uniform depth of the ocean;
ζ = the tide height, reckoned above the free surface;
and $\bar{\zeta} = \dfrac{\Omega}{g}$, where Ω is the tide-producing potential.

Then, subject to certain approximations, the equations for the tidal oscillations are*

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} - 2\omega v \cos\theta = -\frac{g}{a}\,\frac{\partial}{\partial\theta}(\zeta-\bar{\zeta}) \\ &\frac{\partial v}{\partial t} + 2\omega u \cos\theta = -\frac{g}{a\sin\theta}\,\frac{\partial}{\partial\phi}(\zeta-\bar{\zeta}) \\ &\frac{\partial \zeta}{\partial t} = -\frac{h}{a\sin\theta}\left\{\frac{\partial}{\partial\theta}(u\sin\theta)+\frac{\partial v}{\partial\phi}\right\} \end{aligned}\right\}. \qquad (1)$$

$\bar{\zeta}$ has the general form $H_s P_2^s(\cos\theta)\cos(\sigma t+s\phi+\epsilon)$, where P_2^s is a spherical surface harmonic of rank s and order 2. For the three species of tides, s takes the values 0, 1, 2; and for each of these σ has several values.†

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* Lamb, *Hydrodynamics*, p. 314.

† *Ibid.*, pp. 342, 343. For details as to the various values of σ, see Darwin, *Scientific Papers*, Vol. I, pp. 20–25.

Assume
$$\zeta \propto \cos(\sigma t + s\phi + \epsilon),$$
$$u \propto \sin(\sigma t + s\phi + \epsilon),$$
$$v \propto \cos(\sigma t + s\phi + \epsilon).$$

Substitute in equations (1) and solve for u and v. On putting

$$\frac{\sigma^2}{4\omega^2} = f^2, \qquad \frac{\omega^2 a}{g} = m,$$

we find
$$\left.\begin{aligned} u\,.\,4m(f^2-1+\nu^2) &= -\sigma\sqrt{(1-\nu^2)}\,\frac{d\zeta'}{d\nu} - 2\omega s\,\frac{\sqrt{(1-\nu^2)}\,\zeta'}{\nu} \\ v\,.\,4m(f^2-1+\nu^2) &= -\,2\omega\,(1-\nu^2)\,\frac{d\zeta'}{d\nu} - \frac{\sigma s}{\nu}\,\zeta' \\ -\sigma\zeta &= -\,\frac{h}{a\nu}\left[\sqrt{(1-\nu^2)}\,\frac{d}{d\nu}(u\,.\,\nu) - sv\right] \end{aligned}\right\}, \tag{2}$$

where we have changed the variable θ into ν, $= \sin\theta$.

On eliminating u and v, we have, finally, putting

$$\beta = \frac{4\omega^2 a^2}{gh}, \qquad \zeta' = \zeta - \bar{\zeta}.$$

$$\beta\zeta\nu = \sqrt{(1-\nu^2)}\,\frac{d}{d\nu}\left[\frac{-\nu\sqrt{(1-\nu^2)}\,\dfrac{d\zeta'}{d\nu} - \dfrac{s}{f}\sqrt{(1-\nu^2)}\,\zeta'}{f^2-1+\nu^2}\right] + \frac{\dfrac{s}{f}\,(1-\nu^2)\,\dfrac{d\zeta'}{d\nu} + \dfrac{s^2}{\nu}\,\zeta'}{f^2-1+\nu^2}\,. \tag{3}$$

For the free oscillations, which we discuss first, $\bar{\zeta} = 0$, and ζ' is replaced by ζ.

If we suitably rearrange (3) for discussion, we find

$$\nu(1-\nu^2)\,\frac{d^2\zeta}{d\nu^2} - \frac{d\zeta}{d\nu}\left[2\nu^2 - 1 + \frac{2\nu^2(1-\nu^2)}{f^2-1+\nu^2}\right] - \zeta\left[\frac{s}{f}\,\nu + \frac{s}{f}\,\frac{2\nu(1-\nu^2)}{(f^2-1+\nu^2)} + \frac{s^2}{\nu} + \beta\nu(f^2-1+\nu^2)\right] = 0. \tag{4}$$

The coefficients show singularities at

(i) $f^2-1+\nu^2 = 0$, or $\nu = \pm\sqrt{1-f^2}$,

(ii) $1-\nu^2 = 0$, or $\nu = \pm 1$,

(iii) $\nu = 0$,

and (iv) $\nu = \infty$.

With regard to (i), f is unknown. It manifestly depends upon the depth of the sea and the extent of the basin. Moreover, for the same dimensions of the basin, f has an infinity of values. It is impossible, therefore, to treat this singularity as the others. But, on examination, it appears that there is no singularity at all at the points $\nu \pm \sqrt{1-f^2}$. Or, rather, there is a special arrangement of the terms in the equation which prevents infinities occurring at these points.

When
$$f^2-1+\nu^2=0,$$
$$\frac{\sigma}{2\omega}=\pm\cos\theta.$$

If we take equations (1) and remove the time-and-longitude factor, the first pair present the form

$$\left.\begin{aligned}\sigma u-2\omega v\cos\theta &= -\frac{g}{a}\frac{d\zeta}{d\theta}\\ -\sigma v+2\omega u\cos\theta &= \frac{gs}{a\sin\theta}\zeta\end{aligned}\right\}. \qquad (5)$$

On putting $\sigma=2\omega\cos\theta$ in these, we find

$$\sigma(u-v)=-\frac{g}{a}\frac{d\zeta}{d\theta},$$
$$-\sigma(v-u)=\frac{gs}{a\sin\theta}\zeta.$$

That is,
$$\frac{d\zeta}{d\theta}+\frac{s\zeta}{\sin\theta}=0,$$

or
$$\sqrt{(1-\nu^2)}\frac{d\zeta}{d\nu}+\frac{s}{\nu}\zeta=0.$$

Hence, when the denominators of the fractions in (3) vanish, the numerators do also; while the value of the fraction in such a case is a finite quantity—viz., u in the first and v in the second. Exactly the same happens when $\sigma=-2\omega\cos\theta$. Hence this appearance of a singularity may be disregarded.*

Proceeding to (ii) we remark that, in the problem under consideration, ν is always positive. Hence, disregarding the singularity $\nu=-1$, we proceed to find a solution of (4) valid in the vicinity $\nu = 1$. For this purpose let

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* See Poincaré, *Leçons de Méc. Céleste*, t. III, pp. 141, 142. The points $\nu=\pm\sqrt{(1-f^2)}$ are, of course, "apparent singularities."

$$\zeta = (\nu-1)^{\alpha}[A_0+A_1(\nu-1)+A_2(\nu-1)^2+\dots].$$

The indicial equation is

$$\alpha(\alpha-1)+\tfrac{1}{2}\alpha = 0,$$

giving

$$\alpha = 0,\ \tfrac{1}{2}.$$

Hence the solutions suggested are of the character

$$\zeta = A_0+A_1(\nu-1)+A_2(\nu-1)^2+\dots, \tag{6}$$

and

$$\zeta = (\nu-1)^{\frac{1}{2}}\{B_0+B_1(\nu-1)+B_2(\nu-1)^2+\dots\}. \tag{7}$$

Usually the solutions are convergent in the vicinity of the singularity. But it is evident that, unless the coefficients in (7) have special values, $\frac{d\zeta}{d\nu}$ will have an infinity at $\nu = 1$.

For case (iii) the indicial equation is

$$\alpha(\alpha-1)+\alpha-s^2 = 0,$$

whence

$$\alpha = \pm s.$$

The solutions indicated are then of the character

$$\zeta = \nu^{s}\ \{A_0+A_1\nu+A_2\nu^2+\dots\}, \tag{8}$$

and

$$\zeta = \nu^{-s}\{B_0+B_1\nu+B_2\nu^2+\dots\}. \tag{9}$$

In the case of the tides of first species (symmetrical oscillations) $s = 0$. The indicial equation has then equal roots, $\alpha = 0$. The two solutions are then*

$$\zeta = [\nu^{\alpha}\{A_0+A_1\nu+A_2\nu^2+\dots\}]_{\alpha=0},$$

and

$$\zeta = \left[\frac{\partial}{\partial\alpha}\{\nu^{\alpha}(A_0+A_1\nu+A_2\nu^2+\dots)\}\right]_{\alpha=0}.$$

The latter may be written

$$\zeta = \log\nu\,[A_0+A_1\nu+A_2\nu^2+\dots]_{\alpha=0}$$
$$+\left[\frac{\partial}{\partial\alpha}(A_0+A_1\nu+A_2\nu^2+\dots)\right]_{\alpha=0}.$$

This solution, owing to the presence of the logarithm, has a singularity at $\nu=0$. The solution (9) likewise becomes infinite there.[*] Hence they are useless in considering a basin including the pole, though they would be of value in the case of a zonal sea. Solutions (6) and (7) are finite at $\nu=1$; so that it may be concluded that (8) is convergent from $\nu=0$ up to and including $\nu=1$. But, since from (7) $\dfrac{d\zeta}{d\nu}$ probably has a singularity at $\nu=1$, we must investigate fully the convergence of the first derivative.[†]

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* See Forsyth, *Theory of Differential Equations*, Vol. IV, p. 93.

Proceeding to find the relation between the coefficients in the series (8) from equation (3), it will be found advisable to break up the equation by the introduction of an auxiliary quantity X.

Let
$$\nu\frac{d\zeta'}{d\nu}+\frac{s}{f}\zeta'=X(f^2-1+\nu^2). \tag{10}$$

Then (3) becomes
$$\beta f^2\nu^2\zeta=-f^2\nu(1-\nu^2)\frac{dX}{d\nu}+f^2\nu^2X+sf(1-\nu^2)X+s^2\zeta'. \tag{11}$$

From the form of solution indicated in (8), we assume
$$\zeta'=\sum_0^\infty A_n\nu^{s+n},$$
$$\bar{\zeta}=\sum_0^\infty B_n\nu^{s+n},$$
and
$$X=\sum_0^\infty C_n\nu^{s+n};$$
whence
$$\zeta=\zeta'+\bar{\zeta}=\sum_0^\infty (A_n+B_n)\nu^{s+n}.$$

From (10), on substituting and equating coefficients,
$$A_n=\frac{(f^2-1)C_n+C_{n-2}}{n+s+s/f}, \tag{12}$$
and, from (11),
$$\beta f^2(A_{n-2}+B_{n-2})-s^2A_n=\{-f^2(n+s)+sf\}C_n$$
$$+\{f^2(n+s-1)-sf\}C_{n-2}. \tag{13}$$

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[*] The treatment of this solution, since s is an integer, would require to be carried out in a similar manner to the analogous solution of Bessel's equation. See Forsyth, *Theory of Differential Equations*, Vol. IV, p. 101.

[†] I am indebted to Dr. Bromwich for drawing my attention to this fact.

It is simplest, though less convenient afterwards, to eliminate A between (12) and (13). The result is the general relation

$$\begin{aligned}&-C_n\frac{n(2s+n)}{n+s+s/f}\\ &+C_{n-2}\left\{-\frac{\beta(f^2-1)}{n-2+s+s/f}+\frac{(n+s)^2}{n+s+s/f}-1\right\}\\ &-C_{n-4}\frac{\beta}{n-2+s+s/f}=\beta\,.\,B_{n-2}.\end{aligned}\tag{14}$$

In (14) put $n=0$. Then

$$0\,.\,C_0=0 \quad \text{or} \quad C_0 \text{ is arbitrary.}$$

Again, put $n=1$, and we find

$$-C_1\frac{(2s+1)}{1+s+s/f}=0, \quad \text{or} \quad C_1=0.$$

It follows that the series is one of *even* powers of ν only, factored by ν^s; the odd coefficients all vanish. In discussing the solutions of (14) three cases, $s=0, 1, 2$, must be mentioned. For $s=0$ there exist

$$B_0=\tfrac{2}{3}H_0, \quad \text{and} \quad B_2=-H_0.$$

For $s=2$ there exists $B_0=H_2$ only.

In the other case, $s=1$, a different form must be given to the equations owing to the fact that the tide-producing potential is

$$\bar{\zeta}=H_1\nu\sqrt{(1-\nu^2)}.$$

This case is examined separately further on.

We will discuss (14), leaving s general, though having more in mind the value $s=2$, as it is obvious that the simpler case $s=0$ can, by a slight readjustment of quantities, be made to fall within the same analysis.

Let

$$\left.\begin{aligned}L_n&=\left\{\frac{\beta(f^2-1)}{n+s+s/f}-\frac{(n+2+s)^2}{n+2+s+s/f}+1\right\}\frac{n+2+s+s/f}{(n+2)(n+2+2s)}\\ M_n&=\frac{\beta(n+4+s+s/f)}{(n+4)(n+4+2s)(n+2+s+s/f)}\\ &\beta B_0\frac{2+s+s/f}{2(2+2s)}=\kappa C_0\end{aligned}\right\}.\tag{15}$$

Equations (14) then become

$$\left.\begin{aligned} (\kappa+L_0)\,C_0+C_2 &= 0 \\ M_0C_0+L_2C_2+C_4 &= 0 \\ \text{and thenceforward}\quad M_{n-2}C_{n-2}+L_nC_n+C_{n+2} &= 0 \end{aligned}\right\}. \tag{16}$$

In equations (16), when $\kappa=0$, f is unknown. When f is given, κ, which determines C_0, is unknown. Hence they embrace both the free and the forced oscillations. In addition, the quantities C must satisfy the boundary condition. If we suppose the basin we are dealing with bounded by a parallel of latitude $\nu=\nu_1$ we must have there

$$u=0,$$

that is,

$$\Sigma\, C_n \nu_1^{s+n}=0.\text{*} \tag{17}$$

When the ocean covers the whole globe, this equation is replaced by the condition that ζ and u are both finite when $\nu=1$.

Consider the convergence of the series defined by the last equation in (16). It is to be noted that we know nothing of the magnitude of f; it may even be negative. From (16) we have

$$\begin{aligned} \frac{C_{n+2}}{C_n} &= -L_n-\frac{M_{n-2}}{C_n/C_{n-2}} \\ &= \left\{-\frac{\beta(f^2-1)}{n+s+s/f}+\frac{(n+2+s)^2}{n+2+s+s/f}-1\right\}\frac{n+2+s+s/f}{(n+2)(n+2+2s)} \\ &\qquad -\frac{\beta(n+2+s+s/f)}{(n+2)(n+2+2s)(n+s+s/f)}\,\frac{1}{C_n/C_{n-2}}. \end{aligned}$$

It is obvious that in the limit C_{n+2}/C_n tends either to unity or zero. More exactly, if C_n/C_{n-2} is not a small quantity, whatever finite value f may have, for values of n sufficiently large, we may write

$$C_{n+2}/C_n=1+\frac{0}{n}+\frac{\omega_n}{n^2},$$

where ω_n is less than a fixed finite number.

Hence the series $\Sigma\, C_n\nu^{s+n}$ is only convergent so long as $|\nu|<1$.

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* The quantity X or $\Sigma C_n\nu^{s+n}$ is a simple multiple of the velocity u, as is seen by reference to (2) and (10).

Consequently, in order that the series may be valid at the point $\nu = 1$, f (or κ) in (16) must be so determined that

$$\underset{n \to \infty}{L} \frac{C_{n+2}}{C_n} = 0.$$

That is, f (or κ) must be a solution of the convergent infinite continued fraction

$$0 = \kappa + L_0 - \frac{M_0}{L_2 -} \frac{M_2}{L_4 -} \frac{M_4}{L_6 -} \dots \text{ ad inf.} \tag{18}$$

In this case the series $\Sigma C_n \nu^{n+s}$ is valid over the whole globe. But the application of (18) prevents the inclusion of any boundary condition.

If any other values of f (or κ) than those given by (18) be used, $\frac{C_{n+2}}{C_n}$ tends to unity as n increases. Hence the series $\Sigma C_n \nu^{s+n}$ is not valid at $\nu = 1$. Consequently the polar basin for which our series give the tide heights cannot include the equator. For seas including the equator, but not the pole, series in powers of μ, $= \sqrt{(1-\nu^2)}$, are appropriate.

The series we have discussed is, of course, $\Sigma C_n \nu^{s+n}$, and not $\Sigma A_n \nu^{s+n}$, the former being used because of the simplicity of the relation between the coefficients. But there has, as yet, appeared no case in which the series was convergent and its derivative divergent at $\nu = 1$, as was anticipated. Yet, if $\Sigma C_n \nu^{s+n}$ is absolutely convergent then, from (12), $\Sigma A_n \nu^{s+n}$ and its derivative are also convergent within the same circle of convergence. It may be that, owing to a lack of knowledge of the value of f, and a consequent lack of delicacy in the tests, or owing to the relation (12) masking some point, no indication of the expected result has appeared. One case, however, is interesting. For the luni-solar semi-diurnal tide f is rigorously unity and $s = 2$. Hence, from (12)

$$A_n = \frac{C_{n-2}}{n+4}.$$

Also, from (14), $nC_n - \left(\frac{n^2+3n}{n+4}\right) C_{n-2} + \frac{\beta}{n+2} C_{n-4} = 0,$

or
$$n(n+6) A_n - n(n+3) A_{n-2} + \beta A_{n-4} = 0.$$

From this
$$A_n/A_{n-2} = \frac{n+3}{n+6} - \frac{\beta}{n(n+6)} \frac{1}{A_{n-2}/A_{n-4}}.$$

Hence, if A_{n-2}/A_{n-4} is not small, the limiting form of A_n/A_{n-2} is

$$\frac{n+3}{n+6}, \quad \text{or} \quad 1-\frac{3}{n}.$$

This is identical with the limiting form of the ratio of the coefficients of ν^n and ν^{n-2} in the expansion of $(1-\nu^2)^{\frac{1}{2}}$. Hence the terms of the series $\Sigma A_n \nu^{n+s}$ become ultimately comparable with those of $(1-\nu)^{\frac{1}{2}}$, so that we may write

$$\zeta' = L+M(1-\nu^2)^{\frac{1}{2}},$$

where L and M are functions of ν that do not vanish when $\nu = 1$. In this case clearly ζ' is finite at $\nu = 1$, but $\frac{\partial \zeta'}{\partial \nu}$ has an infinity there.*

The series are all undoubtedly convergent with their first derivatives so long as $\nu < 1$. For the future we restrict ourselves to the condition that the basin shall not include nor extend up to the equator.

In seeking the solution for a polar basin we have to solve the equations

$$\left.\begin{array}{r} C_0\nu_1^s+C_2\nu_1^{s+2}+C_4\nu_1^{s+4}+\ldots = 0 \\ (\kappa+L_0)\,C_0+C_2 = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ M_{n-2}\,C_{n-2}+L_n\,C_n+C_{n+2} = 0 \end{array}\right\}. \tag{19}$$

First consider a finite number of unknown quantities, say $(r+1)$. That is, suppose we can find a solution of the original differential equations by giving proper values to $C_0, C_2, \ldots, C_{2r}$. If we suppose f (or κ) known, we have only to express in terms of C_0 the remaining r quantities C. For this purpose we require r equations only, viz., the following :—

$$\left.\begin{array}{r} C_0\nu_1^s+C_2\nu_1^{s+2}+C_4\nu_1^{s+4}+\ldots+C_{2r}\nu_1^{s+2r} = 0 \\ (\kappa+L_0)\,C_0+C_2 = 0 \\ M_0\,C_0+L_2\,C_2+C_4 = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ M_{2r-6}\,C_{2r-6}+L_{2r-4}\,C_{2r-4}+C_{2r-2} = 0 \end{array}\right\}. \tag{20}$$

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* Cf. Lamb, p. 325.

On solving these in the usual way, we have

$$
\begin{aligned}
C_0 \div & \begin{vmatrix}
\nu_1^{s+2} & \nu_1^{s+4} & \nu_1^{s+6} & \dots\dots\dots\dots & & \nu_1^{s+2r} \\
1 & 0 & 0 & \dots\dots\dots\dots & & 0 \\
L_2 & 1 & 0 & \dots\dots\dots\dots & & 0 \\
M_2 & L_4 & 1 & \dots\dots\dots\dots & & 0 \\
 & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & M_{2r-6} & L_{2r-4} \;\; 1 & 0
\end{vmatrix} \\
= -C_2 \div & \,|\cdots| \\
= \quad C_4 \div & \,|\cdots| \\
= \dots & \\
= C_{2r}(-1)^r \div & \begin{vmatrix}
\nu_1^{s} & \nu_1^{s+2} & \nu_1^{s+4} & \dots\dots\dots\dots & & \nu_1^{s+2r-2} \\
\kappa+L_0 & 1 & 0 & \dots\dots\dots\dots & & 0 \\
M_0 & L_2 & 1 & \dots\dots\dots\dots & & 0 \\
0 & M_2 & L_4 & \dots\dots\dots\dots & & 0 \\
0 & \dots & \dots & \dots & & \dots \\
0 & 0 & 0 \;\dots & M_{2r-6} & L_{2r-4} & 1
\end{vmatrix}.
\end{aligned}
$$

Hence, if we write the determinant associated with C_{2r} as D_{2r}, we have

$$(-1)^r . C_{2r} . {\nu_1}^{s+2r} = C_0 D_{2r}.$$

Now the original differential equations can only be satisfied by an infinite series, and, further, since the series $\Sigma C_{2r} {\nu_1}^{s+2r}$ converges, we have as a necessary consequence

$$\mathop{L}_{r\to\infty} C_{2r}\nu_1^{s+2r} = 0.$$

Hence we must have $\qquad \mathop{L}\limits_{r\to\infty} D_{2r} = 0.$

This is the equation that determines the possible values of f (or κ) for a given boundary. Written *in extenso* it is

$$
\begin{vmatrix}
1 & \nu_1^2 & \nu_1^4 & \nu_1^6 & \nu_1^8 & \dots \\
\kappa+L_0 & 1 & 0 & 0 & 0 & .. \\
M_0 & L_2 & 1 & 0 & 0 & \dots \\
0 & M_2 & L_4 & 1 & 0 & \dots \\
0 & 0 & M_4 & L_6 & 1 & \dots \\
\dots & \dots & \dots & \dots & &
\end{vmatrix} = 0. \tag{21}
$$

In calculating the value of this determinant, we may use the sequence

$$\begin{aligned} 0 &= 1-(\kappa+L_0)\Delta_2-M_0\Delta_4, \\ \Delta_2 &= \nu_1^2- \quad L_2\Delta_4 \quad -M_2\Delta_6, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ \Delta_n &= \nu_1^n- \quad L_n\Delta_{n+2} \quad -M_n\Delta_{n+4}\,; \end{aligned}$$

where Δ_n represents the determinant beginning

$$\begin{vmatrix} \nu_1^n & \nu_1^{n+2} & \nu_1^{n+4} & \ldots \\ L_n & 1 & 0 & \ldots \\ M_n & L_{n+2} & 1 & \ldots \\ \ldots & \ldots & \ldots & \end{vmatrix}. \tag{22}$$

In order that this determinant may be used at all we must show that, for a given finite value of f, it converges. The object will then be to find out for what value of f it converges to zero.

From (15) it is easily seen that $|L_n| < 1$ for sufficiently large values of n, and that M_n is a small fraction. Hence

$$|\Delta_n| < \nu^n \begin{vmatrix} 1 & \nu_1^2 & \nu_1^4 & \ldots \\ 1 & 1 & 0 & \ldots \\ M_n & 1 & 1 & \ldots \\ 0 & M_{n+2} & 1 & \ldots \\ \ldots & \ldots & & \ldots \end{vmatrix},$$

if, on expanding, we replace the usual minus signs by plus. From this,

$$\frac{|\Delta_n|}{\nu_1^n} < 1+\nu_1^2+\nu_1^4 \begin{vmatrix} 1 & 1 \\ M_n & 1 \end{vmatrix} +\nu_1^6 \begin{vmatrix} 1 & 1 & 0 \\ M_n & 1 & 1 \\ 0 & M_{n+2} & 1 \end{vmatrix}$$

$$+\nu_1^8 \begin{vmatrix} 1 & 1 & 0 & 0 \\ M_n & 1 & 1 & 0 \\ 0 & M_{n+2} & 1 & 1 \\ 0 & 0 & M_{n+4} & 1 \end{vmatrix} +\ldots.$$

Or,
$$\begin{aligned}\frac{|\Delta_n|}{\nu_1^n} &< 1+\nu_1^2+\nu_1^4(1+M_n)+\nu_1^6(1+M_n+M_{n+2})\\ &\qquad +\nu_1^8(1+M_n+M_{n+2}+M_{n+4}+M_nM_{n+4})\\ &\qquad +\ldots,\\ &< 1+\nu_1^2+\nu_1^4(1+M_n)\\ &\qquad +\nu_1^6(1+M_n)(1+M_{n+2})\\ &\qquad +\nu_1^8(1+M_n)(1+M_{n+2})(1+M_{n+4})\\ &\qquad +\ldots\,. \qquad (23)\end{aligned}$$

The ratio of the $(p+1)$-th term to the p-th of this last series is

$$\nu_1^2(1+M_{n+2p-6}),$$

i.e.,
$$\nu_1^2\left\{1+\frac{\beta(n+2p-2+s+s/f)}{(n+2p+2)(n+2p-2+2s)(n+2p-4+s+s/f)}\right\},$$

i.e., <1, after some fixed value of s, since $\nu<1$.

Hence $\dfrac{|\Delta_n|}{\nu_1^n}$ is less than the sum of an absolutely convergent series, and so has a finite value. It follows at once that the original determinant on the left-hand side of (21) has a finite value for finite values of f.

Again, the series in (23) decreases in sum as n increases, since each coefficient decreases. The limit is $\dfrac{1}{1-\nu_1^2}$. Also the factor ν_1^n decreases to zero as n increases. Hence the series $\Delta_2, \Delta_4, \ldots, \Delta_{2r}$ ultimately decreases to zero in the manner of a geometrical progression.

In the case of the tides of the second species ($s=1$), the form of the tide-producing potential suggests the substitution

$$\zeta' = \sqrt{(1-\nu^2)}\,z',$$

$$\bar{\zeta} = \sqrt{(1-\nu^2)}\,\bar{z},$$

$$\zeta = \sqrt{(1-\nu^2)}\,z.$$

Equations (2) then give, for the velocities,

$$u\,.\,4m(f^2-1+\nu^2) = -\sigma\left\{(1-\nu^2)\frac{dz'}{d\nu}-\nu z'+\frac{2\omega}{\sigma}(1-\nu^2)\frac{z'}{\nu}\right\},$$

$$v\,.\,4m(f^2-1+\nu^2) = -\sqrt{(1-\nu^2)}\left\{2\omega(1-\nu^2)\frac{dz'}{d\nu}-2\omega\nu z'+\sigma\frac{z'}{\nu}\right\}. \qquad (24)$$

And in place of (3) we have

$$\beta z . \nu = -\frac{d}{d\nu}\left[\frac{\nu(1-\nu^2)\dfrac{dz'}{d\nu} - \nu^2 z' + \dfrac{1-\nu^2}{f} z'}{f^2-1+\nu^2}\right]$$

$$+\frac{\dfrac{1-\nu^2}{f}\dfrac{dz'}{d\nu} - \dfrac{\nu z'}{f} + \dfrac{z'}{\nu}}{f^2-1+\nu^2}. \qquad (25)$$

The coefficients in (25) have singularities at $\nu = 0$ and $\nu = \pm 1$. The indicial equation corresponding to a solution in the vicinity of $\nu = 0$, is

$$-\alpha(\alpha-1)-\alpha+1 = 0;$$

whence $$\alpha = \pm 1. \qquad (26)$$

Corresponding to $\nu = +1$ ($\nu = -1$ cannot appear in the problem), we have

$$2\alpha(\alpha-1)+3\alpha = 0;$$

whence $$\alpha = 0, \ -\tfrac{1}{2}.$$

These results might have been inferred, of course, from the results found previously. It seems probable that a discontinuity in the series will occur at $\nu = 1$.

Let $$\nu(1-\nu^2)\frac{dz'}{d\nu} - \nu^2 z' + \frac{1-\nu^2}{f} z' = X(f^2-1+\nu^2). \qquad (27)$$

Equation (25) then becomes

$$f^2\beta z\nu^2 = -\frac{dX}{d\nu} f^2\nu + fX + z'. \qquad (28)$$

As before, assume as a solution in the vicinity of $\nu = 0$,

$$z' = \sum_1^\infty A_n \nu^n,$$

$$\bar{z} = \sum_1^\infty B_n \nu^n,$$

and $$X = \sum_1 C_n \nu^n.$$

On substituting and equating coefficients, (28) gives

$$C_n(f-f^2 n) = f^2\beta . A_{n-2} - A_n + f^2\beta . B_{n-2}.$$

Similarly (27) gives

$$A_n\left(n+\frac{1}{f}\right)-A_{n-2}\left(n-1+\frac{1}{f}\right)=(f^2-1)\,C_n+C_{n-2}.$$

Whence, on eliminating C between these two equations and reducing, the result is

$$A_n+A_{n-2}\left\{\frac{(f^2-1)\beta}{\frac{1}{f}-n}-\frac{(n-2)(n-1)-\frac{1}{f}}{\frac{1}{f}-(n-2)}\right\}\frac{\frac{1}{f}-n}{n^2-1}$$

$$+A_{n-4}\frac{\beta}{\frac{1}{f}-(n-2)}\,\frac{\frac{1}{f}-n}{n^2-1}$$

$$=-\frac{(f^2-1)\beta}{n^2-1}B_{n-2}-\frac{\left(\frac{1}{f}-n\right)\beta}{(n^2-1)\left(\frac{1}{f}-(n-2)\right)}B_{n-4}. \qquad (29)$$

The convergence of the series determined by (29) is discussed in the same manner as before.

$$A_n/A_{n-2}=\left\{-\frac{(f^2-1)\beta}{n-\frac{1}{f}}+\frac{(n-2)(n-1)-\frac{1}{f}}{(n-2)-\frac{1}{f}}\right\}\frac{n-\frac{1}{f}}{n^2-1}$$

$$-\frac{\beta\left(n-\frac{1}{f}\right)}{\left(n+2-\frac{1}{f}\right)(n^2-1)}\,\frac{1}{A_{n-2}/A_{n-4}}.$$

It is possible, as in the last case, for A_n/A_{n-2} to converge to zero. In the other case, when A_{n-2}/A_{n-4} is not small, for sufficiently large values of n,

$$A_n/A_{n-2}=1-\frac{1}{n}+\frac{\omega_n}{n^2}.$$

Hence the series $\Sigma A_n\nu^n$ is convergent only so long as $|\nu|<1$.

Equation (18) gives one type of free oscillations for the case where the ocean covers the whole globe; but, as is well known,[*] there are two types, characterized by symmetry or asymmetry with regard to the equator. Let us find an equation for the other set.

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[*] Hough, *loc. cit.*

The original differential equations (2) and (3) may be satisfied by a solution of the form

$$\zeta' = \sqrt{(1-\nu^2)}\,z', \;\ldots,$$

where z' is a series of powers of ν.

On making such a substitution, we find

$$u\,.\,4m(f^2-1+\nu^2) = -\sigma\left\{(1-\nu^2)\frac{dz'}{d\nu}-\nu z'\right\}-2\omega s(1-\nu^2)\frac{z'}{\nu}.$$

$$v\,.\,4m(f^2-1+\nu^2) = -2\omega\left\{(1-\nu^2)^{\frac{3}{2}}\frac{dz'}{d\nu}-\nu\sqrt{(1-\nu^2)}\,z'\right\}-\frac{\sigma s}{\nu}\sqrt{(1-\nu^2)}\,z',$$

$$\beta z\nu = -\frac{d}{d\nu}\left[\frac{\nu(1-\nu^2)\dfrac{dz'}{d\nu}-\nu^2 z'+(1-\nu^2)\dfrac{s}{f}z'}{f^2-1+\nu^2}\right]$$

$$+\frac{\dfrac{s}{f}(1-\nu^2)\dfrac{dz'}{d\nu}-\dfrac{s}{f}\nu z'+\dfrac{s^2}{\nu}z'}{f^2-1+\nu^2}. \qquad (18a)$$

The equation (18*a*) has apparent singularities at $\nu = \pm\sqrt{(1-f^2)}$, and real singularities at $\nu = \pm 1$, $\nu = 0$.

At $\nu = 0$, the indices are $\pm s$.

Hence we may expect to find a solution in the vicinity of $\nu = 0$, of the type

$$z' = \sum_{n=0}^{\infty} A_n \nu^{s+n}. \qquad (18b)$$

The solution involving the second index, $-s$, includes a logarithm, and hence is infinite at the pole.

On substituting the value (18*b*) in (18*a*) in the same manner as was done previously we arrive at the relation, for free oscillations,

$$A_n\frac{n^2+2ns}{n+s-s/f}+A_{n-2}\left\{\frac{\beta(f^2-1)}{n+s-s/f}-\frac{(n+s-1)(n+s-2)-s/f}{n-2+s-s/f}\right\}$$

$$+A_{n-4}\frac{\beta}{n-2+s-s/f} = 0. \qquad (18c)$$

Equation (29) is included in this as a special case ($s = 1$). The corresponding alternative for the tides of the second species is found by putting $s = 1$ in (14).

Equation (18*c*) differs but little in form from (14); but it is easy to verify that it provides the oscillations omitted in (14). In the case of no rotation, the frequency is given by $\frac{\sigma^2 a^2}{gh} = n(n+1)$, where n may take *any* integral value.

Putting $\omega = 0$ in (18*c*) and (14), it is obvious that the frequencies are given by

$$\frac{\sigma^2 a^2}{gh} = (n+s)(n+s-1)$$

in the first case, and by $\frac{\sigma^2 a^2}{gh} = (n+s-1)(n+s-2)$

in the second case; where $s = 0$ or 2, and n is any *even* integer.* These two statements each supply the other's deficit and complete the whole series of frequencies.

Hence (18*c*), with a corresponding boundary condition, will give a new set of free oscillations in the same manner as those already indicated. The series defined by (18*c*) is subject to the same limitations as that defined by (14), and hence the free oscillations for the ocean covering the whole globe will be given by an infinite continued fraction of the same type as (18), and for which the quantities L and M are given in (18*c*). As, however, these types of motion have no relation to those excited by the tide-producing potential, their interest is much diminished, and I have not proposed to pay them any further attention.

If equation (18*a*) be taken and reduced so as to express oscillations in a plane circular sea, by putting $\nu = x/a$ and making a infinite, we find

$$\frac{1}{x}\frac{d}{dx}\left[x\frac{dz'}{dx}\right] + \left\{(f^2-1)\frac{4\omega^2}{gh} - \frac{s^2}{x^2}\right\} z' = -(f^2-1)\frac{4\omega^2}{gh}\bar{z},$$

which is the familiar form of the equation. It is noteworthy that (3) reduces to precisely the same form. The two sets of oscillations apparently merge into one another as the basin narrows, and finally present only one solution, of the form

$$z = J_s(\kappa x), \qquad \kappa^2 = (f^2-1)\frac{4\omega^2}{gh}.$$

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* The circumstance $s = 0$, $n = 2$ in (14) is discussed later.

The two classes of oscillations in the general case have the forms

$$\sum_{n=0}^{\infty} A_n \nu^{n+s},$$

and

$$\sqrt{(1-\nu^2)} \sum_{n=0}^{\infty} B_n \nu^{n+s}.$$

Since each of the series converges at the equator ($\nu = 1$), the tide height for the second set of oscillations will vanish there, while that of the first will not. This affords a distinction between the two types when expressed in terms of ν.

3. *Tides of the First Species.*

The tides of the first species are determined from the previous formulæ by putting $s = 0$. From (14) we have

$$nC_n + C_{n-2} \left\{ \frac{\beta(f^2-1)}{n-2} - (n-1) \right\} + C_{n-4} \frac{\beta}{n-2} = -\beta . B_{n-2}; \tag{30}$$

together with

$$A_n = \frac{(f^2-1)\, C_n + C_{n-2}}{n}. \tag{31}$$

These equations are not satisfactory when $n = 0$ or 2. From the first when $n = 0$, we have

$$0 . C_0 = 0, \quad \text{or} \quad C_0 \text{ is arbitrary.}$$

But, from (31),

$$A_0 = \frac{(f^2-1)\, C_0}{0},$$

an infinite quantity.

In place of the arbitrary C_0, choose A_0, which is such that

$$nA_n = (f^2-1)\, C_n, \text{ when } n \text{ is zero.} \tag{32}$$

Then, from (30), when $n = 2$,

$$2C_2 + \beta A_0 = 0, \quad \text{or} \quad -\beta B_0, \tag{33}$$

according as the oscillations are free or forced.

The relation (32) requires that C_0 should be zero. Hence, when $n=4$, the condition is

$$4C_4+C_2\left(\frac{\beta(f^2-1)}{2}-3\right)=0, \quad \text{or} \quad -\beta B_2. \tag{34}$$

After these values of n, the relation (30) holds good. The boundary condition is simply $\Sigma C_n \nu_1^n = 0$.

The determinant for free oscillations is

$$\left|\begin{array}{cccccc} 1 & \nu_1^2 & \nu_1^4 & \nu_1^6 & \nu_1^8 & \dots \text{ ad inf.} \\ L_2 & 1 & 0 & 0 & 0 & \dots \\ M_2 & L_4 & 1 & 0 & 0 & \dots \\ 0 & M_4 & L_6 & 1 & 0 & \dots \\ 0 & 0 & M_6 & L_8 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \end{array}\right| = 0\,; \tag{35}$$

where

$$L_n = \frac{\beta(f^2-1)}{n(n+2)} - \frac{n+1}{n+2},$$

$$M_n = \frac{\beta}{(n+2)(n+4)}.$$

In attempting to find the roots of this equation we may take as a rough approximation to the first root the value given by putting all the powers of ν_1 above the second equal to zero. That is

$$L_2 = \frac{1}{\nu_1^2}. \tag{36}$$

Another approximation may be obtained from the original differential equation (3).

Putting $s=0$, it becomes

$$\frac{\sqrt{(1-\nu^2)}}{\nu}\frac{d}{d\nu}\left[\frac{\nu\sqrt{(1-\nu^2)}}{f^2-1+\nu^2}\frac{d\zeta}{d\nu}\right]+\beta\zeta=0.$$

If we put $\nu = x/a$, and then make a indefinitely great, the equation becomes

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{d\zeta}{dx}\right)+(f^2-1)\frac{4\omega^2}{gh}\zeta=0.$$

The solution of this is

$$\zeta = J_0\left\{x\sqrt{\frac{(f^2-1)\,4\omega^2}{gh}}\right\}.$$

The boundary condition then becomes

$$J_0'\left\{\nu_1\sqrt{(f^2-1)\,\frac{4\omega^2a^2}{gh}}\right\} = -J_1\{\nu_1\sqrt{(f^2-1)\beta}\} = 0. \qquad (37)$$

Equation (37) can readily be solved, and the values of f so found used as first approximations to the solution for the lowest roots of (35).

This problem has been carried to a second approximation by Proudman.*

Having determined a value for f, the determinant can then be evaluated by use of the sequence

$$\Delta_{2r} = \nu_1^{2r-2} - L_{2r}\Delta_{2r+2} - M_{2r}\Delta_{2r+4}.$$

The quantity Δ_{2r} eventually becomes equal to ν_1^{2r-2} in value. Hence, for a sufficiently high value of r, we may reject $\Delta_{2r+2}, \Delta_{2r+4}, \ldots$, &c., and putting $\Delta_{2r} = \nu_1^{2r-2}$, evaluate in succession $\Delta_{2r-2}, \Delta_{2r-4}, \ldots, \Delta_2$.

The value of Δ_2 will not be zero. A second and modified value of f is then chosen, and the value of Δ_2 again calculated. By this process of trial and error and final interpolation a value of f may be found that satisfies (35).

Since $\beta = \dfrac{4\omega^2a^2}{gh}$ and $f = \dfrac{\sigma}{2\omega}$, when $\omega = 0$, L_n becomes

$$\frac{\dfrac{a^2}{gh}\sigma^2}{n(n+2)} - \frac{n+1}{n+2},$$

and M_n becomes zero.

Hence, by a very slight modification of the preceding process, the frequencies for the case of no rotation may be calculated.

Following out this process the results given below were obtained. The sequences used in calculating the value of the determinant (35), as they ultimately become the geometric progression $\Sigma\nu_1^{2r}$, converge rapidly for small values of ν_1. But as ν_1 approaches unity the work becomes exceedingly laborious.

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 12 (1913).

TABLE I.—*Lowest frequencies of tides of first species.*

Boundary $\nu_1 = \frac{1}{4}$, *Lat.* 75° 30′.

Depth.	Value of $f^2 = \sigma^2/4\omega^2$.	Period in sidereal time.	Period in case of no rotation.
58,080 feet $\beta = 5$	45·08	1 h. 47 m.	1 h. 48 m.
29,040 ,, $\beta = 10$	23·05	2 h. 30 m.	2 h. 33 m.
14,520 ,, $\beta = 20$	12·02	3 h. 27 m.	3 h. 37 m.
7,260 ,, $\beta = 40$	6·51	4 h. 42 m.	5 h. 6 m.

TABLE II.—*Lowest frequencies of tides of first species.*

Boundary $\nu_1 = \frac{1}{2}$, *Lat.* 60°.

Depth.	Value of $f^2 = \sigma^2/4\omega^2$.	Period in sidereal time.	Period in case of no rotation.
58,080 feet $\beta = 5$	11·74	3 h. 30 m.	3 h. 40 m.
29,040 ,, $\beta = 10$	6·28	4 h. 47 m.	5 h. 12 m.
14,520 ,, $\beta = 20$	3·59	6 h. 20 m.	7 h. 22 m.
7,260 ,, $\beta = 40$	2·25	8 h. 0 m.	10 h. 40 m.

The principal forced tide of the first species is the lunar-fortnightly. For this the tide-producing potential is $H_0(\cos^2\theta - \frac{1}{3})$ or $H_0(\frac{2}{3} - \nu^2)$, and the frequency is $f^2 = {\cdot}00133$. We have, therefore, $B_0 = \frac{2}{3}H_0$ and

$B_2 = -H_0$. Hence, putting $H_0 = \kappa C_2$, we have to solve the equations

$$\left.\begin{aligned} \sum_{n=0}^{n=\infty} C_{2n}\nu_1^{2n} &= 0 \\ C_2 + \frac{\beta}{2} A_0 &= -\frac{\beta}{3} H_0 \\ \left(L_2 - \frac{\beta\kappa}{4}\right) C_2 + C_4 &= 0 \\ \text{and thereafter} \qquad M_{n-4} C_{n-4} + L_{n-2} C_{n-2} + C_n &= 0 \end{aligned}\right\}. \tag{38}$$

In the first of these equations it is to be remembered that, according to our previous arrangement, $C_0 = 0$.

Leaving for a moment the first pair of equations (38), the last set can be solved. We have

$$\begin{aligned} C_2 = -C_4 \div D_2 = C_6 \div D_4 &= -C_8 \div D_6 \\ &= \dots \\ &= (-1)^{r-1} C_{2r} \div D_{2r-2} \\ &= \dots; \end{aligned} \tag{39}$$

where we have written D_{2r} in place of

$$\begin{vmatrix} L_2 - \frac{1}{4}\beta\kappa & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ M_2 & L_4 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & M_4 & L_6 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & M_6 & L_8 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & M_{2r-2} & L_{2r} \end{vmatrix}.$$

The quantities D are related by the succession

$$D_{2r} = L_{2r} D_{2r-2} - M_{2r-2} D_{2r-4}. \tag{40}$$

The method of calculation is the following. The quantities D are calculated in succession by means of (40), the quantity κ being left undetermined. These are then substituted in the boundary equation and a linear equation determining κ results. From the condition

$C_2 = H_0/\kappa$, C_2 is found. All the coefficients C then follow. Finally, A_0 is determined from the second equation of (38), and the remaining A's from (31).

The tide height is then

$$\zeta = H_0(\tfrac{2}{3}-\nu^2)+A_0+\sum_{n=2}^{n=\infty}\left[\frac{(f^2-1)C_n+C_{n-2}}{n}\right]\nu^n.$$

The calculation is possible for any value of ν_1 less than unity; but, of course, it is very tedious owing to the slow convergence when ν_1 is near unity.

The following series for the tide heights were calculated with four-figure logarithms. Only three significant figures have been retained in the results, as, owing to the complexity of the processes, much doubt attaches to the fourth significant figure.

Lunar fortnightly tide. Boundary $\nu_1 = \frac{1}{4}$, Lat. 75° 30′.

Depth 58,080 *feet;* $\beta = 5$.

$$\zeta/H_0 = \{\cdot 0306 - \cdot 962\nu^2 - \cdot 305\nu^4 - \cdot 0102\nu^6 + \cdot 018\nu^8 - \ldots\}\cos(\sigma t+\epsilon),$$

Depth 29,040 *feet;* $\beta = 10$.

$$\zeta/H_0 = \{\cdot 0298 - \cdot 926\nu^2 - \cdot 587\nu^4 - \cdot 152\nu^6 + \cdot 029\nu^8 - \ldots\}\cos(\sigma t+\epsilon).$$

Depth 14,520 *feet;* $\beta = 20$.

$$\zeta/H_0 = \{\cdot 0284 - \cdot 858\nu^2 - 1\cdot 03\nu^4 - \cdot 735\nu^6 - \cdot 056\nu^8 - \ldots\}\cos(\sigma t+\epsilon).$$

Depth 7,260 *feet;* $\beta = 40$.

$$\zeta/H_0 = \{\cdot 0260 - \cdot 740\nu^2 - 1\cdot 62\nu^4 - 1\cdot 89\nu^6 - \cdot 893\nu^8 - \ldots\}\cos(\sigma t+\epsilon).$$

Lunar fortnightly tide. Boundary $\nu_1 = \frac{1}{2}$, Lat. 60°.

Depth 58,080 *feet;* $\beta = 5$.

$$\zeta/H_0 = \{\cdot 117 - \cdot 856\nu^2 - \cdot 285\nu^4 - \cdot 0158\nu^6 + \cdot 0136\nu^8 + \cdot 0091\nu^{10} + \cdot 0056\nu^{12}\ldots\}$$
$$\times\cos(\sigma t+\epsilon).$$

Depth 29,040 *feet* ; $\beta = 10$.

$$\zeta/H_0 = \{\cdot105 - \cdot737\nu^2 - \cdot493\nu^4 - \cdot096\nu^6 + \cdot0170\nu^8 + \cdot0208\nu^{10} + \cdot0133\nu^{12} \ldots\}$$
$$\times \cos(\sigma t + \epsilon).$$

Depth 14,520 *feet* ; $\beta = 20$.

$$\zeta/H_0 = \{\cdot0887 - \cdot557\nu^2 - \cdot751\nu^4 - \cdot358\nu^6 - \cdot0441\nu^8 + \cdot0382\nu^{10} + \cdot0353\nu^{12} \ldots\}$$
$$\times \cos(\sigma t + \epsilon).$$

Depth 7,260 *feet* ; $\beta = 40$.

$$\zeta/H_0 = \{\cdot0668 - \cdot333\nu^2 - \cdot916\nu^4 - \cdot915\nu^6 - \cdot468\nu^8 - \cdot0680\nu^{10} + \cdot0708\nu^{12} \ldots\}$$
$$\times \cos(\sigma t + \epsilon).$$

4. *Tides of Second Species.*

The equations to be solved are (29), together with the boundary condition $u = 0$ at $\nu = \nu_1$. From (24) the latter turns out to be

$$0 = \sum_{n=1}^{\infty} \frac{f^2\beta A_{n-2} - A_n + f^2\beta B_{n-2}}{f - f^2 n} \nu_1^n, \tag{41}$$

or
$$\sum_{n=1}^{\infty} A_n \left\{ \frac{f^2\beta\nu_1^{n+2}}{f - f^2(n+2)} - \frac{\nu_1^n}{f - f^2 n} \right\} = -\sum \frac{f^2\beta B_n}{f - f^2(n+2)} \nu_1^{n+2}. \tag{41$'$}$$

The only value of B occurring in the tide-producing potential is $B_1 = H_1$. Hence, writing, as before,

$$L_n = \left\{ \frac{(f^2-1)\beta}{\frac{1}{f} - (n+2)} - \frac{n(n+1 - \frac{1}{f}}{\frac{1}{f} - n} \right\} \frac{\frac{1}{f} - (n+2)}{(n+2)^2 - 1},$$

$$M_n = \frac{\beta}{\frac{1}{f} - (n+2)} \frac{\frac{1}{f} - (n+4)}{(n+4)^2 - 1},$$

$$a_n = \frac{f^2\beta\nu_1^{n+2}}{f - f^2(n+2)} - \frac{\nu_1^n}{f - f^2 n};$$

the equations to be solved are

$$\left.\begin{aligned} \sum_1^\infty a_n A_n &= -\frac{\beta H_1 \nu_1^3}{\frac{1}{f}-3} \\ L_1 A_1 + A_3 &= -\frac{(f^2-1)\,\beta H_1}{8} \\ M_1 A_1 + L_3 A_3 + A_5 &= -\frac{\left(5-\frac{1}{f}\right)\beta H_1}{\left(3-\frac{1}{f}\right)24} \\ \text{and thereafter}\qquad & \\ M_n A_n + L_{n+2} A_{n+2} + A_{n+4} &= 0 \end{aligned}\right\}. \qquad (42)$$

The determinant for the free oscillations is

$$\begin{vmatrix} a_1 & a_3 & a_5 & a_7 & a_9 & \dots \\ L_1 & 1 & 0 & 0 & 0 & \dots \\ M_1 & L_3 & 1 & 0 & 0 & \dots \\ 0 & M_3 & L_5 & 1 & 0 & \dots \\ 0 & 0 & M_5 & L_7 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \end{vmatrix} = 0.$$

Approximations to the first roots of this equation are found by putting all powers of ν_1 above the third equal to zero. Then

$$a_1 = -\frac{\nu_1}{f-f^2} + \frac{f^2\beta\nu_1^3}{f-f^2 . 3}$$

$$a_3 = -\frac{\nu_1^3}{f-f^2 . 3},$$

the other values of a vanishing.

The equation to be solved is

$$L_1 = \frac{a_1}{a_3},$$

or $$\frac{(f^2-1)\beta}{8} + \frac{\left(\frac{1}{f}-2\right)\left(\frac{1}{f}-3\right)}{\left(\frac{1}{f}-1\right)8} = -\left(\frac{\beta\nu_1^2}{\frac{1}{f}-3} - \frac{1}{f-f^2}\right) \div \frac{\nu_1^2}{f-3f^2}. \qquad (43)$$

This reduces to

$$\frac{9}{8}\beta f^4 - \frac{9}{8}\beta f^3 - f^2\left(\frac{\beta}{8}+\frac{3}{4}+\frac{3}{\nu_1^2}\right)+f\left(\frac{\beta}{8}+\frac{5}{8}+\frac{1}{\nu_1^2}\right)-\frac{1}{8}=0. \quad (44)$$

The solution of this equation exhibits four roots: two large, one positive and one negative; and two smaller, both positive. On making ω zero, the two smaller roots vanish and the larger roots become equal with opposite signs. This is exactly analogous to the results found by Hough.* The oscillations of long period apparently only exist when the rotation is not zero. In the other case they reduce to tidal currents.

In the accompanying table, we have only shewn the first of the smaller roots. The second may be judged to belong to the next complete set of roots. The approximate values only are given. But it is clear that, were the frequencies required for a given basin, they could be exactly evaluated by means of the formulæ given in the same manner as was done in the case of the tides of first species.

TABLE III.—*Lowest frequencies (approximate) of tides of second species. Boundary* $\nu_1 = \frac{1}{4}$, *Lat.* 75° 30′.

Depths.	Frequencies ($\sigma/2\omega$).
58,080 feet $\beta = 5$	3·35 −2·70 ·007
29,010 feet $\beta = 10$	2·51 −1·89 ·0069
14,520 feet $\beta = 20$	1·95 −1·29 ·0065
7,260 feet $\beta = 40$	1·58 − ·91 ·0058

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* *Phil. Trans.* (A), Vol. 191 (1898), p. 159.

TABLE IV.—*Lowest frequencies (approximate) of tides of second species.*

Boundary $\nu_1 = \frac{1}{2}$, *Lat.* 60°.

Depths.	Frequencies ($\sigma/2\omega$).
58,080 feet $\beta = 5$	1·98
	−1·32
	·024
29,040 feet $\beta = 10$	1·58
	− ·93
	·022
14,520 feet $\beta = 20$	1·34
	− ·68
	·018
7,260 feet $\beta = 40$	1·17
	− ·51
	·013

In solving equations (42) for the values of the coefficients in the case of the forced tides, it is worthy of note that, in the case $f = \frac{1}{2}$, which corresponds to the luni-solar diurnal tide K_1, if we disregard the boundary condition, the remaining equations are satisfied by

$$A_3 = A_5 = A_7 = \ldots = 0,$$

and

$$A_1 = -H_1.$$

Hence

$$\zeta = \zeta' + \bar{\zeta} = \nu \sqrt{(1-\nu^2)}(-H_1 + H_1) = 0.$$

This is Laplace's famous result that the diurnal tide for the case of an ocean of constant depth covering the whole globe produces no rise and fall of the surface.

In the case of the other diurnal tides, f, while not rigorously equal to $\frac{1}{2}$, is not far from that value. Hence, as shewn by Hough, the whole of the diurnal tides are small. It becomes, therefore, of importance to see the effect of a boundary in altering these conditions. As the numerical work in any case is very complicated we shall confine our attention to the case of $f = \frac{1}{2}$.

In (42) put $H_1 = \kappa A_1$ as before. The second and third equations then become

$$\left.\begin{aligned}(L_1 - \tfrac{3}{32}\beta\kappa)\,A_1 + A_3 &= 0\\ (M_0 + \tfrac{1}{8}\beta\kappa)\,A_1 + L_3 A_3 + A_5 &= 0\end{aligned}\right\}. \tag{44}$$

These equations can be solved in precisely the same way as was done in the case of the fortnightly tide. The following are the results :—

Luni-solar diurnal tide. Boundary $\nu_1 = \frac{1}{4}$, *Lat.* 75° 30′.

Depth 58,080 *feet* ; $\beta = 5$.

$$\zeta/H_1 = \sqrt{(1-\nu^2)}\,\{{\cdot}895\nu - {\cdot}518\nu^3 - {\cdot}132\nu^5 - {\cdot}122\nu^7 \ldots\}\cos(\sigma t + \phi + \epsilon).$$

Depth 29,040 *feet* ; $\beta = 10$.

$$\zeta/H_1 = \sqrt{(1-\nu^2)}\,\{{\cdot}805\nu - 1{\cdot}12\nu^3 - 2{\cdot}76\nu^5 - 2{\cdot}725\nu^7 - 2{\cdot}348\nu^9 \ldots\}$$
$$\times\cos(\sigma t + \phi + \epsilon).$$

Depth 14,520 *feet* ; $\beta = 20$.

$$\zeta/H_1 = \sqrt{(1-\nu^2)}\,\{{\cdot}648\nu - 2{\cdot}53\nu^3 - 6{\cdot}34\nu^5 - 6{\cdot}43\nu^7 - 5{\cdot}05\nu^9 \ldots\}\cos(\sigma t + \phi + \epsilon).$$

Depth 7,260 *feet* ; $\beta = 40$.

$$\zeta/H_1 = \sqrt{(1-\nu^2)}\,\{{\cdot}703\nu - 4{\cdot}87\nu^3 - 15{\cdot}7\nu^5 - 18{\cdot}3\nu^7 - 13{\cdot}2\nu^9 \ldots\}\cos(\sigma t + \phi + \epsilon).$$

Luni-solar diurnal tide. Boundary $\nu_1 = \frac{1}{2}$, *Lat.* 60°.

Depth 58,080 *feet* ; $\beta = 5$.

$$\zeta/H_1 = \sqrt{(1-\nu^2)}\,\{{\cdot}634\nu - {\cdot}640\nu^3 - 1{\cdot}30\nu^5 - 1{\cdot}25\nu^7 - {\cdot}928\nu^9 - {\cdot}837\nu^4 \ldots\}$$
$$\times\cos(\sigma t + \phi + \epsilon).$$

Depth 29,040 *feet* ; $\beta = 10$.

$$\zeta/H_1 = \sqrt{(1-\nu^2)}\,\{{\cdot}408\nu - 1{\cdot}49\nu^3 - 2{\cdot}84\nu^5 - 2{\cdot}69\nu^7 - 2{\cdot}30\nu^9 - 2{\cdot}022\nu^{11}$$
$$- 1{\cdot}813\nu^{13} \ldots\}\cos(\sigma t + \phi + \epsilon).$$

Depth 14,520 *feet* ; $\beta = 20$.

$$\zeta/H_1 = \sqrt{(1-\nu^2)}\,\{{\cdot}211\nu - 3{\cdot}36\nu^3 - 6{\cdot}82\nu^5 - 6{\cdot}43\nu^7 - 4{\cdot}90\nu^9 - 3{\cdot}86\nu^{11}$$
$$- 3{\cdot}28\nu^{13} \ldots\}\cos(\sigma t + \phi + \epsilon).$$

Depth 7,260 *feet* ; $\beta = 40$.

$$\zeta/H_1 = \sqrt{(1-\nu^2)}\,\{{\cdot}267\nu - 6{\cdot}59\nu^3 - 17{\cdot}6\nu^5 - 19{\cdot}1\nu^7 - 12{\cdot}9\nu^9 - 7{\cdot}18\nu^{11}$$
$$- 4{\cdot}32\nu^{13} \ldots\}\cos(\sigma t + \phi + \epsilon).$$

5. *Tides of Third Species.*

The results for the tides of the third species are derived from the formulæ (12), (15), (16), and (17) by putting $s = 2$. In the tide-producing potential the only term appearing is $B_0 = H_2$. The frequencies are determined from (21) after putting $\kappa = 0$, $s = 2$. As previously, the approximate values of the first set of roots are given by

$$L_0 = \frac{1}{\nu_1^2},$$

or

$$2\beta f^3 - \beta f^2 - f\left(\beta + 12 + \frac{12}{\nu_1^2}\right) + 2 = 0. \tag{45}$$

The roots of this equation exhibit the same characteristics as those of the equation in the second species. There are two large roots, one positive and one negative, and one small positive root. When the rotation disappears the small root becomes zero and the others tend to numerical equality. It is noticeable also that the large roots tend to numerical equality with increasing depth. These two results are consequent upon the reduction in value of β, which is $\frac{4\omega^2 a^2}{gh}$.

TABLE V.—*Lowest frequencies (approximate) of tides of third species. Boundary* $\nu_1 = \frac{1}{4}$, *Lat.* 75° 30′.

Depths.	Frequencies ($\sigma/2\omega$).
58,080 feet $\beta = 5$	4·82 −4·33 ·0096
29,040 feet $\beta = 10$	3·53 −3·03 ·0093
14,520 feet $\beta = 20$	2·64 −2·15 ·0089
7,260 feet $\beta = 40$	2·01 −1·52 ·0082

Table VI.—*Lowest frequencies (approximate) of tides of third species. Boundary* $\nu_1 = \frac{1}{2}$, *Lat.* 60°.

Depths.	Frequencies ($\sigma/2\omega$).
58,080 feet $\beta = 5$	2·80 $-$2·33 ·03
29,040 feet $\beta = 10$	2·12 $-$1·66 ·029
14,520 feet $\beta = 20$	1·69 $-$1·21 ·025
7,260 feet $\beta = 40$	1·39 $-$ ·91 ·020

It is interesting to examine determinant (21) with a view to discovering the possibility of synchronism between the free and forced oscillations of this species. All the semi-diurnal tides have values of f near to 1. For the luni-solar semi-diurnal tide (K_2) f is rigorously unity. If in (21) we put $f = 1$, it becomes an equation in β, the solution of which would give the possible depths at which synchronism could take place. As a first approximation, in (45) put $f = 1$. We have then $\beta =$ about 480 for $\nu_1 = \frac{1}{2}$, and 6490 for $\nu_1 = \frac{1}{4}$.

Using these approximate values we may, by the process of trial and error, find the exact solution of (21). We then arrive at $\beta = 552$ and 7760 respectively. These values correspond to depths 526 feet for $\nu_1 = \frac{1}{2}$, and 38 feet for $\nu_1 = \frac{1}{4}$. The smallness of these depths precludes the possibility of synchronism in oceans of depths comparable with the average depth of the ocean as it is on the earth.

Of course, the equation (21) is satisfied by other values of β. But these will be higher than those already calculated, and will lead to smaller depths still.

The semi-diurnal tides may be calculated from the analytical results previously given by using the proper value of f. We append the expansions for the tide K_2, worked out in the manner suggested.

Luni-solar semi-diurnal tide. Boundary $\nu_1 = \frac{1}{4}$, *Lat.* 75° 30′.

Depth 58,080 *feet* ; $\beta = 5$.

$$\zeta/H_2 = \nu^2 \{1 + \cdot 0522\nu^2 - \cdot 592\nu^4 - \cdot 421\nu^6 - \cdot 258\nu^8 \ldots\} \cos(\sigma t + 2\phi + \epsilon).$$

Depth 29,040 *feet* ; $\beta = 10$.

$$\zeta/H_2 = \nu^2 \{1 + \cdot 104\nu^2 - 1\cdot 06\nu^4 - \cdot 856\nu^6 - \cdot 477\nu^8 \ldots\} \cos(\sigma t + 2\phi + \epsilon).$$

Depth 14,520 *feet* ; $\beta = 20$.

$$\zeta/H_2 = \nu^2 \{1 + \cdot 209\nu^2 - 2\cdot 365\nu^4 - 1\cdot 829\nu^6 - \cdot 665\nu^8 \ldots\} \cos(\sigma t + 2\phi + \epsilon).$$

Depth 7,260 *feet* ; $\beta = 40$.

$$\zeta/H_2 = \nu^2 \{1 + \cdot 419\nu^2 - 4\cdot 75\nu^4 - 3\cdot 76\nu^6 - \cdot 172\nu^8 \ldots\} \cos(\sigma t + 2\phi + \epsilon).$$

Luni-solar semi-diurnal tide. Boundary $\nu_1 = \frac{1}{2}$, *Lat.* 60°.

Depth 58,080 *feet* ; $\beta = 5$.

$$\zeta/H_2 = \nu^2 \{1 + \cdot 213\nu^2 - \cdot 492\nu^4 - \cdot 371\nu^6 - \cdot 244\nu^8 - \cdot 175\nu^{10} - \cdot 137\nu^{12} - \cdot 111\nu^{14} \ldots\} \times \cos(\sigma t + 2\phi + \epsilon).$$

Depth 29,040 *feet* ; $\beta = 10$.

$$\zeta/H_2 = \nu^2 \{1 + \cdot 428\nu^2 - \cdot 982\nu^4 - \cdot 795\nu^6 - \cdot 460\nu^8 - \cdot 290\nu^{10} - \cdot 207\nu^{12} - \cdot 162\nu^{14} \ldots\} \times \cos(\sigma t + 2\phi + \epsilon).$$

Depth 14,520 *feet* ; $\beta = 20$.

$$\zeta/H_2 = \nu^2 \{1 + \cdot 869\nu^2 - 2\cdot 02\nu^4 - 1\cdot 80\nu^6 - \cdot 810\nu^8 - \cdot 314\nu^{10} - \cdot 154\nu^{12} - \cdot 100\nu^{14} \ldots\} \times \cos(\sigma t + 2\phi + \epsilon).$$

Depth 7,260 *feet* ; $\beta = 40$.

$$\zeta/H_2 = \nu^2 \{1 + 1\cdot 83\nu^2 - 3\cdot 86\nu^5 - 4\cdot 53\nu^6 - 1\cdot 26\nu^8 - 1\cdot 87\nu^{10} - 1\cdot 21\nu^{12} - \cdot 658\nu^{14} \ldots\} \times \cos(\sigma t + 2\phi + \epsilon).$$

6. *Conclusion.*

An examination of the expressions for the tide heights in a polar basin shows that they do not differ in form from those for an ocean wholly covering the globe; the differences of the tides are merely in magnitude. However, the following remarks seem worthy of attention :—

(i) As the table below shows, the fortnightly tide is very much reduced in magnitude. Indeed, it is minute when compared with the value given by the equilibrium theory. The table exhibits a comparison between the heights of the tides at the pole and at the boundary as given by this theory, as given by the theory for an ocean covering the globe completely, and as given by the equilibrium theory. The entries in columns three and six are taken, for depths giving $\beta = 10$ and $\beta = 40$, from Darwin's paper,[*] and for depth $\beta = 5$ from Lamb.[†] Unfortunately, Lamb uses $f = 0$, though this apparently makes only a small difference in the result. Hough [‡] has given the most complete expressions, but his results include the self-attraction of the water particles; and, as this has been omitted throughout the present paper, a comparison with them would be misleading.

Boundary $\nu_1 = \frac{1}{4}$, *Lat.* 75° 30′.

	Tide height at *Pole.*			Tide height at *Boundary.*		
	Present theory.	Theory of complete globe.	Equilibrium theory.	Present theory.	Theory of complete globe.	Equilibrium theory.
$\beta = 5$	·0306H_0	·434H_0	·6667H_0	−·0307H_0	−·401H_0	·6042H_0
$\beta = 10$	·0298H_0	·314H_0	·6667H_0	−·0305H_0	− 255H_0	·6042H_0
$\beta = 20$	·0284H_0	——	·6667H_0	−·0293H_0	——	·6042H_0
$\beta = 40$	·0260H_0	·104H_0	·6667H_0	−·0270H_0	−·1023H_0	·6042H_0

Boundary $\nu_1 = \frac{1}{2}$, *Lat.* 60°.

	Tide height at *Pole.*			Tide height at *Boundary.*		
	Present theory.	Theory of complete globe.	Equilibrium theory.	Present theory.	Theory of complete globe.	Equilibrium theory.
$\beta = 5$	·117H_0	·434H_0	·6667H_0	−·115H_0	−·302H_0	·4167H_0
$\beta = 10$	·105H_0	·314H_0	·6667H_0	−·112H_0	−·144H_0	·4167H_0
$\beta = 20$	·0887H_0	——	·6667H_0	−·103H_0	——	·4167H_0
$\beta = 40$	·0668H_0	·104H_0	·6667H_0	−·0899H_0	−·117H_0	·4167H_0

N.B.—The quantity H_0 used here $= -H'$ in Lamb, and $-E$ in Darwin.

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[*] *Proc. Roy. Soc.*, Vol. XLI (1886), p. 337; *Collected Works*, Vol. I, p. 370.

[†] *Hydrodynamics*, p. 321.

[‡] *Loc. cit.*, p. 240.

It is to be noticed that all three theories give the tide in the same phase at the pole, but at the boundary the two dynamical theories give the opposite phase from that of the equilibrium theory.

The tide heights required by the present theory are so small compared with the other two that it seems worth while to verify the work by reference to the problem of a plane circular rotating sheet of water, which should present results not much different from those given by the smaller of the two basins just discussed.

Using the notation of Lamb,* the equations for the case of symmetry about the axis $(s = 0)$ are

$$\frac{\partial^2\zeta'}{\partial r^2} + \frac{1}{r}\frac{\partial\zeta'}{\partial r} - \kappa^2\bar{\zeta} = 0, \qquad \kappa^2 = \frac{4\omega^2}{gh};$$

and the boundary condition

$$\frac{\partial\zeta'}{\partial r} = 0 \quad \text{at} \quad r = r_1.$$

In the first equation we have put

$$f^2 = \frac{\sigma^2}{4\omega^2} = 0$$

as a close approximation.

The tide-producing potential will be now

$$\bar{\zeta} = H_0\left(\tfrac{2}{3} - \nu_1^2\frac{r^2}{r_1^2}\right).$$

ν_1 being the boundary taken in the spherical problem.

The solution is

$$\frac{\zeta}{H_0} = \frac{4\nu_1^2}{\kappa^2 r_1^2} - \frac{2\nu_1^2}{\kappa r_1 I_1(\kappa r_1)} I_0(\kappa r).$$

So that the tide height at the pole is

$$\left\{\frac{4\nu_1}{\kappa^2 r_1^2} - \frac{2\nu_1^2}{\kappa r_1 I_1(\kappa r_1)}\right\} H_0;$$

at the boundary it is
$$\left\{\frac{4\nu_1^2}{\kappa^2 r_1^2} - \frac{2\nu_1^2}{\kappa r_1}\frac{I_0(\kappa r_1)}{I_1(\kappa r_1)}\right\} H_0.$$

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* P. 304.

Remembering that

$$\kappa^2 r_1^2 = \frac{4\omega^2 r_1^2}{gh} = \frac{4\omega^2 a^2}{gh}\,\frac{r_1^2}{a^2} = \beta\nu_1^2\,;$$

we can, with the aid of tables * of the functions $I_s(z)$, evaluate the above expressions. The results are, for $\nu_1 = \frac{1}{4}$,

	Pole.	*Boundary.*
$\beta = 5$,	$\cdot032H_0$	$-\cdot031H_0$,
$\beta = 20$,	$\cdot029H_0$	$-\cdot030H_0$.

These figures completely verify the results given in the previous tables and establish the extreme smallness of the fortnightly tide in such a basin as that considered. It is also worthy of notice how very far from the truth the equilibrium tide is.

(ii) The diurnal and semi-diurnal tides both vanish at the pole in agreement with the equilibrium theory. At the boundary each shows positive values if the basin is of small extent. But the diurnal tide, for larger basins, rapidly becomes largely negative. The values at the boundary are given in the following table :—

	Diurnal tides.		Semi-diurnal tides.	
	$\nu_1 = \frac{1}{4}$	$\nu_1 = \frac{1}{2}$	$\nu_1 = \frac{1}{4}$	$\nu_1 = \frac{1}{2}$
$\beta = 5$	$\cdot212H_1$	$\cdot167H_1$	$\cdot0625H_2$	$\cdot254H_2$
$\beta = 10$	$\cdot178H_1$	$-\ \cdot084H_1$	$\cdot0626H_2$	$\cdot257H_2$
$\beta = 20$	$\cdot114H_1$	$-\ \cdot502H_1$	$\cdot0627H_2$	$\cdot265H_2$
$\beta = 40$	$\cdot082H_1$	$-1\cdot24H_1$	$\cdot0629H_2$	$\cdot285H_2$

		$\nu_1 = \frac{1}{4}$	$\nu_1 = \frac{1}{2}$
Equilibrium tide	Diurnal	$\cdot242H_1$	$\cdot433H_1$
	Semi-diurnal ...	$\cdot0625H_2$	$\cdot25H_2$

It is noteworthy that the semi-diurnal tide has, in each of the cases worked out, a value very near that given by the equilibrium theory. In the case of the *plane* circular basin there is exact equality.*

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* *Brit. Assoc. Reports*, 1889.

When the ocean completely covers the globe the luni-solar diurnal tide vanishes entirely, and the other diurnal tides, except at an approach to synchronism with one of the free modes, are small. In the last table the numbers for the diurnal tides are large in most cases. I have therefore made a comparison of the diurnals and semi-diurnals at the various depths to ascertain their relative importance. From Darwin's tables † $H_1/H_2 = 4{\cdot}5$ about. It is at once seen that in all cases the diurnal tide is the greater, having from $1\frac{1}{2}$ to about 20 times the range of the semi-diurnal in the examples given. This remarkable effect of a boundary in increasing a tide from zero to such importance can readily be verified by reference again to the plane circular rotating sea.

Near the pole the tide producing potential may be written $H_1\nu_1 . r/r_1$ for the diurnal tides, and $H_2\nu_1^2(r/r_1)^2$ for the semi-diurnals; where r_1 is the radius of the basin.

Using Lamb's results,‡ the tide elevation at the edge ($r = r_1$) will be, for the diurnal tide,

$$\frac{3I_1(z)}{I_1(z)+zI_0(z)} H_1\nu_1,$$

where

$$z = \frac{\nu_1}{2}\sqrt{3\beta}.$$

The semi-diurnal dynamical height is the same as the equilibrium height, viz., $H_2\nu_1^2$.

On evaluating these quantities for the usual values of β and $\nu_1 = \frac{1}{4}$, we have

$$\frac{\text{Diurnal height}}{\text{Semi-diurnal height}} = 17{\cdot}7 \text{ for } \beta = 5,$$

$$= 17{\cdot}3 \text{ for } \beta = 10,$$

$$= 16{\cdot}7 \text{ for } \beta = 20.$$

$$= 15{\cdot}6 \text{ for } \beta = 40.$$

These results verify completely those given by the general theory.

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* Lamb, p. 309 (27), and p. 311 (12).

† Darwin, *Collected Scientific Papers*, Vol. I, pp. 20, 21.

‡ *Hydrodynamics*, p. 309.

In his discussions of the tidal observations [*] of the "Discovery" and "Nimrod" Antarctic Expeditions, Sir G. Darwin has drawn attention to the fact that the diurnal tides are relatively of much more importance than the semi-diurnals in high latitudes. And while it is not suggested that there is any resemblance between the Antarctic Ocean and the basins considered in this paper, yet it is clear that the exaggeration of the diurnal tides can be accounted for in a dynamical theory. The Arctic Ocean would be nearer in form to the oceans we have been considering, but the observations of the tides seem less complete. Harris gives two sets of reductions,[†] but in them the semi-diurnal greatly preponderates. He gives reasons[‡] for thinking, however, that the Arctic tides are derived from the Atlantic Ocean, and not generated in the limits of the Arctic Ocean itself.

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[*] *Collected Scientific Papers*, Vol. I, p. 372; Vol. IV, p. 473.

[†] *Manual of Tides*, Vol. IV*a*, p. 664.

[‡] *Ibid.*, Vol. IV*b*, p. 386.

ON A MODIFIED FORM OF PURE RECIPROCANTS POSSESSING THE PROPERTY THAT THE ALGEBRAICAL SUM OF THE COEFFICIENTS IS ZERO

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Prof. Sylvester at the close of his Sixteenth Lecture on the "Theory of Reciprocants," remarks :*—"It will be already seen from an inspection of the fundamental forms that there is no law for the coefficients of reciprocants akin to that of their algebraical sum being zero in invariants."

I wish to shew in this short communication that the forms may be so modified that the vanishing of the algebraical sum of the coefficients is in evidence.

Denoting with Sylvester dy/dx by t and dx/dy by τ, also

$$\frac{1}{(s+2)!}\,\frac{d^{s+2}y}{dx^{s+2}} \text{ by } a_s, \qquad \frac{1}{(s+2)!}\,\frac{d^{s+2}x}{dy^{s+2}} \text{ by } \alpha_s,$$

we have the well known formulæ

$$\alpha_0 = -a_0 \div t^3,$$

$$\alpha_1 = (-a_1 t + 2a_0^2) \div t^5,$$

$$\alpha_2 = (-a_2 t^2 + 5a_0 a_1 t - 5a_0^3) \div t^7,$$

$$\alpha_3 = \{-a_3 t^3 + (6a_0 a_2 + 3a_1^2)\, t^2 - 28a_0^2 a_1 t + 14a_0^4\} \div t^9,$$

$$\alpha_4 = \{-a_4 t^4 + (7a_0 a_3 + 7a_1 a_2)\, t^3 - (28a_0^2 a_2 + 28a_0 a_1^2)\, t^2 + 84a_0^3 a_1 t - 42a_0^5\} \div t^{11},\dagger \quad \ldots .$$

In these formulæ we may interchange the sets of symbols

$$t_1 a_0,\ a_1,\ \ldots, \qquad \tau_1 \alpha_0,\ \alpha_1,\ \ldots .$$

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[*] *Mathematical Papers*, Vol. IV, p. 398.

[†] Observe that in the *Mathematical Papers*, Vol. IV, p. 311, the term $28a_0 a_1^2$ is erroneously printed $28a_1^2$.

A pure reciprocant is a homogeneous and isobaric function of

$$a_0,\ a_1,\ a_2,\ \dots,$$

which is equal, the sign being disregarded, to the same function of

$$\alpha_0,\ \alpha_1,\ \alpha_2,\ \dots,$$

multiplied by some power of t.

If we form from the above relations any such function of $\alpha_0, \alpha_1, \alpha_2, \dots,$ each term of the function, when expressed in terms of $t, a_0, a_1, a_2, \dots,$ gives rise to a term which is simply a power of a_0 multiplied by a power of t, and in the combination of terms which stands for a pure reciprocant these powers of a_0 must in combination vanish.

Ex. gr.,

$$\alpha_0^2\alpha_3 - 3\alpha_0\alpha_1\alpha_2 + 2\alpha_1^3 = \dots + \left(-\frac{a_0}{t^3}\right)^2\left(14\,\frac{a_0^4}{t^9}\right)$$

$$-3\left(-\frac{a_0}{t^3}\right)\left(2\,\frac{a_0^2}{t^5}\right)\left(-5\,\frac{a_0^3}{t^7}\right) + 2\left(2\,\frac{a_0^2}{t^5}\right)^3,$$

and since the right-hand side is equal to $a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$ with a negative sign and a power of t, it is clear that the terms written on the right-hand side must vanish identically.

Hence, if $f(a_0, a_1, a_2, a_3, \dots)$ be a pure reciprocant,

$$f(-a_0,\ 2a_0^2,\ -5a_0^3,\ 14a_0^4,\ 42a_0^5,\ \dots)$$

must vanish identically.

The general a_0 term here is

$$(-)^s\,\frac{1}{s}\binom{2s}{s-1}\,a_0^s.$$

Hence, if we write $\quad a_s = (-)^{s+1}\,\dfrac{1}{s+1}\dbinom{2s+2}{s}\,b_s,$

that is to say $\quad b_s = (-)^{s+1}\,\dfrac{(s+1)!}{(2s+2)!}\,\dfrac{d^{s+2}y}{dx^{s+2}},$

and denote the pure reciprocant by

$$\phi(b_0,\ b_1,\ b_2,\ \dots),$$

the form is such that the algebraical sum of the coefficients vanishes.

If a term of the reciprocant be as regards the literal portion

$$a_0^{\lambda_0}a_1^{\lambda_1}a_2^{\lambda_2}\dots,$$

this, by the substitution, acquires the sign

$$(-)^{\lambda_0+\lambda_2+\lambda_4+\dots},$$

which is the same for *every* term because

$$(-)^{\lambda_0+\lambda_2+\lambda_4+\dots} = (-)^{(\lambda_0+\lambda_1+\lambda_2+\dots)-(\lambda_1+2\lambda_2+3\lambda_3+4\lambda_4+\dots)},$$

and the right-hand side is constant for every term of the reciprocant. Hence we may take the substitution to be

$$a_s = \frac{1}{s+1}\binom{2s+2}{s} b_s,$$

and then the algebraical sum of the coefficients vanishes. Otherwise we may assert that a pure reciprocant vanishes on putting

$$a_s = \frac{1}{s+1}\binom{2s+2}{s}.$$

The same result is deducible from the writer's theorem given in the *Transactions of the Cambridge Philosophical Society*, Vol. XXI, No. 6, pp. 143–170. The paper is entitled "The Operator Reciprocants of Sylvester's Theory of Reciprocants," and the theorem will be found on p. 166.

It states that the transformation

$$a_0 = \frac{1}{c_0},$$

$$a_1 = -\frac{2c_1}{c_0^3},$$

$$a_2 = -\frac{3c_2}{c_0^4} + \frac{8c_1^2}{c_0^5},$$

$$a_3 = -\frac{4c_3}{c_0^5} + \frac{30c_1c_2}{c_0^6} - \frac{40c_1^3}{c_0^7},$$

&c.,

converts all pure reciprocants in the elements

$$a_0,\ a_1,\ a_2,\ a_3,\ \dots,$$

into seminvariants in the elements

$$c_0,\ c_1,\ c_2,\ c_3,\ \dots.$$

The reader will observe that on the dexter of the above relations the sums of the coefficients are respectively

$$1,\ -2,\ +5,\ -14,\ +42,\ \ldots,\ (-)^s \frac{1}{s+1}\binom{2s+2}{s},$$

and that knowing that in seminvariants the sum of the coefficients vanishes, we at once proceed to the theorem of this paper.

The substitution of

$$\frac{1}{s+1}\binom{2s+2}{s} b_s \text{ for } a_s$$

in pure reciprocants leads to a remarkable simplification in the numerical values of the coefficients which must be of importance to any investigator who desires to study the mutual relations of the forms.

Below are given (1) Sylvester's forms, (2) the forms which arise by the modification which has been explained.

$$\begin{cases} a_0 \\ b_0 \end{cases}$$

$$\begin{cases} 4a_0a_2-5a_1^2 \\ b_0b_2-b_1^2 \end{cases}$$

$$\begin{cases} a_0^2a_3-3a_0a_1a_2+2a_1^3 \\ 7b_0^2b_3-15b_0b_1b_2+8b_1^3 \end{cases}$$

$$\begin{cases} 50a_0^2a_4-175a_0a_1a_3+28a_0a_2^2+105a_1^2a_2 \\ 3b_0^2b_4-7b_0b_1b_2+b_0b_2^2+3b_1^2b_2 \end{cases}$$

$$\begin{cases} 800a_0^2a_2a_4-875a_0^2a_3^2-1000a_0a_1^2a_4+2450a_0a_1a_2a_3-1344a_0a_2^3-35a_1^2a_2^2 \\ 48b_0^2b_2b_4-49b_0^2b_3^2-48b_0b_1^2b_4+98b_0b_1b_2b_3-48b_0b_2^3-b_1^2b_2^2 \end{cases}$$

$$\begin{cases} 125a_0^3a_3^2-750a_0^2a_1a_2a_3+256a_0^2a_2^3+165a_0a_1^2a_2^2+500a_1^3a_3-300a_1^4a_2 \\ 49b_0^3b_0^2-210b_0^2b_1b_2b_3+64b_0^2b_2^3+33b_0b_1^2b_2^2+112b_1^3b_3-48b_1^4b_2 \end{cases}$$

$$\left\{\begin{array}{l} \quad 625a_0^3a_4^2-4375a_0^2a_1a_3a_4-49700a_0^2a_2^2a_4+55125a_0^2a_2a_3^2+128625a_0a_1^2a_2a_4 \\ -61250a_0a_1^2a_3^2-156800a_0a_1a_2^2a_3+84868a_0a_2^4 \\ -78750a_1^4a_4 \quad +183750a_1^3a_2a_3 \quad -102165a_1^2a_2^3 \\ \quad 9b_0^3b_4^2-42b_0^2b_1b_3b_4-426b_0^2b_2^2b_4+441b_0^2b_2b_3^2+882b_0b_1^2b_2b_4 \\ -392b_0b_1^2b_3^2-896b_0b_1b_2^2b_3+433b_0b_2^4 \\ -432b_1^4b_4 \quad +840b_1^3b_2b_3 \quad -417b_1^2b_2^3 \end{array}\right.$$

ON DARBOUX'S METHOD OF SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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1. The following paper arose out of an attempt to answer a question proposed by Prof. Forsyth in the lecture on "The Present State of Partial Differential Equations of the Second Order, as regards Formal Integration," delivered before the Fourth International Congress of Mathematicians,* at Rome, in April 1908. The quotations given below will indicate the scope of the inquiry.

(*a*) "An intermediate integral may be 'general,' as involving an arbitrary function, after the idea of Monge; it may be 'complete,' as involving two arbitrary constants; it may be 'particular,' or 'singular,' or 'special.' Whether general or complete, M. Darboux's method suffices for its construction, and the result easily allows its character to be discerned. When it is singular it can be constructed by direct operations on the given equation itself. *When it is particular or special we have at present to depend almost upon fortuitous chance for its discovery.*"

(*b*) "There seems very great importance in *the discrimination of those classes of equations which are amenable to M. Darboux's method in a finite number of operations.*"

(*c*) "Equations might be imagined as possessing unquadratured primitives expressible only in non-finite terms. . . . If M. Darboux's method has a limiting form when the number of operations which it requires in practical application is imagined as being capable of unlimited increase, the equations amenable to the limiting form of the method would be of the kind indicated."

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* The same suggestions occur in Prof. Forsyth's Presidential Address to the London Mathematical Society, November 1906.

As regards (*c*), the only reason for its introduction here is to point out the necessity of very careful handling in following out the idea suggested. A very simple example will suffice to show this.

The equation $$x^{4n/(2n-1)}r+2x^{(2n+1)/(2n-1)}p=a^2t, \tag{1}$$

is soluble by Darboux's method (most easily, however, by Laplace's method), there being two integrable combinations of each set of characteristics, one in each involving x and y only, the other involving the n-th order characteristics. In fact, if

$$\xi=(2n-1)\,ax^{-1/(2n-1)}+y, \qquad \eta=(2n-1)\,ax^{-1/(2n-1)}-y,$$

it is readily seen that the general solution may be expressed in the form

$$z=\phi+\psi-\frac{n-1}{2n-2}(\xi+\eta)(\phi'+\psi')+\frac{(n-1)(n-2)}{(2n-2)(2n-3)}\,\frac{(\xi+\eta)^2}{2!}(\phi''+\psi'')-\dots$$
$$\dots+(-)^{n-1}\frac{(n-1)!}{(2n-2)!}(\xi+\eta)^{n-1}\left\{\phi^{(n-1)}+\psi^{(n-1)}\right\}, \tag{2}$$

where ϕ and ψ are arbitrary functions of ξ and η respectively.

It is evident that both the equation (1) and the solution (2) tend to a definite limit as n tends to infinity, but the limit of (2) is plainly

$$z=\phi\left(\xi-\frac{\xi+\eta}{2}\right)+\psi\left(\eta-\frac{\xi+\eta}{2}\right)=\phi(y)+\psi(-y),$$

which does not satisfy nor in any way suggest the solution of the limit of (1), namely,

$$x^2r+2xp=a^2t.$$

2. The attempt to reply to the quotation (*b*) has, as was perhaps to be expected, met with but slight success, but the nature of the few results obtained leads one to hope that the complete answer may be given in the not very remote future. We are, incidentally, enabled to reply to the italicised portion of (*a*). It should be noticed that, in the remark last referred to, Prof. Forsyth was speaking of intermediate integrals of the second and higher orders, for Ampère's method * suffices to discover all intermediate integrals, of whatever kind, of the first order.

The method here followed is nothing more than a slight modification of the Monge-Ampère method of obtaining first order integrals, or of Darboux's method of obtaining intermediate integrals of higher order. It is perfectly general except in its application to the determination of first order integrals, in which case it applies only to equations of the form

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* Goursat (*loc. cit.*, p. 77 *infra*), t. 1, § 100.

$$f(x, y, z, p, q, r+ms, s+mt) = 0, \tag{3}$$

where m is a function of x, y, z, p, q; this includes the "linear" equation

$$Rr+Ss+Tt = V, \tag{4}$$

for, if m is a root of the characteristic equation

$$Rm^2-Sm+T = 0,$$

equation (4) may be re-written as

$$mR(r+ms)+T(s+mt) = mV,$$

which is of the form (3).

To obtain those first order integrals of (3) which belong to the system of characteristics for which the root of the critical equation is m, let $u =$ constant be an integral of

$$dp+m\,dq = 0,$$

in which x, y, z are treated as constants. Assume as a first order integral of (3) the equation

$$u+v = 0, \tag{5}$$

where v is a function of x, y, and z only.

If μ is an integrating factor of $dp+m\,dq = 0$ we may take

$$\frac{\partial u}{\partial p} = \mu, \quad \frac{\partial u}{\partial q} = m\mu,$$

and differentiating (5) with respect to x, and with respect to y, we shall have

$$\mu(r+ms)+\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z} = 0,$$

$$\mu(s+mt)+\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z} = 0.$$

Substituting these values of $r+ms$, $s+mt$ in (3), we shall have a first order equation to determine v, which, since v does not contain p or q, will, in general, be equivalent to several simultaneous equations for v. If these equations have any common solution it will give, on substitution in (5), a first order integral of (3).

As simple examples we may take the following :—

(i) $$r-a^2t = a^2[z+x(p-aq)].$$

Here we take $p \pm aq = v$, whence

$$\frac{\partial v}{\partial x} \mp a\frac{\partial v}{\partial y} + (p \mp aq)\frac{\partial v}{\partial z} = a^2[z+x(p-aq)],$$

i.e., $$\frac{\partial v}{\partial x} \mp a\frac{\partial v}{\partial y} + (v \mp 2aq)\frac{\partial v}{\partial z} = a^2[z+x(v-2aq)],$$

$$\text{or } a^2(z+vx).$$

The equation cannot be satisfied if we take the lower sign, but with the upper sign it leads to

$$\frac{\partial v}{\partial x} - a\frac{\partial v}{\partial y} = a^2z, \qquad \frac{\partial v}{\partial z} = a^2x,$$

whence the *general integral*

$$p+aq = v = a^2zx + F(y+ax).$$

(ii) $$(s-at)(p-aq) = x(r-as)(s-at) + y(s-at)^2 + b^2.$$

Here $$p-aq = v,$$

where v may, in this particular case, be regarded as a function of x and y only.

We have $$s-at = \frac{\partial v}{\partial y}, \qquad r-as = \frac{\partial v}{\partial x},$$

whence $$v\frac{\partial v}{\partial y} = x\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + y\left(\frac{\partial v}{\partial y}\right)^2 + b^2,$$

i.e., $$v - x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y} = b^2 \Big/ \frac{\partial v}{\partial y},$$

leading to the *complete integral*

$$p-aq = v = hx+ky+b^2/k,$$

where h and k are arbitrary constants.

(iii) $$p(2qr+ps)(2qs+pt) + x(2qr+ps) + y(2qs+pt) = pq,$$

here $$p^2q = v,$$

whence $$\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=v,$$

leading to the *complete integral*

$$p^2q=v=ax+by+ab,$$

and the *singular integral* $$p^2q+xy=0.$$

(iv) $$(rq+sp)(sq+tp)=z.$$

Here we put $$pq=v,$$

whence $$\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\left(p\frac{\partial v}{\partial x}+q\frac{\partial v}{\partial y}\right)+v\left(\frac{\partial v}{\partial z}\right)^2=z,$$

necessitating $$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0,\qquad v\left(\frac{\partial v}{\partial z}\right)^2=z,$$

i.e., $$p^{\frac{3}{2}}q^{\frac{3}{2}}=v^{\frac{3}{2}}=a^{\frac{3}{2}}+z^{\frac{3}{2}},$$

a *particular integral*, involving one arbitrary constant.

The equation $$(rq+sp)(sq+tp)=f(z)$$

may evidently be treated in the same way.

(v) $$r+zt=p^2/z.$$

Here $$m=\pm\,\iota z^{\frac{1}{2}},$$

and we take $$p\pm\iota z^{\frac{1}{2}}q+v=0, \tag{6}$$

whence we find

$$\frac{p^2}{z}+\tfrac{1}{2}q^2\pm\iota z^{-\frac{1}{2}}pq\mp\iota z^{\frac{1}{2}}\left(\frac{\partial v}{\partial x}+q\frac{\partial v}{\partial z}\right)+\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}=0.$$

Substituting from (6) for q, and simplifying, we find

$$p\left(2\frac{\partial v}{\partial z}-\frac{3}{2}\,\frac{v}{z}\right)+\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial z}\mp\iota z^{\frac{1}{2}}\frac{\partial v}{\partial y}-\frac{v^2}{2z}=0,$$

in which the coefficient of p and the term independent of p must be separately zero. From the first of the two equations so obtained we have

$$v=z^{\frac{3}{4}}f(x,\,y),$$

and therefore, from the second, we see that the only possible solution is $v = 0$. In this case, therefore, $p \pm \iota z^{\frac{1}{2}} q = 0$, or, in one equation,

$$p^2 + zq^2 = 0$$

is a *special integral* of the given equation.

It is, of course, to be understood that all the results above given might have been obtained by the more general method of Ampère, and that they are to be taken merely as illustrative of the method to be applied in the subsequent portions of this paper. It will, however, be found that, in any case to which it applies, the method is considerably shorter than that of Ampère.

3. In extending the above method so as to obtain intermediate integrals of the second and higher orders it becomes general; that is to say, it applies to the equation

$$r + f(x, y, z, p, q, s, t) = 0, \tag{7}$$

and to both systems of characteristics.

Let m_1 and m_2 be the roots of the characteristic equation, so that

$$\left. \begin{aligned} m_1 + m_2 &= \frac{\partial f}{\partial s} \\ m_1 m_2 &= \frac{\partial f}{\partial t} \end{aligned} \right\}. \tag{8}$$

If p_{lm} denote $\dfrac{\partial^{l+m} z}{\partial x^l \partial y^m}$, the characteristics of the n-th order are

$$dp_{n-h,\,h} = (p_{n-h+1,\,h} + m_1 p_{n-h,\,h+1})\,dx \quad (h = 0, 1, 2, \ldots, n),$$

where, for the second system, we must replace m_1 by m_2. These may be combined to form the n equations,

$$dp_{n-h,\,h} + m_2\, dp_{n-h-1,\,h+1} = \left(p_{n-h+1,\,h} + \frac{\partial f}{\partial s} p_{n-h,\,h+1} + \frac{\partial f}{\partial t} p_{n-h-1,\,h+2}\right) dx$$
$$(h = 0, 1, 2, \ldots, n-1), \tag{9}$$

in which the left-hand side may, on account of (7), be freed from derivatives of order higher than n.

Let $F(x, y, z, p, q, r, s, t, \ldots, p_{n0}, \ldots, p_{0n}) = 0$ be any equation resulting from an integrable combination of the system (9) and all the preceding characteristics. Then, by means of (7) and its derivatives, r and all the derivatives p_{hk}, where h is greater than unity, may be eliminated from F. We may therefore assume that F involves no derivative of this form, so that the only derivatives of the n-th order contained in F will be $p_{1,\,n-1}$ and p_{0n}.

These derivatives can only arise from the characteristic equation

$$dp_{1,\,n-1}+m_2\,dp_{0n}=\left(p_{2,\,n-1}+\frac{\partial f}{\partial s}\,p_{1n}+\frac{\partial f}{\partial t}\,p_{0,\,n+1}\right)dx,$$

from which the terms of order $n+1$ disappear on account of (7). We therefore, as in the simpler case of the search for first order integrals, integrate

$$dp_{1,\,n-1}+m_2\,dp_{0n}=0, \tag{10}$$

as if all derivatives of order lower than n were constants. If $n=2$, equation (10), becomes

$$ds+m_2\,dt=0,$$

of which the integral is

$$\phi\equiv u+v=0, \tag{11}$$

where

$$\frac{\partial u}{\partial s}=\mu,\qquad \frac{\partial u}{\partial t}=m_2\mu,$$

and v is a function of x, y, z, p and q.

If, however, $n>2$ the integral of (10) is

$$\phi\equiv p_{1,\,n-1}+m_1p_{0n}+v=0, \tag{12}$$

where, for convenience, we have written m_1 instead of m_2, and where v is a function of $x, y, z, p, q, s, t, p_{12}, p_{03}, \cdots, p_{1,\,n-2}, p_{0,\,n-1}$.

To determine the form of the function v, if such exists, we use the conditions that (11) or (12) should be in involution with (7). With the notation of Goursat's *Leçons sur l'Intégration des Equations aux dérivées partielles du Second Ordre*, Tome II, p. 83, these conditions are

$$\frac{\partial\phi}{\partial p_{0n}}-m_1\frac{\partial\phi}{\partial p_{1,\,n-1}}=0,$$

$$\left(\frac{d\phi}{dx}\right)+m_2\left(\frac{d\phi}{dy}\right)-\frac{\partial\phi}{\partial p_{1,\,n-1}}\left(\frac{d^{n-1}f}{dy^{n-1}}\right)=0, \tag{13}$$

of which the first is an identity and the second leads to a first order equation for v, which, since v does not contain $p_{1,n-1}$ or p_{0n}, must be equivalent to two or more equations which must simultaneously be satisfied. If these equations have any common solution, the result, when substituted in (12) or (11), will give an intermediate integral of order n.

It will usually be found that this modification of Darboux's method will lead to intermediate integrals of a given order rather more readily than the method as usually given; for example, in Forsyth's *Theory of Differential Equations*, Vol. vi. And, further, the modification will lead to particular or special intermediate integrals of any order when such exist. As an example of the former statement we may refer to the equation solved in the next section; as an example of the latter consider the equation

$$rt-s^2+a^2(1+p^2+q^2)^2 = 0. \tag{14}$$

Here
$$m_1 = [-s+a(1+p^2+q^2)]/t,$$
$$m_2 = [-s-a(1+p^2+q^2)]/t.$$

According to rule we integrate $ds+m_2\,dt = 0$, obtaining

$$[s+a(1+p^2+q^2)]/t = v = r/[s-a(1+p^2+q^2)],$$

where v is a function of x, y, z, p, q.

The second condition of involution leads, after reduction, to

$$2a^2v(1+p^2+q^2)(q+pv)+[s+a(1+p^2+q^2)]\left[\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}+a(1+p^2+q^2)\frac{\partial v}{\partial q}\right]$$
$$+v[s-a(1+p^2+q^2)]\left[a(1+p^2+q^2)\frac{\partial v}{\partial p}-\frac{\partial v}{\partial y}-q\frac{\partial v}{\partial z}\right] = 0,$$

whence the two equations

$$\frac{1}{a}\frac{\partial v}{\partial y}+\frac{q}{a}\frac{\partial v}{\partial z}+q+pv = (1+p^2+q^2)\frac{\partial v}{\partial p},$$
$$\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial y}+(p-qv)\frac{\partial v}{\partial z}+a(1+p^2+q^2)\left(\frac{\partial v}{\partial q}+v\frac{\partial v}{\partial p}\right) = 0.$$

From the first we find

$$v = \frac{pq}{1+q^2}+(1+p^2+q^2)^{\frac{1}{2}}f(x, u, w, q),$$

where $u = z-qy$ and $w = ay\sqrt{(1+q^2)}-\tan^{-1}[p/\sqrt{(1+q^2)}]$;

whence, substituting in the second, we find

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial w} = 0,$$
$$2qf+(1+q^2)\frac{\partial f}{\partial q} = 0,$$
$$f^2+1/(1+q^2)^2 = 0,$$

which are satisfied by $f = \pm\iota/(1+q^2)$, leading to Serret's well known *special integral*

$$1+v^2+(p-qv)^2 = 0.$$

It will also be found, in the course of the work, that the equations for v are satisfied if $1+p^2+q^2 = 0$, an obvious special integral of the first order. In general all integrable combinations (other than special integrals) of characteristics of lower orders will appear in the search for an integral of order n. A simple example is given in the next section.

When $n = 2$ the number of independent equations which have to be satisfied by v is indefinite, depending on the nature of the given equation; when $n = 3$ the number is easily seen to be three, but, as will be shown immediately, one of these is an identity; and when n is greater than 3 the number of independent conditions is two.

In (13), if $n > 2$, $\partial\phi/\partial p_{1,n-1} = 1$, $(d\phi/dx)$ and $(d\phi/dy)$ contain $p_{1,n-1}$ and p_{0n} linearly only, while the terms of highest order in $(d^{n-1}f/dy^{n-1})$ are linear in $p_{1,n-1}$ and p_{0n} if $n > 3$, but quadratic in p_{12} and p_{03} if $n = 3$.

The derivative $p_{1,n-1}$ may be eliminated from (13) by means of (12); hence, if $n = 3$, equation (13) contains p_{03}^2, p_{03}, and a term independent of p_{03}, leading to *three* simultaneous equations for v, while, if $n > 3$, equation (13) contains p_{0n} and a term independent of p_{0n}, leading to *two* equations for v.

4. *The Equations which must be Satisfied by v when n = 3.*

We shall, at least for the present, omit the consideration of first and second order integrals, and shall pass on to the determination of the equations which must be satisfied by an integral of the third or higher order, on the one hand because of the difficulty of arriving at definite conclusions, and on the other hand because first and second order integrals have already been dealt with in many special cases by several investigators,[*] to the almost entire neglect, so far as I am aware, of integrals of higher order, where one must, of course, make an exception in favour of Laplace's linear equation and the equations treated by Lie, namely, $s = f(z)$ and equation (14).

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[*] *E.g.*, F. De Boer has considered the equation $s = f(r, t)$, *Archives Néerlandaises*, t. XXVII, and Goursat has considered those cases (he has, however, somehow overlooked one such case) in which $s = f(x, y, z, p, q)$ is integrable at the second order, *Annales de la Faculté des Sciences de l'Université de Toulouse*, 2e Série, t. I (1899). In this paper Goursat has briefly indicated a method of attacking the problem of integrals of order higher than the second for this particular equation. For some more general results see Goursat, "Sur une classe d'équations aux dérivées partielles du second ordre, et sur la théorie des intégrales intermédiaires," *Acta Mathematica*, t. XIX (1895).

The equations which must be satisfied by v in the case of a third order integral will differ from those which must be satisfied in the case of an integral of higher order. We therefore take third order integrals first.

Let $\dfrac{d}{dy} \equiv \dfrac{\partial}{\partial y} + q\dfrac{\partial}{\partial z} + s\dfrac{\partial}{\partial p} + t\dfrac{\partial}{\partial q} + \ldots + p_{1n}\dfrac{\partial}{\partial p_{1,\,n-1}} + p_{0,\,n+1}\dfrac{\partial}{\partial p_{0n}}$,

so that if f is the function of equation (7),

$$\frac{df}{dy} = \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + s\frac{\partial f}{\partial p} + t\frac{\partial f}{\partial q} + p_{12}\frac{\partial f}{\partial s} + p_{03}\frac{\partial f}{\partial t}.$$

Then we shall have $\qquad p_{2n} = -\dfrac{d^n f}{dy^n}$,

$$\left(\frac{d^n f}{dy^n}\right) = \frac{d^n f}{dy^n} - \frac{\partial f}{\partial s}p_{1,\,n+1} - \frac{\partial f}{\partial t}p_{0,\,n+2}.$$

Further, let

$$\frac{d}{dx} \equiv \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - f\frac{\partial}{\partial p} + s\frac{\partial}{\partial q} - \ldots - \frac{d^{n-1}f}{dy^{n-1}}\,\frac{\partial}{\partial p_{1,\,n-1}} + p_{1n}\frac{\partial}{\partial p_{0n}}.$$

In the search for intermediate integrals of the third order, equations (12) and (13) become

$$\phi \equiv p_{12} + m_1 p_{03} + v = 0, \tag{15}$$

$$p_{03}\left(\frac{dm_1}{dx} + m_2\frac{dm_1}{dy}\right) + \frac{dv}{dx} + m_2\frac{dv}{dy} = \left(\frac{d^2 f}{dy^2}\right). \tag{16}$$

On expanding this equation, and substituting for p_{12} from (15), it will be found that the coefficient of p_{03}^2 is (remembering that v is a function of x, y, z, p, q, s, t only)

$$m_1^2\frac{\partial^2 f}{\partial s^2} - 2m_1\frac{\partial^2 f}{\partial s\,\partial t} + \frac{\partial^2 f}{\partial t^2} + (m_1 - m_2)\left(\frac{\partial m_1}{\partial t} - m_1\frac{\partial m_1}{\partial s}\right).$$

But this, from equations (8), is readily seen to vanish, whatever the function f. Hence the equations for v reduce to two, namely, the coefficient of p_{03} and the term independent of p_{03} separately equated to zero. The explicit forms of these equations are

$$(m_1-m_2)\left(m_1\frac{\partial v}{\partial s}-\frac{\partial v}{\partial t}\right)+\left(\frac{dm_1}{dx}\right)+m_2\left(\frac{dm_1}{dy}\right)-\frac{\partial m_1}{\partial s}\left(\frac{df}{dy}\right)+v\left(m_1\frac{\partial m_1}{\partial s}-\frac{\partial m_1}{\partial t}\right)$$
$$-2v\left(m_1\frac{\partial^2 f}{\partial s^2}-\frac{\partial^2 f}{\partial s\,\partial t}\right)+2m_1\left(\frac{d}{dy}\right)\frac{\partial f}{\partial s}-2\left(\frac{d}{dy}\right)\frac{\partial f}{\partial t}+m_1\frac{\partial f}{\partial p}-\frac{\partial f}{\partial q}=0, \tag{17}$$

and
$$v\left(m_1\frac{\partial v}{\partial s}-\frac{\partial v}{\partial t}\right)-\frac{\partial v}{\partial s}\left(\frac{df}{dy}\right)+\left(\frac{dv}{dx}\right)+m_2\left(\frac{dv}{dy}\right)-\left[\frac{d^2 f}{dy^2}\right]$$
$$+2v\left(\frac{d}{dy}\right)\frac{\partial f}{\partial s}+v\frac{\partial f}{\partial p}-v^2\frac{\partial^2 f}{\partial s^2}=0, \tag{18}$$

where in equation (18) we have introduced a notation which we shall again have occasion to use subsequently, namely,

$$\left[\frac{d^n f}{dy^n}\right]=\left(\frac{d^n f}{dy^n}\right)-\text{terms of order } n+1.$$

It has already been remarked that if, at any stage, an integrable combination of characteristics has been missed, it will appear as an integral of the equations for v in the search for an intermediate integral of higher order. As an example let us search for third order integrals of

$$rs+p=0,$$

taking $m_1=0$, $m_2=-p/s^2$.

Here equation (17) is

$$-\frac{p}{s^2}\frac{\partial v}{\partial t}=0,$$

showing that v is independent of t, while (18) leads to

$$\frac{\partial v}{\partial z}=\frac{\partial v}{\partial q}=0,$$

$$\frac{v}{s}+\frac{v^2 p}{s^3}+\frac{\partial v}{\partial s}+\frac{2p}{s}\frac{\partial v}{\partial p}+\frac{p}{s^2}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}=0.$$

The subsidiary equations are

$$\frac{ds}{s^2}=\frac{dp}{2ps}=\frac{dy}{p}=\frac{dx}{-s^2}=\frac{dv}{-sv-\dfrac{v^2 p}{s}},$$

of which
$$p/s^2=A, \quad x+s=B, \quad y-p/s=C,$$

are particular integrals. These are the three well known integrals of the second order. The equations also lead to the third order integral

$$\frac{s^3 p_{12}}{s^2 - pp_{12}} = F\left(\frac{p}{s^2},\ x+s,\ y-\frac{p}{s}\right).$$

There is, of course, a first order integral, obtained by putting $p = v$.

As an example of the method here used, let us find third order integrals of the equation

$$r-t-4p/x = 0,$$

for which $m_1 = 1$, $m_2 = -1$, $f = -t-4p/x$.

Equation (17) is now $\quad \dfrac{\partial v}{\partial s} - \dfrac{\partial v}{\partial t} - \dfrac{2}{x} = 0,$

whence $$v = -2t/x + \psi(x,\ y,\ z,\ p,\ q,\ s+t).$$

Equation (18) is

$$\left(\frac{dv}{dx}\right) - \left(\frac{dv}{dy}\right) + \left(v + \frac{4s}{x}\right)\frac{\partial v}{\partial s} - v\left(\frac{\partial v}{\partial t} + \frac{4}{x}\right) = 0,$$

or, if $u = s+t$, we find that ψ must satisfy the equations

$$\frac{\partial \psi}{\partial p} - \frac{\partial \psi}{\partial q} - \frac{2}{x}\,\frac{\partial \psi}{\partial u} + \frac{3}{x^2} = 0,$$

$$\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} + (p-q)\frac{\partial \psi}{\partial z} - u\left(\frac{\partial \psi}{\partial p} - \frac{\partial \psi}{\partial q}\right) + \frac{4p}{x}\,\frac{\partial \psi}{\partial p} + \frac{4u}{x}\,\frac{\partial \psi}{\partial u} - \frac{2\psi}{x} = 0.$$

From the first of these two equations, we have

$$\psi = \frac{3}{x^2}\,(p+xu) + F(a,\ b,\ x,\ y,\ z),$$

where $a = p+q$, $b = 2p+xu$.

Substituting in the second equation, we find

$$x\frac{\partial F}{\partial z} + 2\frac{\partial F}{\partial a} + \frac{\partial F}{\partial b} = 0,$$

$$\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} - a\frac{\partial F}{\partial z} + \frac{3b}{x}\,\frac{\partial F}{\partial b} - \frac{2F}{x} = 0.$$

There is an obvious solution for which

$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial a} = \frac{\partial F}{\partial b} = 0,$$

$$\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} - \frac{2F}{x} = 0,$$

and it is easy to see that this is the only solution. Hence

$$F = x^2\phi(x+y),$$

and a third order integral of the given equation is

$$p_{12}+p_{03}+\frac{3s+t}{x}+\frac{3p}{x^2}+x^2\phi(x+y) = 0.$$

By changing the sign of y we see that, for the system in which $m_1 = -1$ $m_2 = 1$, an integral is

$$p_{12}-p_{03}-\frac{3s-t}{x}+\frac{3p}{x^2}+x^2\psi(x-y) = 0.$$

It will be found that the work here given is considerably shorter than that required for the solution of the same equation by Darboux's ordinary process as given on pp. 357–9 of Forsyth's *Theory of Differential Equations*, Vol. VI, though it is not yet quite so simple as Laplace's solution.

5. *The Equations which must be Satisfied by r when $n > 3$.*

Before proceeding to the explicit forms of the equations which must be satisfied by the r of equation (12) we must obtain the coefficients of p_{1n} and $p_{0,\,n+1}$ in $\dfrac{d^n f}{dy^n}$. Let A_n denote the coefficient of p_{1n}, B_n that of $p_{0,\,n+1}$. We then have

$$\left(\frac{d^n f}{dy^n}\right) = \left[\frac{d^n f}{dy^n}\right]+A_n\,p_{1n}+B_n\,p_{0,\,n+1}.$$

Differentiating this with regard to y we find

$$\left(\frac{d^{n+1} f}{dy^{n+1}}\right) = \frac{d}{dy}\left[\frac{d^n f}{dy^n}\right]+\frac{dA_n}{dy}\,p_{1n}+\frac{dB_n}{dy}\,p_{0,\,n+1}+\left(A_n+\frac{d}{dy}\,\frac{\partial f}{\partial s}\right)p_{1,\,n+1}$$
$$+\left(B_n+\frac{d}{dy}\,\frac{\partial f}{\partial t}\right)p_{0,\,n+2},$$

whence $$A_{n+1} = A_n+\frac{d}{dy}\,\frac{\partial f}{\partial s}.$$

But, by actual calculation,

$$A_3 = \frac{\partial f}{\partial p}+3\,\frac{d}{dy}\,\frac{\partial f}{\partial s};$$

$$\left.\begin{aligned} &\text{therefore} && A_n = \frac{\partial f}{\partial p} + n\frac{d}{dy}\frac{\partial f}{\partial s} \\ &\text{and similarly,} && B_n = \frac{\partial f}{\partial q} + n\frac{d}{dy}\frac{\partial f}{\partial t} \end{aligned}\right\}. \tag{19}$$

The general equations (12) and (13) may now be written in the form

$$p_{1,\,n-1} + m_1 p_{0n} + v = 0, \tag{20}$$

where v is a function of x, y, z, p, q, s, t, ..., $p_{1,\,n-2}$, $p_{0,\,n-1}$, and

$$\begin{aligned} p_{0n}\left(\frac{dm_1}{dx} + m_2\frac{dm_1}{dy}\right) + \left(\frac{dv}{dx}\right) + m_2\left(\frac{dv}{dy}\right) - \left(\frac{d^{n-2}f}{dy^{n-2}}\right)\frac{\partial v}{\partial p_{1,\,n-2}} \\ - \left(p_{1,\,n-1}\frac{\partial f}{\partial s} + p_{0n}\frac{\partial f}{\partial t}\right)\frac{\partial v}{\partial p_{1,\,n-2}} + p_{1,\,n-1}\frac{\partial v}{\partial p_{0,\,n-1}} \\ + m_2\left(p_{1,\,n-1}\frac{\partial v}{\partial p_{1,\,n-2}} + p_{0n}\frac{\partial v}{\partial p_{0,\,n-1}}\right) \\ = \left[\frac{d^{n-1}f}{dy^{n-1}}\right] + \left(\frac{\partial f}{\partial p} + (n-1)\frac{d}{dy}\frac{\partial f}{\partial s}\right)p_{1,\,n-1} + \left(\frac{\partial f}{\partial q} + (n-1)\frac{d}{dy}\frac{\partial f}{\partial t}\right)p_{0n}. \end{aligned} \tag{21}$$

Substituting for $p_{1,\,n-1}$ its value $-m_1 p_{0n} - v$, the coefficient of p_{0n} in (21) leads to the equation

$$\begin{aligned} (m_1 - m_2)\left(m_1\frac{\partial v}{\partial p_{1,\,n-2}} - \frac{\partial v}{\partial p_{0,\,n-1}}\right) + \frac{dm_1}{dx} + m_2\frac{dm_1}{dy} + m_1\frac{\partial f}{\partial p} - \frac{\partial f}{\partial q} \\ + (n-1)\left(m_1\frac{d}{dy}\frac{\partial f}{\partial s} - \frac{d}{dy}\frac{\partial f}{\partial t}\right) = 0, \end{aligned} \tag{22}$$

while the term independent of p_{0n} leads to

$$\begin{aligned} v\left(m_1\frac{\partial v}{\partial p_{1,\,n-2}} - \frac{\partial v}{\partial p_{0,\,n-1}}\right) - \left(\frac{d^{n-2}f}{dy^{n-2}}\right)\frac{\partial v}{\partial p_{1,\,n-2}} + \left(\frac{dv}{dx}\right) + m_2\left(\frac{dv}{dy}\right) \\ + v\left(\frac{\partial f}{\partial p} + (n-1)\frac{d}{dy}\frac{\partial f}{\partial s}\right) = \left[\frac{d^{n-1}f}{dy^{n-1}}\right]. \end{aligned} \tag{23}$$

In the particular case when $m_1 = m_2$, v disappears from (22), which therefore gives directly a condition which must be satisfied by f in order that there should be even a particular integral of order n, for (23) now reduces to a single equation, so that there will always be a value of v provided that the condition to which (22) reduces is satisfied.

When $m_1 = m_2 = m$, say, $= \frac{1}{2}\frac{\partial f}{\partial s} = \left(\frac{\partial f}{\partial t}\right)^{\frac{1}{2}}$, we have

$$m\frac{d}{dy}\frac{\partial f}{\partial s} = m\left(\frac{\partial f}{\partial t}\right)^{-\frac{1}{2}}\frac{d}{dy}\frac{\partial f}{\partial t} = \frac{d}{dy}\frac{\partial f}{\partial t};$$

and equation (22) becomes

$$\left(\frac{dm}{dx}\right)+m\left(\frac{dm}{dy}\right)+m\frac{\partial f}{\partial p}-\frac{\partial f}{\partial q}-\left(\frac{df}{dy}\right)\frac{\partial m}{\partial s}+p_{12}\left[\left(m-\frac{\partial f}{\partial s}\right)\frac{\partial m}{\partial s}+\frac{\partial m}{\partial t}\right]$$
$$+p_{03}\left(m\frac{\partial m}{\partial t}-\frac{\partial f}{\partial t}\frac{\partial m}{\partial s}\right) = 0.$$

The coefficients of p_{12} and p_{03} are easily seen to be zero; hence the conditions which must be satisfied by f are

$$\frac{1}{2}\frac{\partial f}{\partial s} = \left(\frac{\partial f}{\partial t}\right)^{\frac{1}{2}} = m,$$

$$\left(\frac{dm}{dx}\right)+m\left(\frac{dm}{dy}\right)-\left(\frac{df}{dy}\right)\frac{\partial m}{\partial s}+m\frac{\partial f}{\partial p}-\frac{\partial f}{\partial q} = 0. \tag{24}$$

This last condition is independent of n, and it will be found that equation (17) reduces to exactly the same form when $m_1 = m_2$. Whence it follows that if the roots of the characteristic equation

$$m^2-\frac{\partial f}{\partial s}m+\frac{\partial f}{\partial t} = 0,$$

are equal, equation (7) cannot be solved by Darboux's method unless the condition (24) is satisfied, and if the condition is satisfied there is certainly an intermediate integral of the third order.

In particular, if $m = 0$, the condition (24) reduces to $\frac{\partial f}{\partial q} = 0$; hence no equation of the form

$$r+f(x, y, z, p, q) = 0$$

can be solved by Darboux's method if f contains q. If f does not contain q, there is, of course, a first order integral.

The general form of equation for which $m_1 = m_2$ is obtained by eliminating m from

$$r+2ms+m^2t+2\phi\,(m, x, y, z, p, q) = 0, \tag{25}$$

$$s+mt+\frac{\partial\phi}{\partial m} = 0. \tag{26}$$

Here $m_1 = m_2 = m$.

The condition that this equation should be soluble by Darboux's method is

$$\left(\frac{dm}{dx}\right)+m\left(\frac{dm}{dy}\right)-\left(\frac{df}{dy}\right)\frac{\partial m}{\partial s}+m\frac{\partial f}{\partial p}-\frac{\partial f}{\partial q}=0.$$

And we have
$$1+\left(t+\frac{\partial^2\phi}{\partial m^2}\right)\frac{\partial m}{\partial s}=0,$$

$$\left(t+\frac{\partial^2\phi}{\partial m^2}\right)\left(\frac{dm}{dy}\right)+\frac{\partial^2\phi}{\partial m\,\partial y}+q\,\frac{\partial^2\phi}{\partial m\,\partial z}+s\,\frac{\partial^2\phi}{\partial m\,\partial p}+t\,\frac{\partial^2\phi}{\partial m\,\partial q}=0,$$

$$\left(t+\frac{\partial^2\phi}{\partial m^2}\right)\left(\frac{dm}{dx}\right)+\frac{\partial^2\phi}{\partial m\,\partial x}+p\,\frac{\partial^2\phi}{\partial m\,\partial z}-(2ms+m^2t+2\phi)\,\frac{\partial^2\phi}{\partial m\,\partial p}+s\,\frac{\partial^2\phi}{\partial m\,\partial q}=0,$$

$$\frac{\partial f}{\partial p}=2\,\frac{\partial\phi}{\partial p},\qquad \frac{\partial f}{\partial q}=2\,\frac{\partial\phi}{\partial q}.$$

Substituting these values in (24), and eliminating s by means of (26), we find that t also disappears, so that there is but a single equation for ϕ, namely,

$$2\,\frac{\partial^2\phi}{\partial m^2}\left(m\,\frac{\partial\phi}{\partial p}-\frac{\partial\phi}{\partial q}\right)-\frac{\partial^2\phi}{\partial m\,\partial x}-m\,\frac{\partial^2\phi}{\partial m\,\partial y}-(p+mq)\,\frac{\partial^2\phi}{\partial m\,\partial z}+2\phi\,\frac{\partial^2\phi}{\partial m\,\partial p}$$
$$-\frac{\partial\phi}{\partial m}\left(m\,\frac{\partial^2\phi}{\partial m\,\partial p}-\frac{\partial^2\phi}{\partial m\,\partial q}\right)+2\left(\frac{\partial\phi}{\partial y}+q\,\frac{\partial\phi}{\partial z}-\frac{\partial\phi}{\partial p}\,\frac{\partial\phi}{\partial m}\right)=0. \quad (27)$$

Thus the equation resulting from the elimination of m between (25) and (26) can be solved by Darboux's method if, and only if, equation (27) is satisfied.

It is here proved only that there is an intermediate integral of the n-th order, where n is not less than three. It is known that there is an intermediate integral of the second order when this condition is satisfied (Goursat, *loc. cit.* in Section 3, Vol. II, p. 167 ; see also Vol. I, Section 93).

6. The explicit form of the conditions that equations (22) and (23) [*] should have at least one common integral may be written down by the theorem given in Forsyth's *Theory of Differential Equations*, Vol. I, Section 173 (which will here be referred to as "§ 173"). For this purpose

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[*] Conditions entirely similar to those here given might, of course, have been written down in the case of the familiar equations (13). The rectangular array would then have contained one more row.

we re-write (22) and (23) in the forms *

$$\Delta_1 u = \frac{\partial u}{\partial p_{0,\,n-1}} - m_1 \frac{\partial u}{\partial p_{1,\,n-2}} + a\frac{\partial u}{\partial v} = 0,$$

$$\Delta_2 u = \frac{\partial u}{\partial x} + m_2 \frac{\partial u}{\partial y} + (m_2 q + p)\frac{\partial u}{\partial z}$$

$$+ \sum_{k=1}^{n-2} \left\{ \left(m_2 p_{1k} - \frac{d^{k-1} f}{dy^{k-1}}\right) \frac{\partial u}{\partial p_{1,\,k-1}} + (m_2 p_{0,\,k+1} + p_{1k}) \frac{\partial u}{\partial p_{0k}} \right\}$$

$$- \left(\frac{d^{n-2} f}{dy^{n-2}}\right) \frac{\partial u}{\partial p_{1,\,n-2}} + (bv + c)\frac{\partial u}{\partial v} = 0,$$

where †

$$(m_1 - m_2)\, a = \frac{dm_1}{dx} + m_2 \frac{dm_1}{dy} + m_1 \frac{\partial f}{\partial p} - \frac{\partial f}{\partial q} + (n-1)\left(m_1 \frac{d}{dy}\frac{\partial f}{\partial s} - \frac{d}{dy}\frac{\partial f}{\partial t}\right),$$

$$b = a - \frac{\partial f}{\partial p} - (n-1)\frac{d}{dy}\frac{\partial f}{\partial s},$$

$$c = \left[\frac{d^{n-1} f}{dy^{n-1}}\right],$$

and a new dependent variable u, has been introduced by putting

$$\frac{\partial v}{\partial \xi} = -\frac{\partial u}{\partial \xi} \Big/ \frac{\partial u}{\partial v},$$

where ξ is any one of the independent variables.

Let
$$\delta = \frac{\partial}{\partial p_{0,\,n-1}} - m_1 \frac{\partial}{\partial p_{1,\,n-2}},$$

$$\Delta_{12} u = \Delta_1 \Delta_2 u - \Delta_2 \Delta_1 u,$$

$${}_1\Delta_{12} u = \Delta_1 \Delta_{12} u - \Delta_{12} \Delta_1 u, \ \ldots ;$$

then the constituents of the first column in the rectangular array of § 173 are the coefficients of $\partial u/\partial \xi$ in $\Delta_{12} u$, the constituents of the second column are the coefficients in ${}_1\Delta_{12} u$, and so on.

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* We assume that $m_1 \neq m_2$ in what follows. The case of $m_1 = m_2$ has long been completely solved.

† It may be observed that a cannot vanish if $n \neq 2$.

The first two columns of the array are readily seen to be*

$$\begin{array}{cc}
0 & 0 \\
0 & 0 \\
\cdot & \cdot \\
\cdot & \cdot \\
0 & 0 \\
m_1(m_1-m_2) & 0 \\
-(m_1-m_2) & 0 \\
\dfrac{dm_1}{dx}+m_2\dfrac{dm_1}{dy}-\delta\,(d^{n-2}f/dy^{n-2}) & \delta\left(\dfrac{dm_1}{dx}+m_2\dfrac{dm_1}{dy}\right)+(m_1-m_2)\left(m_1\dfrac{\partial m_1}{\partial s}-\dfrac{\partial m_1}{\partial t}\right)-\delta^2\left(\dfrac{d^{n-2}f}{dy^{n-2}}\right) \\
v\delta b+\delta c+ab-\Delta_2 a & \delta\,(ab)-\delta\Delta_2 a-\Delta_{12}a+\delta^2 c+a\delta b.
\end{array}$$

If n is greater than 4, the second column consists entirely of zeros, and when $n=4$ the next to last term is plainly

$$2(m_1-m_2)\left(m\frac{\partial m_1}{\partial s}-\frac{\partial m_1}{\partial t}\right)-2\left(\frac{\partial^2 f}{\partial t^2}-2m_1\frac{\partial^2 f}{\partial s\,\partial t}+m_1^2\frac{\partial^2 f}{\partial s^2}\right)=0\,;$$

while the last term, after a somewhat laborious reduction, is found also to vanish, so that in every case, provided only that $n>3$, the second column of the array consists entirely of zeros. It is, perhaps, the chief result of the paper that we are able definitely to prove that all fourth order integrals of (7) are such that the equation ${}_1\Delta_{12}u=0$ is an identity.

In the general case there are $2n$ rows in the array and a number of columns which depends upon n, but which it is difficult to specify except for particular values of n. The required conditions for the existence of a general intermediate integral of the n-th order are that all the determinants of $2n-1$ rows and columns which can be formed from every selection of $2n-1$ columns from those of the array should vanish. For a particular integral it is sufficient that all the determinants of $2n$ rows and columns should vanish. There will, in general, be a very large number of such conditions, but many of the determinants will vanish identically owing to the large number of zeros which occur, so that, for this and other reasons, the number of *independent* conditions will not be very great. The conditions for the existence of a common integral to equations (22) and (23) are, as here given, impracticable. In any given case it would be simpler to form the equations themselves and to deal with them directly; but I cannot bring myself to believe that the difficulties in the way of a complete solution of the problem are such as to defy investigation, even in the present state of analysis.

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* The form of the array is somewhat different for $n=3$.

DIFFRACTION OF TIDAL WAVES ON FLAT ROTATING SHEETS OF WATER

By J. PROUDMAN.

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1. The methods of the following paper are very similar to those already well known for two-dimensional problems in the diffraction of sound and electric waves. The results, however, are different owing to the different boundary conditions. Also, for the diffraction by objects whose linear dimensions are small compared with a wave length, the interest in the results lies mainly in their form near the objects themselves, which is not usually the case for sound and electric waves.

We shall only consider free tidal motion of sheets of water of uniform depth.

A complete solution is obtained for the case of the diffraction of a plane wave by a circular island, but the remaining solutions are all approximations. They are based on Lord Rayleigh's approximate theory of diffraction,* and the method of conjugate functions is introduced so that Schwarz's method for conformal transformations becomes available. The number of problems which can thus be approximately solved (at least symbolically) is quite large, but, as examples, we shall only consider those of the diffraction of a plane wave by an elliptic island, by a semi-elliptic cape, by a rectangular bay, and by a passage between one sea and another.

General Equations.

2. Suppose the sheet of water to be rotating with constant angular velocity ω about an axis perpendicular to its plane, and let the depth

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* "On the Passage of Waves through Apertures in Plane Screens, and Allied Problems," *Phil. Mag.* (5), Vol. XLIII, p. 259 (1897), [*Sc. Papers*, Vol. IV, p. 283]. Also, "On the Incidence of Aerial and Electric Waves upon Small Obstacles . . .," *Phil. Mag.* (5), Vol. XLIV, p. 28 (1897) [*Sc. Papers*, Vol. IV, p. 305].

of the water (as rotating in free relative equilibrium) be uniform and equal to h. Let ζ denote the elevation of the free surface at any time.

Then, for a disturbance in which the time t only enters through the factor $e^{i\sigma t}$, we have the equation *

$$(\nabla_1^2+\kappa^2)\,\zeta = 0, \tag{1}$$

there being no disturbing force. Here $\nabla_1^2 = \partial^2/\partial x^2+\partial^2/\partial y^2$ in Cartesian coordinates, and

$$\kappa^2 = \frac{\sigma^2-4\omega^2}{gh}, \tag{2}$$

g being the acceleration due to gravity.

The boundary condition is given by

$$i\sigma\frac{\partial\zeta}{\partial n}+2\omega\frac{\partial\zeta}{\partial s} = 0, \tag{3}$$

where $\partial/\partial n$ denotes differentiation along the outward drawn normal to the boundary, and $\partial/\partial s$ along the positive direction of the arc. We exclude the cases in which $\sigma = 0$ and $\sigma^2 = 4\omega^2$.

When we use polar coordinates r, θ, normal solutions of (1) are given by

$$J_m(\kappa r)\,e^{\pm im\theta}, \qquad D_m(\kappa r)\,e^{\pm im\theta},$$

m being any constant. Here $J_m(\kappa r)$, $D_m(\kappa r)$ are Bessel's functions of the first and second kind respectively, the latter being taken to be the form appropriate for the disturbances produced by a finite object in an infinite sea.

For small values of κr, taking only principal parts, we have

$$J_0(\kappa r) = 1, \qquad J_m(\kappa r) = \frac{(\kappa r)^m}{2^m m!}, \tag{4}$$

$$D_0(\kappa r) = -\frac{2}{\pi}\left\{\log\tfrac{1}{2}\kappa r+\gamma+\tfrac{1}{2}i\pi\right\}, \qquad D_m(\kappa r) = \frac{2^m(m-1)!}{\pi(\kappa r)^m}, \tag{5}$$

γ being Euler's constant. These, with the associated functions of θ, all form two-dimensional harmonic functions.

Diffraction of a Plane Wave by a Circular Island.

3. Let the primary wave be given in Cartesians by

$$\zeta = \zeta_0 e^{i\sigma t} \equiv e^{i(\kappa x+\sigma t)}, \tag{6}$$

which obviously satisfies the equation (1).

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* See Lamb, *Hydrodynamics*, 3rd ed., p. 303.

Transforming to polar coordinates, we have

$$\zeta_0 = e^{i\kappa r\cos\theta} = J_0(\kappa r) + 2\sum_{n=1}^{\infty} i^n J_n(\kappa r)\cos n\theta. \tag{7}$$

Let us assume that the disturbance produced by the island is given by $\zeta = \zeta_1 e^{i\sigma t}$, where

$$\zeta_1 = A_0 D_0(\kappa r) + 2\sum_{n=1}^{\infty} i^n D_n(\kappa r)\{A_n\cos n\theta + B_n \sin n\theta\}, \tag{8}$$

A_n, B_n being constants whose values are to be determined.

If the shore of the island be given by $r = a$, the boundary condition will be

$$\left(i\sigma\frac{\partial}{\partial r} + 2\omega\frac{\partial}{a\partial\theta}\right)(\zeta_0 + \zeta_1) = 0, \tag{9}$$

for $r = a$, and all values of θ. Substituting in this from (7) and (8), we obtain

$$i\sigma\kappa J_n'(\kappa a)\cos n\theta - \frac{2\omega n}{a}J_n(\kappa a)\sin n\theta + i\sigma\kappa D_n'(\kappa a)\{A_n\cos n\theta + B_n\sin n\theta\}$$

$$-\frac{2\omega n}{a}D_n(\kappa a)\{A_n\sin n\theta - B_n\cos n\theta\} = 0, \tag{10}$$

which holds for all the values of n, including zero, if we take $B_0 = 0$.

On equating to zero the coefficients of $\cos n\theta$ and $\sin n\theta$ in (10) we obtain

$$i\sigma\kappa D_n'(\kappa a)A_n + \frac{2\omega n}{a}D_n(\kappa a)B_n + i\sigma\kappa J_n'(\kappa a) = 0,$$

$$\frac{2\omega n}{a}D_n(\kappa a)A_n - i\sigma\kappa D_n'(\kappa a)B_n + \frac{2\omega n}{a}J_n(\kappa a) = 0.$$

Solving these algebraically, we obtain

$$A_n = -\frac{\sigma^2(\kappa a)^2 J_n'(\kappa a)D_n'(\kappa a) - 4\omega^2 n^2 J_n(\kappa a)D_n(\kappa a)}{\sigma^2(\kappa a)^2 D_n'^2(\kappa a) - 4\omega^2 n^2 D_n^2(\kappa a)}, \tag{11}$$

$$B_n = -i\frac{2\omega n\sigma\kappa a\{J_n(\kappa a)D_n'(\kappa a) - J_n'(\kappa a)D_n(\kappa a)\}}{\sigma^2(\kappa a)^2 D_n'^2(\kappa a) - 4\omega^2 n^2 D_n^2(\kappa a)}, \tag{12}$$

for all the values of n, including zero.

We therefore have

$$-\zeta_1 = \frac{J_0'(\kappa a)}{D_0'(\kappa a)}D_0(\kappa r) + 2\sum_{n=1}^{\infty}\frac{i^n}{\sigma^2(\kappa a)^2 D_n'^2(\kappa a) - 4\omega^2 n^2 D_n^2(\kappa a)}$$

$$\times D_n(\kappa r)\left[\{\sigma^2(\kappa a)^2 J_n'(\kappa a)D_n'(\kappa a) - 4\omega^2 n^2 J_n(\kappa a)D_n(\kappa a)\}\cos n\theta\right.$$

$$\left.+2\omega n\sigma\kappa a\{J_n(\kappa a)D_n'(\kappa a) - J_n'(\kappa a)D_n(\kappa a\}i\sin n\theta\right]. \tag{13}$$

Now let us take the principal part of this for points near the island when κa is very small. We have, on substituting from (4) and (5),

$$\zeta_1 = \frac{\kappa a}{\sigma^2-4\omega^2}\,\frac{a}{r}\,\{(\sigma^2+4\omega^2)\, i\cos\theta - 4\omega\sigma\sin\theta\}. \tag{14}$$

To the same approximation the total elevation is given by

$$\zeta = \left[1+\left\{\kappa r+\kappa a\,\frac{\sigma^2+4\omega^2}{\sigma^2-4\omega^2}\,\frac{a}{r}\right\} i\cos\theta - \frac{4\omega\sigma\kappa a}{\sigma^2-4\omega^2}\,\frac{a}{r}\sin\theta\right] e^{i\sigma t}, \tag{15}$$

the real part of which may be written

$$\zeta = \left\{1-\frac{4\omega\sigma\kappa a}{\sigma^2-4\omega^2}\,\frac{a}{r}\sin\theta\right\}\cos(\sigma t-\epsilon), \tag{16}$$

where
$$\epsilon = -\left\{\kappa r+\kappa a\,\frac{\sigma^2+4\omega^2}{\sigma^2-4\omega^2}\,\frac{a}{r}\right\}\cos\theta. \tag{17}$$

Similarly written, and to the same order of approximation, the primary wave is given by

$$\zeta = \cos(\sigma t-\epsilon_0), \quad \epsilon_0 = -\kappa r\cos\theta. \tag{18}$$

The change in the height of high tide can be obtained from (16), and the change in phase from (17).

Approximate Theory.

4. Lord Rayleigh's approximate theory of diffraction depends on the fact that over a region the linear dimensions of which are small compared with a wave-length, a solution of (1) is sensibly a two-dimensional harmonic function. Over such a region, therefore, in the present application, we treat ζ as such a function, and proceed to satisfy the boundary conditions. This is facilitated by the use of conjugate functions.

Suppose that $\zeta = \zeta_1 + i\zeta_2$, where ζ_1, ζ_2 are real. Take $\zeta' = \zeta_1' + i\zeta_2'$, where ζ_1', ζ_2' are real and such that $\zeta_1 + i\zeta_1'$, $\zeta_2 + i\zeta_2'$ are functions of $x+iy$, x, y being Cartesian coordinates in the plane of the sheet of water. ζ' is now determined except to an arbitrary additive constant.

At the boundary we shall have

$$\frac{\partial\zeta_1}{\partial n} = \frac{\partial\zeta_1'}{\partial s}, \qquad \frac{\partial\zeta_2}{\partial n} = \frac{\partial\zeta_2'}{\partial s},$$

and consequently $$i\sigma\frac{\partial\zeta}{\partial n} + 2\omega\frac{\partial\zeta}{\partial s} = i\sigma\frac{\partial\zeta'}{\partial s} + 2\omega\frac{\partial\zeta}{\partial s} = 0.$$

If then we take

$$\psi = i\sigma\zeta' + 2\omega\zeta, \tag{19}$$

ψ will be a two-dimensional harmonic function which is constant along a coast-line. Let us also take

$$\phi = i\sigma\zeta - 2\omega\zeta', \tag{20}$$

so that if $\phi = \phi_1 + i\phi_2$, and $\psi = \psi_1 + i\psi_2$, when ϕ_1, ϕ_2 ψ_1, ψ_2 are real, we shall have

$$\phi_1 = -\sigma\zeta_2 - 2\omega\zeta_1', \qquad \phi_2 = \sigma\zeta_1 - 2\omega\zeta_2',$$

$$\psi_1 = -\sigma\zeta_2' + 2\omega\zeta_1, \qquad \psi_2 = \sigma\zeta_1' + 2\omega\zeta_2.$$

We then see that $\phi_1 + i\psi_1$, $\phi_2 + i\psi_2$ are functions of $x+iy$, and that when either ϕ or ψ is known the other is determinate except to an arbitrary additive constant.

From (19) and (20) we have the relation

$$\zeta = -\frac{1}{\sigma^2 - 4\omega^2}(i\sigma\phi + 2\omega\psi). \tag{21}$$

Let us use ζ_0, ζ_0', ϕ_0, ψ_0 to denote the respective values of ζ, ζ', ϕ, ψ for the primary wave only.

We require further conditions for the functions $\phi - \phi_0$, $\psi - \psi_0$, and these are to be determined by the principal parts near the diffracting object of the possible forms for the complete expression of the disturbance.

We shall assume that for the seas in question the possible forms for $\phi - \phi_0$ are such as vanish at infinity or else take the form

$$\log(\tfrac{1}{2}\kappa r) + \gamma + \tfrac{1}{2}i\pi. \tag{22}$$

The form (22) will only be required when the effect at a distance is that due to a source.

Of course, the consideration of forms at infinity is only auxiliary, the results being applicable only over a very small region.

On the above principles the expression (14) can be very easily reproduced.

Diffraction by an Elliptic Island.

5. Let the equation of the shore of the island be given by $\xi = a$, where

$$x = c\cosh\xi\cos\eta, \quad y = c\sinh\xi\sin\eta,$$

and let the primary wave be given by

$$\zeta = e^{i\kappa(lx+my)+i\sigma t}, \tag{23}$$

where $l^2+m^2 = 1$.

Then, in the neighbourhood of the island, we have

$$\zeta_0 = 1+i\kappa(lx+my), \quad \zeta_0' = i\kappa(ly-mx),$$

on dropping the time factor, so that

$$\phi_0 = i\sigma+(i\sigma l+2\omega m)i\kappa x+(i\sigma m-2\omega l)i\kappa y, \tag{24}$$

$$\psi_0 = 2\omega+(i\sigma l+2\omega m)i\kappa y-(i\sigma m-2\omega l)i\kappa x. \tag{25}$$

In terms of ξ, η, ψ_0 becomes

$$\psi_0 = 2\omega+(i\sigma l+2\omega m)i\kappa c \sinh\xi \sin\eta-(i\sigma m-2\omega l)i\kappa c\cosh\xi\cos\eta.$$

For the total disturbance we shall then have

$$\psi = 2\omega+i\kappa c(i\sigma l+2\omega m)(\sinh\xi-e^{a-\xi}\sinh a)\sin\eta$$
$$-i\kappa c(i\sigma m-2\omega l)(\cosh\xi-e^{a-\xi}\cosh a)\cos\eta,$$

since this is constant over $\xi = a$, and differs from ψ_0 by a function which tends to vanish as $\xi\to\infty$.

Simplifying, we have

$$\psi = 2\omega+i\kappa c e^{a}(i\sigma l+2\omega m)\sinh(\xi-a)\sin\eta$$
$$-i\kappa c e^{a}(i\sigma m-2\omega l)\sinh(\xi-a)\cos\eta, \tag{26}$$

and then ϕ must be given by

$$\phi = i\sigma+i\kappa c e^{a}(i\sigma l+2\omega m)\cosh(\xi-a)\cos\eta$$
$$+i\kappa c e^{a}(i\sigma m-2\omega l)\cosh(\xi-a)\sin\eta, \tag{27}$$

in accordance with § 4.

Substituting from these into (21), we obtain, after a little reduction,

$$\zeta = 1-\frac{\kappa c e^{a}}{\sigma^2-4\omega^2}\Big\{2\omega\sigma e^{a-\xi}(l\sin\eta-m\cos\eta)$$
$$-i[\sigma^2\cosh(\xi-a)-4\omega^2\sinh(\xi-a)](l\cos\eta+m\sin\eta)\Big\}. \tag{28}$$

Restoring the time factor, and taking only the real part, we have, to our order of approximation,

$$\zeta = \left\{1-\frac{2\omega\sigma\kappa c e^{2a-\xi}}{\sigma^2-4\omega^2}(l\sin\eta-m\cos\eta)\right\}\cos(\sigma t-\epsilon), \tag{29}$$

where $\epsilon = -\dfrac{\kappa c e^{a}}{\sigma^2-4\omega^2}\{\sigma^2\cosh(\xi-a)-4\omega^2\sinh(\xi-a)\}(l\cos\eta-m\sin\eta)$. (30)

Similarly written, the primary wave is

$$\zeta = \cos(\sigma t-\epsilon_0), \qquad \epsilon_0 = -\kappa c(l\cosh\xi\cos\eta+m\sinh\xi\sin\eta). \tag{31}$$

Diffraction by a Semi-Elliptic Cape.

6. We consider a cape projecting from a straight coast. Let the equation of the shore of the cape be given by $\xi = a$, where

$$x = c\sinh\xi\cos\eta, \qquad y = c\cosh\xi\sin\eta,$$

for $0 \leqslant \eta \leqslant \pi$, the equation of the coast-line being $y = 0$ or $\eta = 0$ and π.

Let the primary wave be given by

$$\zeta = e^{-\kappa\sqrt{(\sigma^2-4\omega^2)}.(i\sigma x+2\omega y)+i\sigma t}, \tag{32}$$ *

which satisfies (1) and gives no velocity perpendicular to the straight coast-line.

Near the cape we have, on dropping the time factor,

$$\zeta_0 = 1-\frac{\kappa}{\sqrt{(\sigma^2-4\omega^2)}}(i\sigma x+2\omega y), \qquad \zeta'_0 = -\frac{\kappa}{\sqrt{(\sigma^2-4\omega^2)}}(i\sigma y-2\omega x),$$

giving $\qquad \phi_0 = i\sigma+\sqrt{(\sigma^2-4\omega^2)}\,\kappa x, \qquad \psi_0 = 2\omega+\sqrt{(\sigma^2-4\omega^2)}\,\kappa y.$ (33)

As the change in ϕ produced by the cape will be real, and the change in ψ will vanish on the straight coast, we see from (21) that on the cape and the neighbouring coast the high tide will have its original coastal height.

In terms of ξ, η, ψ_0 becomes

$$\psi_0 = 2\omega+\sqrt{(\sigma^2-4\omega^2)}\,\kappa c\cosh\xi\sin\eta,$$

and then for the total disturbance it is easily seen that we have

$$\psi = 2\omega+\sqrt{(\sigma^2-4\omega^2)}\,\kappa c e^{a}\sinh(\xi-a)\sin\eta, \tag{34}$$

$$\phi = i\sigma+\sqrt{(\sigma^2-4\omega^2)}\,\kappa c e^{a}\cosh(\xi-a)\cos\eta. \tag{35}$$

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[*] This is the solution first given by Lord Kelvin with reference to a rotating canal. See Lamb, *l.c.*

Substituting from these into (21) we obtain

$$\zeta = 1 - \frac{\kappa c e^{\alpha}}{\sqrt{(\sigma^2 - 4\omega^2)}} \left\{ i\sigma \cosh(\xi - \alpha) \cos\eta + 2\omega \sinh(\xi - \alpha) \sin\eta \right\}. \qquad (36)$$

Restoring the time factor and taking only the real part, we have, to our order of approximation,

$$\zeta = \left\{ 1 - \frac{2\omega\kappa c e^{\alpha}}{\sqrt{(\sigma^2 - 4\omega^2)}} \sinh(\xi - \alpha) \sin\eta \right\} \cos(\sigma t - \epsilon), \qquad (37)$$

where

$$\epsilon = \frac{\sigma\kappa c e^{\alpha}}{\sqrt{(\sigma^2 - 4\omega^2)}} \cosh(\xi - \alpha) \cos\eta. \qquad (38)$$

Similarly written, the primary wave is

$$\zeta = \left\{ 1 - \frac{2\omega\kappa c}{\sqrt{(\sigma^2 - 4\omega^2)}} \cosh\xi \sin\eta \right\} \cos(\sigma t - \epsilon_0), \qquad (39)$$

where

$$\epsilon_0 = \frac{\sigma\kappa c}{\sqrt{(\sigma^2 - 4\omega^2)}} \sinh\xi \cos\eta. \qquad (40)$$

Coast-lines with Projecting Corners.

7. If we take $\alpha = 0$ in the preceding section, we have a formal solution for the case of a straight narrow *promontory* of length c projecting perpendicularly from a straight coast.

The expressions found, however, lead to an infinite velocity for the water at the projecting end of the promontory, owing to the vanishing there of the Jacobian $\partial(x, y)/\partial(\xi, \eta)$. The question then arises whether the formal solution gives an approximate solution of the physical problem except in the immediate neighbourhood of the end of the promontory, or whether it fails altogether. The question applies to all cases in which an acute-angled portion of land projects into the sea.*

Below are given arguments, tending to show, that with a more stringent condition on the size of the motion in the incident wave, the formal solutions will be approximate representations of the physical facts, except in the immediate neighbourhood of the projecting sharp corners.

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* This question was raised by the referee. It would appear to apply also to certain solutions of problems in the diffraction of sound waves given in the papers of Lord Rayleigh already quoted, and to be then answerable in a manner similar to that of the present section.

Except in these neighbourhoods, the formal solutions only undergo small changes in character owing to a small change in the shape of the disturbing object. This remains true even though a round corner changes into a sharp point, as in the case of the promontory.

We assume that the same continuity holds for the actual physical state.

If therefore we can show that it is possible to round off the sharp corners, without materially altering the general configuration of the object, but making the curvatures everywhere small enough for the formal solution in the modified case to be a true representation of the physical facts in this case, we may perhaps assume that the formal solution in the original case (*i.e.*, with the sharp corners), gives an approximate representation of the physical facts in that case, except in the neighbourhood of the corners.

Taking a primary wave of the general type

$$\zeta = Ae^{i(\sigma t+\kappa x)},$$

it is necessary for the validity of the equations of § 2 that the velocity of the water, which is proportional to the gradient of ζ, shall be small compared with $(gh)^{\frac{1}{2}}$. This requires that A/h shall be small.*

Now let c denote the linear order of magnitude of the disturbing object, and a the smallest radius of curvature of the coast-line. We must then have κc small, while at the points of greatest curvature on the coast-line, the order of magnitude of the velocity will bear to that of the primary wave the ratio c/a.

We must then have Ac/ha small, while the purpose of the present section requires that a/c shall also be small. Together, these conditions require that $(A/h)^{\frac{1}{2}}$ shall be small.

If then $(A/h)^{\frac{1}{2}}$ and not merely A/h may be considered as small, the modification mentioned above will be a possible one. If, however, $(A/h)^{\frac{1}{2}}$ cannot be considered as small, the partial justification of the present section breaks down.

In the remaining sections we shall assume that the motion in the primary wave is small enough to satisfy the condition just obtained.

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* *Cf.*, Lamb, *l.c.*, p. 243.

Diffraction by a Rectangular Bay.

8. Let the three sides of the bay be given in Cartesians by $x = \pm a$, $y = 0$; and the remaining coast-line by the portions of the line $y = b$ for which $|x| > a$ (Fig. 1).

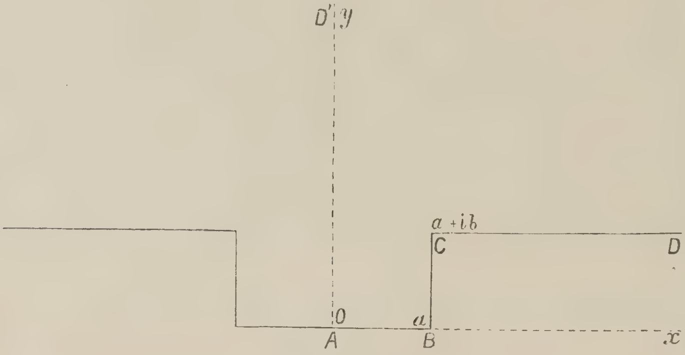

FIG. 1.—z plane.

Let us take the same primary wave as in § 6, so that again

$$\phi_0 = i\sigma + \sqrt{(\sigma^2 - 4\omega^2)}\, \kappa x, \qquad \psi_0 = 2\omega + \sqrt{(\sigma^2 - 4\omega^2)}\, \kappa y. \tag{33}$$

Let us also write $$w = \frac{1}{\sqrt{(\sigma^2 - 4\omega^2)}} \{\phi + i\psi - i(\sigma + 2\omega)\}. \tag{41}$$

Then w will be a function of $x + iy$ which is purely imaginary on $x = 0$, while the imaginary part will be equal to $i\kappa b$ on the coast-line, and to $i\kappa y$ at infinity. The correspondence between w and z is shown in Figs. 1 and 2, where corresponding points are similarly lettered.

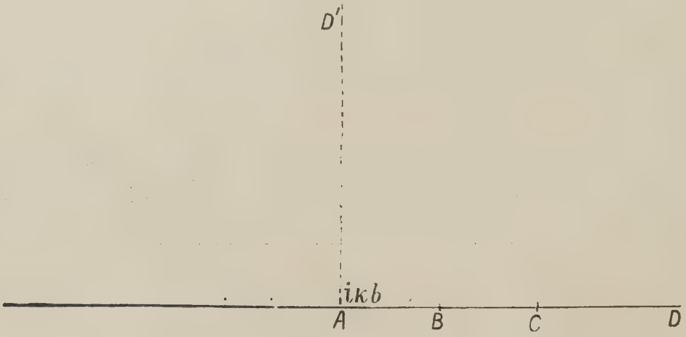

FIG. 2.—w plane.

Taking an auxiliary t plane, to the upper half of which both the regions

$ABCDD'A$ of Figs. 1 and 2 correspond, we may use Schwarz's transformation.

Supposing the points A, B, C, D to correspond respectively to $t = 0,\ 1,\ 1/k^2,\ \infty$, where k has to be found, we have

$$dz = \frac{M(1-k^2t)^{\frac{1}{2}}\,dt}{t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}, \qquad dw = \frac{N\,dt}{t^{\frac{1}{2}}}, \tag{42}$$

M, N being constants whose values are to be determined.

Since $dw/dz = \kappa$ at infinity, we have $N = M\kappa k$.

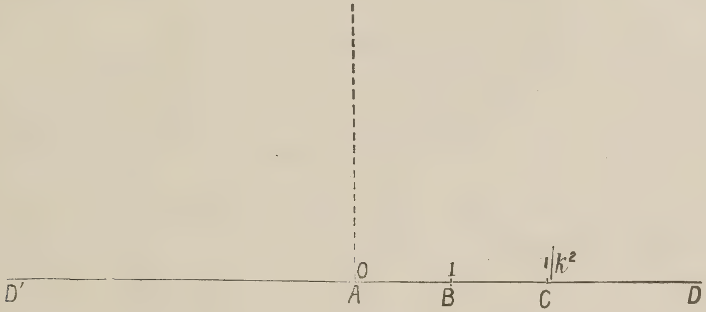

FIG. 3.—t plane.

Let us now write $t^{\frac{1}{2}} = \operatorname{sn} u$, the ordinary Jacobian elliptic function. We shall have

$$w = \iota\kappa b + 2M\kappa k \operatorname{sn} u, \tag{43}$$

and

$$dz = 2M \operatorname{dn}^2 u\, du,$$

giving

$$z = 2ME(u). \tag{44}$$

Here $E(u)$ is the elliptic integral usually so denoted, and the corresponding part of the u plane is shown in Fig. 4.

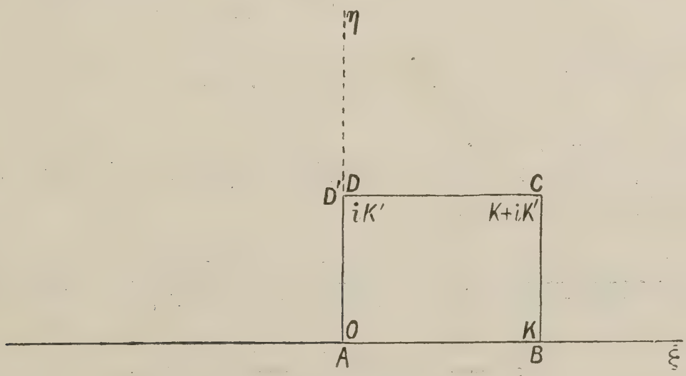

FIG. 4.—u plane.

Now from the correspondences for B and C, we have

$$a = 2ME(K) = 2ME,$$

$$a+ib = 2ME(K+iK') = 2M\{E+i(K'-E')\},$$

K, K', E, E' being the constants usually so denoted in the theory of elliptic functions.

Thus $$M = a/2E,$$

and $$\frac{K'-E'}{E} = \frac{b}{a}, \tag{45}$$

the latter determining the modulus k.

If we take $u = \xi+i\eta$, where ξ, η are real, we shall have

$$w = i\kappa b+\kappa a\,\frac{k}{E}\,\frac{\operatorname{sn}\xi\operatorname{dn}\eta}{1-\operatorname{dn}^2\xi\operatorname{sn}^2\eta}+i\kappa a\,\frac{k}{E}\,\frac{\operatorname{cn}\xi\operatorname{dn}\xi\operatorname{sn}\eta\operatorname{cn}\eta}{1-\operatorname{dn}^2\xi\operatorname{sn}^2\eta}, \tag{46}$$

from the ordinary addition formula, and the imaginary transformation, k being the modulus for the functions of ξ, and k' that for the functions of η.

For the tidal problem we have therefore

$$\phi = i\sigma+\sqrt{(\sigma^2-4\omega^2)}\,\kappa a\,\frac{k}{E}\,\frac{\operatorname{sn}\xi\operatorname{dn}\eta}{1-\operatorname{dn}^2\xi\operatorname{sn}^2\eta}, \tag{47}$$

$$\psi = 2\omega+\sqrt{(\sigma^2-4\omega^2)}\left\{\kappa b+\kappa a\,\frac{k}{E}\,\frac{\operatorname{cn}\xi\operatorname{dn}\xi\operatorname{sn}\eta\operatorname{cn}\eta}{1-\operatorname{dn}^2\xi\operatorname{sn}^2\eta}\right\}, \tag{48}$$

and consequently

$$\zeta = 1-\frac{1}{\sqrt{(\sigma^2-4\omega^2)}}\left\{i\sigma\kappa a\,\frac{k}{E}\,\frac{\operatorname{sn}\xi\operatorname{dn}\eta}{1-\operatorname{dn}^2\xi\operatorname{sn}^2\eta}+2\omega\kappa b\right.$$
$$\left.+2\omega\kappa a\,\frac{k}{E}\,\frac{\operatorname{cn}\xi\operatorname{dn}\xi\operatorname{sn}\eta\operatorname{cn}\eta}{1-\operatorname{dn}^2\xi\operatorname{sn}^2\eta}\right\}, \tag{49}$$

ξ, η being given by $$x+iy = \frac{a}{E}\,E(\xi+i\eta),$$

and the modulus k by (45).

Restoring the time factor, and taking only the real part, we have, to our order of approximation,

$$\zeta = \left\{1-\frac{2\omega}{\sqrt{(\sigma^2-4\omega^2)}}\left(\kappa b+\kappa a\,\frac{k}{E}\,\frac{\operatorname{cn}\xi\operatorname{dn}\xi\operatorname{sn}\eta\operatorname{cn}\eta}{1-\operatorname{dn}^2\xi\operatorname{sn}^2\eta}\right)\right\}\cos(\sigma t-\epsilon), \tag{50}$$

where $$\epsilon = \frac{\sigma\kappa a}{\sqrt{(\sigma^2-4\omega^2)}}\,\frac{k}{E}\,\frac{\operatorname{sn}\xi\operatorname{dn}\eta}{1-\operatorname{dn}^2\xi\operatorname{sn}^2\eta}. \tag{51}$$

It is interesting to notice such particular cases as are easily evaluated. The changes in phase at A and B are instances.

At A, $\xi = \eta = 0$, and therefore $\epsilon = 0$, so that there is no change in phase.

At B we have, for ϵ,

$$\frac{\sigma\kappa a}{\sqrt{(\sigma^2-4\omega^2)}}\,\frac{k}{E},$$

as against

$$\frac{\sigma\kappa a}{\sqrt{(\sigma^2-4\omega^2)}},$$

for the primary wave.

Passage between Two Seas.

9. Let the two seas be given by the upper and lower portions of the z plane, the passage lying between the two points $z = \pm c$, c being real.

Let us take

$$x = c\cosh\xi\cos\eta, \qquad y = c\sinh\xi\sin\eta,$$

and suppose that ξ ranges from $-\infty$ to ∞, while η is restricted to lie between 0 and π. Then the coast-lines are given by each side of $\eta = 0$ and of $\eta = \pi$, and the passage by $\xi = 0$.

Let us take the primary wave to be given by (32) for the upper sea only. Then we have

$$\phi_0 = i\sigma + \sqrt{(\sigma^2-4\omega^2)}\,\kappa c\cosh\xi\cos\eta, \quad \psi_0 = 2\omega + \sqrt{(\sigma^2-4\omega^2)}\,\kappa c\sinh\xi\sin\eta, \tag{52}$$

for the upper sea, while over the lower sea we may conveniently take $\zeta_0' = 2\omega/i\sigma$, so that

$$\phi_0 = -\frac{4\omega^2}{i\sigma}, \quad \psi_0 = 2\omega. \tag{53}$$

When $|\xi|$ is large, we have

$$\log\tfrac{1}{2}\kappa r = |\xi| + \log(\tfrac{1}{4}\kappa c),$$

so that at infinity the only effect of the passage on the value of ϕ must be a constant multiple of

$$|\xi| + \log(\tfrac{1}{4}\kappa c) + \gamma + \tfrac{1}{2}i\pi.$$

It is then easily seen that the solution is given by

$$\phi = -\frac{1}{2i\sigma}(\sigma^2+4\omega^2) + \frac{1}{2i\sigma}(\sigma^2-4\omega^2)\frac{\xi}{\log(\frac{1}{4}\kappa c)+\gamma+\frac{1}{2}i\pi} + \tfrac{1}{2}\sqrt{(\sigma^2-4\omega^2)}\,\kappa c e^{\xi}\cos\eta, \tag{54}$$

$$\psi = 2\omega + \frac{1}{2i\sigma}(\sigma^2-4\omega^2)\frac{\eta}{\log(\frac{1}{4}\kappa c)+\gamma+\frac{1}{2}i\pi} + \tfrac{1}{2}\sqrt{(\sigma^2-4\omega^2)}\,\kappa c e^{\xi}\sin\eta. \tag{55}$$

These give

$$\zeta = \tfrac{1}{2}\left[1 - \frac{\xi + \dfrac{2\omega}{i\sigma}\eta}{\log(\frac{1}{4}\kappa c) + \gamma + \frac{1}{2}i\pi} - \frac{\kappa c e^{\xi}}{\sqrt{(\sigma^2 - 4\omega^2)}}\,(i\sigma\cos\eta + 2\omega\sin\eta)\right]e^{i\sigma t}. \qquad (56)$$

In the middle of the passage we have $\xi = 0$, $\eta = \frac{1}{2}\pi$, and consequently

$$\zeta = \tfrac{1}{2}\left[1 - \frac{\omega}{i\sigma}\,\frac{\pi}{\log(\frac{1}{4}\kappa c) + \gamma + \frac{1}{2}i\pi} - \frac{2\omega\kappa c}{\sqrt{(\sigma^2 - 4\omega^2)}}\right]e^{i\sigma t},$$

the height of high tide being changed in the ratio

$$1 : \tfrac{1}{2}\left[1 + \frac{\omega}{2\sigma}\,\frac{\pi^2}{\{\log(\frac{1}{4}\kappa c) + \gamma\}^2 + \frac{1}{4}\pi^2} - \frac{2\omega\kappa c}{\sqrt{(\sigma^2 - 4\omega^2)}}\right],$$

and the phase being changed by

$$\frac{\pi\omega}{\sigma}\,\frac{\log(\frac{1}{4}\kappa c) + \gamma}{\{\log(\frac{1}{4}\kappa c) + \gamma\}^2 + \frac{1}{4}\pi^2},$$

providing this be small.

DIFFRACTION AT A STRAIGHT EDGE

By H. M. MACDONALD.

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THE object of the following note is to shew how the solution of the problem of diffraction at a straight edge may be obtained in a simple manner that does not involve the use of special analytical functions. It has been proved* that, at a point in the neighbourhood of the boundary of the geometrical shadow, the components of the electric and magnetic forces are in a ratio L to the components of the electric and magnetic forces in the incident waves, where

$$L = 2^{-\frac{1}{2}} e^{\frac{1}{4}\pi i} \int_{u_0}^{\infty} e^{-\frac{1}{2}\pi i u^2}\, du,$$

provided the point is at a distance from the surface of the obstacle of higher order than $\rho\,(\kappa\rho)^{-\frac{1}{3}}$, where ρ^{-1} is the curvature of the surface of the obstacle in the direction of the ray at the curve of contact of the geometrical shadow, and $2\pi/\kappa$ is the wave length. When the obstacle is a plane screen, ρ vanishes at an edge, and the above result suggests that the relation between the actual components of the electric and magnetic forces, and the corresponding components in the incident waves is of the same form near to the screen. Taking the case of plane waves incident on a perfectly absorbing screen, bounded by a straight edge, when the electric force Z_1 in the incident waves is parallel to the edge of the screen, the value of the electric force Z at any point will be given by

$$Z = 2^{-\frac{1}{2}} e^{i\kappa(x\cos\theta_1 + y\sin\theta_1 + Vt) + \frac{1}{4}\pi i} \int_{-\infty}^{u_0} e^{-\frac{1}{2}\pi i u^2}\, du,$$

where κr is not necessarily large, and

$$Z_1 = e^{i\kappa(x\cos\theta_1 + y\sin\theta_1 + Vt)},$$

the origin of the axes of reference is on the edge, the axis of z is the edge, the axis of y is perpendicular to the screen, the axis of x is in the plane of the screen perpendicular to the edge, and θ_1 is the angle the direction of the incident waves makes with the plane of the screen. Now Z satisfies the differential equation

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* *Phil. Trans.*, Ser. A, Vol. 212, p. 334, 1912.

$$\frac{\partial^2 Z}{\partial x^2}+\frac{\partial^2 Z}{\partial y^2}=\frac{1}{V^2}\frac{\partial^2 Z}{\partial t^2},$$

therefore

$$2i\kappa\left(\frac{\partial u_0}{\partial x}\cos\theta_1+\frac{\partial u_0}{\partial y}\sin\theta_1\right)+\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}-\pi i u_0\left\{\left(\frac{\partial u_0}{\partial x}\right)^2+\left(\frac{\partial u_0}{\partial y}\right)^2\right\}=0;$$

and writing $x=r\cos\theta,\quad y=r\sin\theta,$

this becomes

$$\frac{\partial^2 u_0}{\partial r^2}+\frac{1}{r}\frac{\partial u_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_0}{\partial \theta^2}-\pi i u_0\left\{\left(\frac{\partial u_0}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial u_0}{\partial \theta}\right)^2\right\}$$
$$+2i\kappa\left\{\frac{\partial u_0}{\partial r}\cos(\theta-\theta_1)-\frac{1}{r}\frac{\partial u_0}{\partial \theta}\sin(\theta-\theta_1)\right\}=0.$$

Taking u_0 to be a real quantity, it follows that

$$\frac{\partial^2 u_0}{\partial r^2}+\frac{1}{r}\frac{\partial u_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_0}{\partial \theta^2}=0,$$

$$\frac{\partial u_0}{\partial r}\cos(\theta-\theta_1)-\frac{1}{r}\frac{\partial u_0}{\partial \theta}\sin(\theta-\theta_1)=\frac{\pi}{2\kappa}u_0\left\{\left(\frac{\partial u_0}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial u_0}{\partial \theta}\right)^2\right\};$$

from the second of these equations it appears that u_0 is of the form $(\kappa r)^{\frac{1}{2}}v_0$, hence from the first of the equations

$$u_0=B(\kappa r)^{\frac{1}{2}}\cos\tfrac{1}{2}(\theta-\epsilon),$$

and substituting in the second equation

$$\tfrac{1}{2}\left(\frac{\kappa}{r}\right)^{\frac{1}{2}}\left[\cos\tfrac{1}{2}(\theta-\epsilon)\cos(\theta-\theta_1)+\sin\tfrac{1}{2}(\theta-\epsilon)\sin(\theta-\theta_1)\right]$$
$$=\tfrac{1}{2}\pi\left(\frac{\kappa}{r}\right)^{\frac{1}{2}}\tfrac{1}{4}B^2\cos\tfrac{1}{2}(\theta-\epsilon),$$

that is, $\cos\tfrac{1}{2}(\theta+\epsilon-2\theta_1)=\tfrac{1}{4}\pi B^2\cos\tfrac{1}{2}(\theta-\epsilon),$

therefore $\epsilon=\theta_1,\quad B=2\pi^{-\frac{1}{2}},$

and $u_0=2(\kappa r)^{\frac{1}{2}}\pi^{-\frac{1}{2}}\cos\tfrac{1}{2}(\theta-\theta_1),$

whence $Z=2^{-\frac{1}{2}}e^{i\kappa(x\cos\theta_1+y\sin\theta_1+Vt)+\frac{1}{4}\pi i}\int_{-\infty}^{u_0}e^{-\frac{1}{2}\pi i u^2}\,du,$

gives the value of Z at any point when the screen is perfectly absorbing.

To obtain the solution for a perfectly reflecting screen, the value of the electric force in the reflected waves on the upper side of the screen at a great distance from the screen tends to the value

$$e^{i\kappa(x\cos\theta_1-y\sin\theta_1+Vt)},$$

and therefore the corresponding term in the solution is obtained by changing the sign of θ_1 in the previous result; hence the value of the electric force at any point for the reflected waves is

$$2^{-\frac{1}{2}}e^{i\kappa(x\cos\theta_1-y\sin\theta_1+Vt)+\frac{1}{4}\pi i}\int_{-\infty}^{u_1}e^{-\frac{1}{2}\pi iu^2}\,du,$$

where $$u_1=2(\kappa r)^{\frac{1}{2}}\,\pi^{-\frac{1}{2}}\cos\tfrac{1}{2}(\theta+\theta_1),$$

and, since Z vanishes at the surface of a perfectly reflecting screen, the complete solution is given by

$$\begin{aligned}Z = \;& 2^{-\frac{1}{2}}e^{i\kappa(x\cos\theta_1+y\sin\theta_1+Vt)+\frac{1}{4}\pi i}\int_{-\infty}^{u_0}e^{-\frac{1}{2}\pi iu^2}\,du\\ &-2^{-\frac{1}{2}}e^{i\kappa(x\cos\theta_1-y\sin\theta_1+Vt)+\frac{1}{4}\pi i}\int_{-\infty}^{u_1}e^{-\frac{1}{2}\pi iu^2}\,du,\end{aligned}$$

where u_0, u_1 have the values given above.

In the same way, when the magnetic force γ_1 in the incident wave is parallel to the edge of the screen, and the screen is perfectly absorbing, the magnetic force γ at any point is given by

$$\gamma=2^{-\frac{1}{2}}e^{i\kappa(x\cos\theta_1+y\sin\theta_1+Vt)+\frac{1}{4}\pi i}\int_{-\infty}^{u_0}e^{-\frac{1}{2}\pi iu^2}\,du,$$

where $$\gamma_1=e^{i\kappa(x\cos\theta_1+y\sin\theta_1+Vt)}.$$

When the screen is perfectly reflecting, the magnetic force at any point is given by

$$\begin{aligned}\gamma = \;& 2^{-\frac{1}{2}}e^{i\kappa(x\cos\theta_1+y\sin\theta_1+Vt)+\frac{1}{4}\pi i}\int_{-\infty}^{u_0}e^{-\frac{1}{2}\pi iu^2}\,du\\ &+2^{-\frac{1}{2}}e^{i\kappa(x\cos\theta_1-y\sin\theta_1+Vt)+\frac{1}{4}\pi i}\int_{-\infty}^{u_0}e^{-\frac{1}{2}\pi iu^2}\,du.\end{aligned}$$

ON A CONFIGURATION OF 21 POINTS AND 21 LINES WHICH ARISES FROM THE COMPLETE QUADRILATERAL AND DETERMINES THE GROUP OF 168 PLANE COLLINEATIONS

By W. BURNSIDE.

[Read November 12th, 1914.]

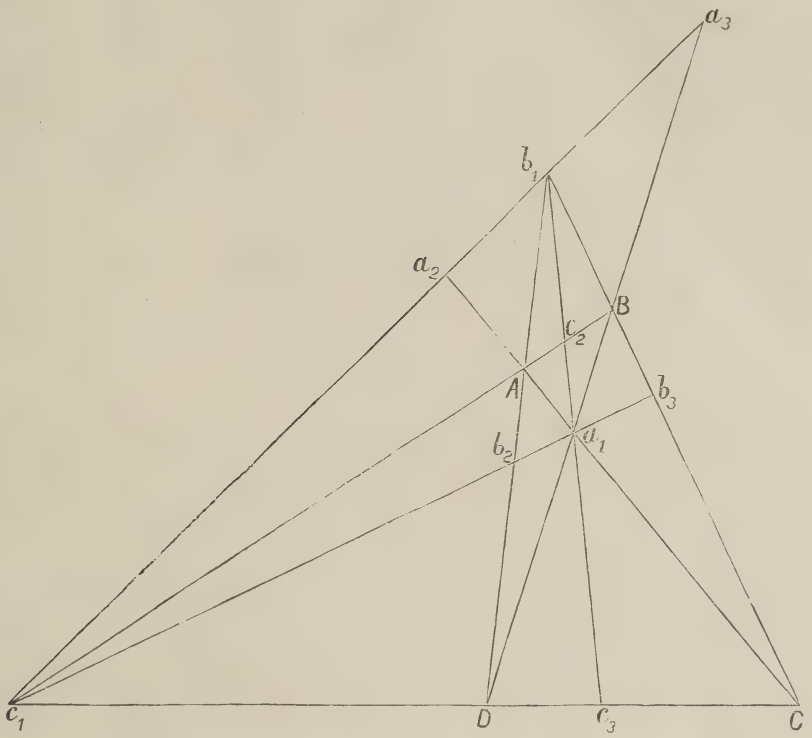
A PROJECTIVE transformation, or collineation, in the plane is completely determined when the new positions A', B', C', D' of four points A, B, C, D are given, it being understood that no three of either set lie on a straight line. Such a collineation may be specified by the symbol

$$\begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \end{pmatrix}.$$

In particular to each permutation of a set of 4 points there corresponds a distinct collineation; and the 24 collineations that so arise constitute a group. This group will permute the points of the plane, in general, in sets of 24. The accompanying figure makes it clear that the 3 points a_1, b_1, c_1 are permuted by the group, as also are the 6 points a_2, a_3, b_2, b_3, c_2, c_3. Moreover, if α_2, α_3 is the pair of points which divide both a_2, a_3 and b_1, c_1 harmonically; and β_2, β_3 and γ_2, γ_3 the corresponding pairs for b_2, b_3, c_1, a_1 and c_2, c_3, a_1, b_1; then the 6 points α_2, α_3, β_2, β_3, γ_2, γ_3 (which are all imaginary when A, B, C, D are real) are permuted by the group. The 8 imaginary fixed points of the collineations of order three are permuted by the group. Any point, other than those enumerated, which lies on one of the 9 lines of the figure, belongs to a set of 12 which are permuted by the group. All other points belong to sets of 24.

Disregarding the points A, B, C, D, the figure consists of 9 lines and 9 points, the latter lying 4 by 4 on 3 of the lines, and 2 by 2 on the other 6. If we combine with the 9 points a set of 12 which arises, under the collineations of the group, from an arbitrarily chosen point d_1 on a_3a_1, we arrive at a set of 21 points which lie 4 by 4 on the 9 lines of the figure.

The individual points of the set of 12 may be most conveniently specified by the collineations which change d_1 into each of them, the collineations themselves being denoted by the corresponding permutations



of A, B, C, D. The following Table I indicates both how the points arise from d_1, and the lines of the figure on which they lie. It also has the advantage of indicating immediately certain collinearities between the original 9 points and the new set of 12. Thus d_3 arises from d_2 by the collineation (AC). This is a perspective of which a_2 is the fixed point; and therefore $a_2 d_2 d_3$ is a line. The complete set of such collinearities derived from the original figure and Table I is given in Tables II (α) and II (β).

TABLE I.

d_1	e_1	f_1	g_1
identity, (AC);	$(AC)(BD)$, (BD);	$(AD)(BC)$, $(ABCD)$;	$(AB)(CD)$, $(ADCB)$;
on a_3a_1		on a_1a_2	
d_2	f_2	e_2	g_2
(AB), (ACB);	$(ACBD)$, (ABD);	$(ADBC)$, (BCD);	(CD), (ADC);
on b_1b_2		on b_1b_3	
d_3	g_3	f_3	e_3
(BC), (ABC);	$(ABDC)$, (BDC);	(AD), (ACD);	$(ACDB)$, (ADB)
on c_1c_3		on c_1c_2	

TABLE II (α).

$a_2 b_2 c_3$	$a_1 d_2 e_2$	$a_1 f_2 g_2$	$b_1 d_1 f_1$	$b_1 e_1 g_1$	$c_1 d_1 g_1$	$c_1 e_1 f_1$
$a_2 b_3 c_2$	$a_1 d_3 e_3$	$a_1 f_3 g_3$	$b_1 d_3 f_3$	$b_1 e_3 g_3$	$c_1 d_2 g_2$	$c_1 e_2 f_2$
$a_3 b_2 c_2$	$a_3 d_2 e_3$	$a_2 f_2 g_3$	$b_2 d_1 f_3$	$b_3 e_1 g_3$	$c_3 d_1 g_2$	$c_2 e_1 f_2$
$a_3 b_3 c_3$	$a_3 d_3 e_2$	$a_2 f_3 g_2$	$b_2 d_3 f_1$	$b_3 e_3 g_1$	$c_3 d_2 g_1$	$c_2 e_2 f_1$

TABLE II (β).

$a_2 d_2 d_3$	$b_2 e_3 e_1$	$c_2 d_1 d_2$
$a_2 e_2 e_3$	$b_2 g_3 g_1$	$c_2 g_1 g_2$
$a_3 f_2 f_3$	$b_3 d_3 d_1$	$c_3 e_1 e_2$
$a_3 g_2 g_3$	$b_3 f_3 f_1$	$c_3 f_1 f_2$

The sets of 21 points made up of the original 9 points and the 12 derived from d_1 lie 3 by 3 on the 28 lines of Table II (α), while these lines pass 4 by 4 through the 21 points. So long as d_1 is taken arbitrarily the 12 lines of Table II (β) will each contain only 3 of the points.

Now as d_1 moves on $a_3 a_1$, e_1 and d_2 describe projective ranges on $a_3 a_1$ and $b_1 b_2$, and therefore the rays $a_2 d_2 d_3$ and $a_2 e_1$ describe projective pencils. Two projective pencils, with a common vertex, have two self-corresponding rays. Choose for d_1 one of the two positions for which $a_2 d_2 d_3$ and $a_2 e_1$ coincide. Then a_2, d_2, d_3, e_1 are collinear. Moreover, since a_2 is the fixed point of the perspective (AC) of order two, while e_1 lies on $a_3 a_1$ which is the fixed line of the perspective, the points d_2, d_3 divide the points a_2, e_1 harmonically. The collineation $(AC)(BD)$ is seen from Table I to permute d_1 with e_1, d_2 with e_2, d_3 with e_3, and to leave a_2 unchanged. Hence when a_2, d_2, d_3, e_1 are collinear, so are a_2, e_2, e_3, d_1.

It follows that when the position of d_1 is chosen as above, each of the 12 lines of Table II (β) passes through 4 of the 21 points.

Combining these with the original 9 lines, a set of 21 lines arises on which the 21 points lie 4 by 4. The actual collinearities are given in

TABLE III.

$a_2 a_3 b_1 c_1$	$b_2 b_3 c_1 a_1$	$c_2 c_3 a_1 b_1$	$d_2 d_3 a_2 e_1$
$a_3 a_1 d_1 e_1$	$b_3 b_1 e_2 g_2$	$c_3 c_1 d_3 g_3$	$d_3 d_1 b_3 f_2$
$a_1 a_2 f_1 g_1$	$b_1 b_2 d_2 f_2$	$c_1 c_2 e_3 f_3$	$d_1 d_2 c_2 g_3$

$e_2 e_3 a_2 d_1$	$f_2 f_3 a_3 g_1$	$g_2 g_3 a_3 f_1$
$e_3 e_1 b_2 g_2$	$f_3 f_1 b_3 d_2$	$g_3 g_1 b_2 e_2$
$e_1 e_2 c_3 f_3$	$f_1 f_2 c_3 e_3$	$g_1 g_2 c_2 d_3$

Moreover in each collinearity the two points with the same letter divide the other two points harmonically. It is found on an examination of Table III that the 21 lines pass 4 by 4 through the 21 points; and that the straight line joining any two of the 21 points belongs either to Table II (a) or to Table III.

This configuration is invariant for the group of 24 collineations that arises from the permutations of A, B, C, D. In this there are 9 perspectives of order two, viz., those whose fixed points and lines are a_1, a_2a_3; a_2, a_3a_1; ...; c_3, c_1c_2. Table III suggests that the configuration is also invariant for the 12 perspectives of order two whose fixed points and lines are d_1, d_2d_3; d_2, d_3d_1; ...; g_3, g_1g_2.

The first of these leaves d_1, d_2, d_3, a_2, e_1 unchanged and in virtue of the collinearities $d_1d_2c_2g_3$, $d_3d_1b_3f_2$, $a_3a_1d_1e_1$, $e_2e_3a_2d_1$ it permutes c_2 with g_3, b_3 with f_2, a_1 with a_3, and e_2 with e_3. Since it permutes c_2 with g_3 and leaves d_3 unchanged, it permutes the lines $g_1g_2c_2d_3$ and $c_3c_1d_3g_3$. Now, from Table II (a), $c_1d_1g_1$ and $c_3d_1g_2$ are lines. Hence the collineation permutes c_1 with g_1 and c_3 with g_2. Again the perspective permutes $f_3f_1b_3d_2$ and $b_1b_2d_2f_2$, while, from Table II (a), $b_1d_1f_1$ and $b_2d_1f_3$ are lines. Hence it permutes b_1 with f_1 and b_2 with f_3. The perspective of order two of which d_1 is the fixed point and d_2d_3 the fixed line gives therefore the following permutation of the 21 points

$$(a_1a_3)\,a_2\,(b_1f_1)(b_2f_3)(b_3f_2)(c_1g_1)(c_2g_3)(c_3g_2)\,d_1d_2d_3e_1\,(e_2e_3).$$

The 21 perspectives of order two which arise from Table III generate a group of collineations for which the configuration is invariant. The order and nature of this group of collineations is perhaps most readily determined by a consideration of the 7 triangles

$$a_1a_2a_3,\ b_1b_2b_3,\ c_1c_2c_3,\ d_1d_2d_3,\ e_1e_2e_3,\ f_1f_2f_3,\ g_1g_2g_3$$

into which the 21 points fall. These triangles are permuted by each of the 21 perspectives. The a-triangle is changed into itself by the 9 perspectives of which the fixed points and lines are

a_1, a_2a_3; a_2, a_3a_1; a_3, a_1a_2; b_1, b_2b_3; c_1, c_2c_3; d_1, d_2d_3; e_1, e_2e_3; f_1, f_2f_3; g_1, g_2g_3;

and is changed by

c_2, c_3c_1; c_3, c_1c_2; into the b-triangle,
b_2, b_3b_1; b_3, b_1b_2; ,, c-triangle,
e_2, e_3e_1; e_3, e_1e_2; ,, d-triangle,
d_2, d_3d_1; d_3, d_1d_2; ,, e-triangle,
g_2, g_3g_1; g_3, g_1g_2; ,, f-triangle,
f_2, f_3f_1; f_3, f_1f_2; ,, g-triangle.

Moreover, the 9 perspectives which leave the a-triangle unchanged permute the other six as follows, viz.,

$$bc\,(de)(fg),\ \ (bc)\,de\,(fg),\ \ (bc)(de)\,fg,\ \ bc\,(df)(eg),\ \ bc\,(dg)(ef),\ \ (bf)(cg)\,de,$$

$$(bg)(cf)\,de,\ \ (bd)(ce)\,fg,\ \ (be)(cd)\,fg.$$

These statements all follow from the 21 collinearities. It is easy to verify that the preceding permutations generate a group of order 24; and that a collineation which changes each of the 7 triangles into itself leaves every point unchanged. Hence the order of the above group of collineations for which the configuration is invariant is 168; and it is simply isomorphic with the permutation-group of degree 7 of this order.

Returning to the figure, the triangle $a_1b_1c_1$, formed by 3 of the 21 points, is invariant for the group of 24 collineations arising from the permutations of A, B, C, D. Hence it must be one of 7 triangles permuted by the group. The complete set is found to be

$$a_1b_1c_1,\ \ a_2d_1e_1,\ \ a_3f_1g_1,\ \ b_2e_2g_2,\ \ b_3d_2f_2,\ \ c_2d_3g_3,\ \ c_3e_3f_3.$$

Each pair of these, and each pair of the other set of 7 triangles, is in perspective.

Just as the 21 points lie 3 by 3 on the 28 lines of Table II (a), so the 21 lines pass 3 by 3 through a set of 28 points. These are the points that arise from A, B, C, D under the collineations of the group; and they lie 4 by 4 on the 21 lines of the configuration.

The question of the reality of the lines and points considered in no way affects the reasoning by which the existence of the configuration and the group of 168 collineations has been established. When the points A, B, C, D are real, the 12 points $d_1, d_2, \ldots, g_3$ are necessarily imaginary.

ON THE REDUCTION OF SETS OF INTERVALS

By W. H. YOUNG *and* GRACE CHISHOLM YOUNG.

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1. It has been already pointed out that, before sets of intervals were studied for their own sake, various writers had had occasion to make use of them, and had in this way virtually obtained the Heine-Borel theorem without, however, enunciating it. In particular this is true of Cantor, Heine, Dini, and Darboux, in their proofs of the theorem that the property of uniform continuity, first considered by Heine, is possessed by every continuous function.

We do not intend to enter further into these matters here.* It is, however, not only interesting but of some importance to notice that a slight modification of Heine's proof enables us to obtain a theorem, including the Heine-Borel theorem as a special case. We have indeed only to retain that which is essential in Heine's argument and reject that which is accessory.

The new theorem† so obtained is not only more general than the Heine-Borel theorem ; it leads to results unobtainable by application of the Heine-Borel theorem alone, and in particular to a number of theorems relating to intervals which have been at the base of much of the work in the modern theory of derivates and their integrals. The new theorem accordingly may well be destined to introduce order into the somewhat heterogeneous collection of theorems of the type of the Heine-Borel theorem which have been formulated and employed from time to time by writers on the Theory of Functions of a Real Variable.

We propose in the present paper to state the theorem in question, and to modify Heine's reasoning in such a way as to render it applicable for the purposes of proof. For the convenience of readers, we give Heine's own brief demonstration *verbatim*. We then pass to the deduction as corollaries of the theorems relating to sets of intervals above referred to. It will be noticed that nowhere are transfinite numbers employed.

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* The reader may refer to a footnote in a paper entitled "Note on Overlapping Regions," by W. H. Young (1912), *Messenger of Mathematics*, pp. 126, 127.

† See below, § 3.

2. As is well known, a continuous function $f(x)$ has the property of uniform continuity; that is to say, given any positive quantity e, we can find a positive quantity d, so that, in any interval of length $\leqslant d$ in the fundamental segment, the oscillation of the function $\leqslant 2e$; in symbols it comes to the same thing if we say

$$|f(x+h)-f(x)| \leqslant e,$$

provided only $$-d \leqslant h \leqslant d.$$

We proceed to reproduce Heine's proof of this property in the original.*

"Bezeichnet $3e$ eine beliebige Grösse, so existiert eine solche Zahl, dass von $x=a$ bis zu ihr hin $f(x)-f(a)$ absolut $\leqslant 3e$ ist. Ein Wert der dies leistet, ist der grösste und macht zugleich $f(x)-f(a)-3e=0$. Dieser Wert sei x_1. In ähnlicher Art findet man eine Zahl x_2 als die grösste, welche bewirkt, dass von $x=x_1$ bis $x=x_2$ immer $|f(x)-f(x_1)| \leqslant 3e$ bleibt. So fährt man fort; kommt man nach einer endlichen Anzahl n von Operationen zu $x_n=b$, oder findet, dass $|f(x)-f(x_{n-1})|$ von $x=x_{n-1}$ bis $x=b$ noch nicht $3e$ überschreitet, so ist der Satz bewiesen.

"Es bleibt noch der Fall übrig, dass kein n existiert, dass also die Grössen $x_1, x_2, \ldots$ eine unendliche Reihe von wachsenden Grössen bilden, die unter b liegen. Diese Reihe wäre dann eine Zahlenreihe, deren Zahlzeichen X sei: hervorzuheben ist ihre Eigenschaft, nach der für jedes n die Gleichung besteht, $f(x_{n+1})-f(x_n)=3e$. Nun sei η_0 von der Beschaffenheit, dass $f(X)$ sich von $f(X-\eta)$ um weniger als e unterscheidet, so lange $\eta < \eta_0$. Zwischen die Zahlen $X-\eta_0$ und X mögen von der Zahlenreihe $x_n, x_{n+1}, \ldots$, etc., fallen, so dass $|f(x_{n+1})-f(x_n)|$ kleiner als $3e$ wäre, während andereseits es $3e$ sein müsste. Die zu Grunde liegende Annahme ist daher unmöglich, und die Funktion ist eine gleichmässig continuirliche."

The beginning of this proof contains the germ of a very powerful mode of treating sets of intervals. Heine has, corresponding to each point x of the closed interval (a, b), a set of intervals on the right and on the left of x, with x as common end-point; these are the intervals for which

$$-3e \leqslant f(x+h)-f(x) \leqslant 3e \quad (0 < h)$$

on the right, and $$-3e \leqslant f(x)-f(x-h) \leqslant 3e \quad (0 < h)$$

on the left.

These intervals, which, for fixed x, form a set of potency c, since they include all the intervals with x as end-point inside any one of them,

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* H. Heine, Oct., 1871, *Jour. für reine und angewandte Mathematik*, Vol. LXXIV, p. 188.

Heine replaces by an unique interval on each side of the point x, namely the smallest interval with x as end-point containing all the given intervals on the appropriate side. We may call these intervals r_x and l_x. These intervals have the property that some interval with x as end-point, whose other end-point is outside r_x but as near to it as we please, is not an interval of the given set associated with the point x; and similarly on the left there is such an interval, not associated with x and containing l_x.

Having so reduced our intervals to an unique pair at each point, Heine starts with the left-hand point of the interval, the point a, and takes the corresponding interval r_a. Let x_1 be its right-hand end-point; he then takes r_{x_1}, which abuts at x_1 with r_a, and so on. He points out that, if this process comes to an end after a finite number of stages, that is, if we arrive at a point x_n which is either b or lies to the right of b, the theorem is proved. We have therefore only to discuss the possibility of the alternative hypothesis, that no such number n exists.

Suppose this is the case, then the abutting intervals $r_a, r_{x_1}, r_{x_2}, \ldots$ form an infinite set of abutting intervals, and their end-points $a, x_1, x_2, \ldots$ form a monotone sequence, proceeding towards the right. They therefore define uniquely a limiting point X, to the right of them all.

So far Heine's reasoning has been perfectly general and of a masterly character. At this point, however, he descends to the artifice of a proof *ad hoc*. The real point is hidden. This point consists in the fact that inside any interval whatever with X as right-hand end-point there is an infinite number of the points x_i, say

$$x_n, \; x_{n+1}, \; x_{n+2}, \; \ldots;$$

and therefore there is certainly an interval (x_n, y_n), where y_n lies between x_{n+1} and x_{n+2}, which is *not* an interval of the given set for either of its end-points x_n and y_n. Indeed it has been already pointed out that this is a consequence of the characteristic of the unique interval r_{x_n}.

On the other hand, there is an interval with X as right-hand end-point which is such that every interval inside it is an interval of the given set. This is not in general a property of sets of intervals associated with the points of an interval in the way we have been considering. It is a special property of the set of intervals which Heine was contemplating. It belongs, in fact, to the greatest interval throughout which

$$-e \leqslant f(X) - f(X-h) \leqslant e,$$

in virtue of the fact that

$$f(X-h_n) - f(X-h_{n+1}) = f(X) - f(X-h_{n+1}) - f(X) + f(X-h_n).$$

In fact there was no occasion to take $3e$, Heine might have taken $2e$; he took $3e$ because he was using the special property that the increment over any one of his unique intervals r_x and l_x was exactly $3e$, a property due to the continuity of $f(x)$. The introduction of this special property instead of the general one that as near as we please to the point x_{n+1} there is a point y_n such that the increment of $f(x)$ over (x_n, y_n) is *greater* than $3e$, must, from our present point of view, be regarded as an artistic fault in Heine's proof.

The property which we have assumed in this latter part of Heine's proof is then that there is one of the given intervals on the left of X such that every interval inside it belongs to the given set. A particular consequence of this property is that the interval $r_{x'}$ corresponding to any point x' in this neighbourhood of the point X, reaches at least to X; and this consequence is alone needed in the proof.

3. We are thus led to enunciate the following theorem.[*] The proof of the theorem is taken almost verbatim from Heine, as will be seen by comparison with the above extract. For convenience of reference and classification the theorem, though more general than any contemplated or required by Heine, is therefore called the Heine-Young theorem, and a similar nomenclature is adopted for the simpler corollaries from it.

It should be noticed that the proof of the theorem and of its first corollary is independent of the principle of arbitrary choice, and that it is shown how in an unique manner the set of intervals required is to be constructed.

THEOREM (*The Heine-Young Theorem*).—*If with each point of a closed segment* (a, b) *we have associated a pair*[†] *of intervals* r_x *and* l_x, *such that*

(i) x *is the left-hand end-point of* r_x *and the right-hand end-point of* l_x;

(ii) *if* x' *is an internal point of* l_x, *then* x *is an internal or end-point of* $r_{x'}$;

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[*] This theorem was first enunciated by W. H. Young in his course of lectures on "Integration and the Theory of Sets of Points," 1913, at the University of Liverpool. It is here published for the first time. The second corollary, which is here called the Heine-Lusin theorem, has recently been used explicitly by Lusin—indicating that it may be proved by Lebesgue's device (see below, § 11)—for the purpose of proving the important result that *a continuous function cannot have at every point of a set of positive content an infinite differential coefficient.* Lusin remarks that his theorem is not a special case of the Heine-Borel theorem, even in its generalised form. He does not, however, observe that it is virtually this theorem, rather than the Heine-Borel theorem, which is proved, but not enunciated, by Heine.

[†] In the case of the end-points a and b, the intervals r_b and l_a are not used, and need not exist.

then we can find a finite number of the intervals r_x, *abutting end-to-end, and covering over the whole segment* (a, b).*

By hypothesis there is an interval r_a, with a as left-hand end-point; let its right-hand end-point be x_1. Similarly there is an interval r_{x_1}, with x_1 as left-hand end-point. Thus we proceed; if after a finite number n of these operations, we arrive at $x_n = b$, or if we find that on arriving at x_{n-1} the corresponding interval $r_{x_{n-1}}$ contains b, the theorem is proved.

There remains only the case in which no such integer n exists, so that the points $x_1, x_2, \ldots$ form an infinite monotone sequence each lying to the right of the preceding, and all to the left of b. Let X be the limiting point. There is then an interval l_X with X as right-hand end-point. Inside l_X there will fall all but a finite number of the points of the sequence, since X is the limiting point and the sequence is monotone and on the left of X. Let $x_n, x_{n+1}, \ldots$ lie inside l_X. Then, since x_n is an internal or end-point of l_X, r_{x_n} must reach at least as far as X, by the hypothesis (2). But this is not the case, since it reaches only to x_{n+1}. Thus the original supposition is untenable, and the succession of points $x_1, x_2, \ldots$ cannot be infinite. This proves the theorem.

COR. 1 (*The Heine-Young Theorem for a half-open interval*).—*If the interval* (a, b) *is a half open interval, open on the right, that is, if the conditions of the theorem hold for every point except the point* b, *so that the interval* l_b *does not exist, the theorem is still true if we substitute a countably infinite number of the intervals* r_x *for a finite number of them.*

In fact the reasoning of the above proof shows that the point X can in this case only be the point b.

4. We can now prove the following theorem, due to Lusin, as an immediate corollary.

COR. 2 (*The Heine-Lusin*† *Theorem*).—*If associated with every point of a closed interval* (a, b) *we have all the intervals with* x *as end-point in a certain neighbourhood of the point* x *on both sides*,‡ *then a finite*

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[*] It should be noticed that, though the conditions of the theorem refer to both l_x and r_x, the filling up of the segment is effected by the intervals r_x alone. The existence of the intervals l_x constitutes a restriction on the generality of the set of intervals r_x.

[†] See footnote * on preceding page. N. Lusin, "Sur un théorème fondamentale du calcul intégral," 1911, *Recueil de la Société mathématique de Moscou*, Vol. XXVIII, 2, in Russian.

[‡] As before, the neighbourhoods of a and b may be taken to be only on that side of the point in question which is towards the interval (a, b).

number of these intervals can be found, abutting end-to-end, and covering in the segment (a, b).

In fact we only have to take for r_x (as Heine does), the smallest interval with x as left-hand end-point containing all those of the given intervals which have x for end-point, whether originally associated with x or not, and for l_x the given neighbourhood on the left of x. The conditions of the theorem are then satisfied.

Therefore there is a finite number of the intervals r_x covering up (a, b), and abutting end-to-end. Let the points of division be

$$a,\ x_1,\ \ldots,\ x_{n-1},\ b.$$

We may evidently assume that x_n coincides with b, for we may suppose that none of the given neighbourhoods reach further than b; the intervals on the right associated with b do not enter into the reasoning, and may be non-existent.

We may therefore choose an interval of the given set with x_{n-1} as left-hand end-point reaching either to b or to within the given neighbourhood of b on the left. Thus, if this interval does not reach b, we may choose in addition the interval abutting with it and reaching to b, since this is one of the given intervals associated with the point b. Similarly we proceed in each of the n divisions, and so obtain the required finite set of the given intervals.

It is to be observed that it is only in the latter part of this proof that the principle of arbitrary choice is assumed, and this only a finite number of times.

5. COR. 3 (*The Heine-Borel** *Theorem*).—*If corresponding to each point x of a closed segment (a, b) we are given one or more intervals containing the point x as internal point, then we can find a finite number of these intervals, so that each point of the closed segment (a, b) is an internal point (not an end-point) of one of these intervals.*

In fact if we replace the given intervals corresponding to the point x by all the intervals with x as end-point contained each in one of the given intervals, we shall have the conditions of Cor. 1. Having chosen a finite number of these new intervals in accordance with that corollary, we only need to replace each by one of the original intervals containing it, to obtain that which was required.

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* E. Borel enunciated and proved this theorem in a note at the end of his "Thèse," republished in the *Ann. de l'Ecole Norm.* (3), Vol. XII (1895), pp. 50 *seq.* It has been variously proved since both by the author and others.

This again only requires a finite number of arbitrary choices.

The next corollary shows the power of the original theorem, as more general than any required or tacitly proved by Heine.

It is by means of this Lemma, or Lemmas of the same form, that we are enabled to deal with the question of the integration of derivates, as well as that of the points at which the derivates are infinite, whereas the Heine-Lusin and Heine-Borel theorem are inadequate for this purpose, except in the case when a differential coefficient exists everywhere.

COR. 4 (*Young's First Lemma for a Closed Segment*).—*If with each point*[*] *of a closed interval* (a, b) *as left-hand end-point we are given an interval, or several intervals forming a finite or infinite set, we can find a finite number of these, nowhere overlapping, and such that the content of the complementary intervals is less than any pre-assigned positive quantity* e.

Let G denote the content of the whole set of intervals, and let the integer n be so chosen that

$$2G/n < e. \tag{1}$$

Let us elongate each of the given intervals $(x, x+h)$ on the left by one n-th of its length. Now let us define as r_x the smallest interval with x as left-hand end-point containing all the original intervals which in their elongated form[†] contain x as internal point. Let us define l_x as the interval with x as right-hand end-point of length k_x/n, where k_x denotes the upper bound of the lengths of the original intervals with x as left-hand end-point.

Then, if x is any internal point of l_x,

$$n(x-x') < k_x.$$

Therefore there is one of the original intervals with x as left-hand end-point which in its elongated form reaches beyond the point x'. Thus $r_{x'}$ contains x as internal point.

We may therefore apply the Heine-Young theorem, and obtain a finite number of the intervals r_x, abutting end-to-end and covering up the whole of (a, b).

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[*] An exceptional closed set of content zero clearly makes no difference.

[†] It is hardly necessary to point out that intervals, if any, which originally contained x as *internal* point are not here included; such an interval could, of course, not lie in r_x, whose left-hand end-point is x.

Let the points of division be

$$a,\ x_1,\ x_2,\ \ldots,\ x_{n-2},\ x_{n-1},\ x_n.$$

Then, since x_n is b or lies to the right of it, we can by the definition of $r_{x_{n-1}}$, which is (x_{n-1}, x_n), find one of the original intervals, which, in its elongated form enclosed x_{n-1}, and whose right-hand end-point lies to the right of b or at most at a distance $(x_n - x_{n-1})/n$ to the *left* of b. This interval therefore covers up the whole interval (x_{n-1}, b), excepting perhaps two complementary intervals, whose length is less than $2(x_n - x_{n-1})/n$.

Similarly we choose an interval in each of the n segments between the points $a, x_1, \ldots, x_n$. Thus we get n chosen intervals, nowhere overlapping, and whose complementary intervals have a content less than

$$\frac{2}{n}\left\{(x_1-a)+(x_2-x_1)+\ldots\right\} = \frac{2}{n}(x_n-a),$$

which is less than G/n, that is, less than e. This proves the corollary.

7. It will be noted that in the Heine-Borel theorem we are able to replace the given sets of intervals corresponding to the points x by the set of *all* the completely open intervals containing the point x and lying in a certain neighbourhood of the point x, namely the interval made up of r_x and l_x. Such sets of intervals were considered by W. H. Young.[*] He introduced the terms "tile" and "point of attachment" for such an interval and its corresponding point, and, in the case in which to the point of attachment we have corresponding *all* the intervals in a certain neighbourhood of it, he speaks of a "tile which may be chipped as much as we please." Chipping a tile is, of course, removing a sub-interval containing an end-point and not containing the point of attachment.

The first theorem given by W. H. Young for sets of tiles was originally called an extension of the Heine-Borel theorem;[†] in virtue of the second theorem given by the same author it may be called "The Tile

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[*] "On an Extension of the Heine-Borel Theorem," 1904, *Messenger of Mathematics*, "The Tile Theorem," 1904, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 67–69.

[†] See next footnote.

Theorem for a closed Segment," and is again a corollary from our theorem :—

COR. 5 (*The Tile Theorem for a Closed Segment*).—*Given a set of tiles, which may be chipped as much as we please, and whose points of attachment fill up a closed segment* (a, b), *then we can determine a finite number of the tiles having the following properties :—*

(i) *The length of each tile* d_{P_i} *is less than* e.

(ii) *Each point of* (a, b) *is covered by one or more of the chosen tiles* d_{P_i}.

(iii) *The point of attachment* P_i *of the title* d_{P_i} *is not covered by any other of the chosen tiles, but only by its own tile* d_{P_i}.

(iv) *The sum of the tiles differs from the content of* (a, b) *by less than* e'.

Here e and e' are any assigned positive quantities.

In order to satisfy the condition (i) we omit at once all tiles of length $\geqslant e$.

Let us take for r_x the smallest interval with x as left-hand end-point, containing all those parts of tiles to the right of the point x, such that the tile contains x, and its point of attachment is x or lies to the right of x.

Let l_x be defined as the smallest interval with x as right-hand end-point containing all those parts of tiles to the left of the point x whose point of attachment is x. Then, if x' is internal to l_x, there is a tile with x as point of attachment containing x' : hence, since x lies to the right of x', $r_{x'}$ contains this tile, and therefore contains x.

The conditions of the Heine-Young theorem are thus satisfied.

There is therefore a finite number of the intervals r_x covering up (a, b) and abutting end-to-end. Let the points of division be

$$a, x_1, \ldots, x_{n-1}, x_n.$$

Now, since $r_{x_{n-1}}$ is (x_{n-1}, x_n), and contains b as internal point, or as the end-point x_n, we can find in (x_{n-1}, x_n) a tile containing x_{n-1} and whose point of attachment is either x_{n-1} or lies in the completely open interval (x_{n-1}, x_n) (but cannot coincide with x_n), and which reaches as near as we please to x_n. Thus, if b does not coincide with x_n, this tile covers both x_{n-1} and b. But if b coincides with x_n, we can make this tile reach so near to b that the abutting interval required to reach b is part of one of the tiles associated with b. Thus we get one or two abutting tiles covering x_{n-1} and b.

Again, since $r_{x_{n-2}}$ is (x_{n-2}, x_{n-1}), there is in it a tile whose point of attachment is either x_{n-2} or lies in the completely open interval (x_{n-2}, x_{n-1}), but does not coincide with x_{n-1}, and which reaches so near to x_{n-1} that the abutting interval required to reach x_n is part of the tile first chosen. We may then, by suitably chipping off from the first tile a part to the left of x_{n-1}, and therefore to the left of its point of attachment, ensure that the first tile does not contain the point of attachment of the second tile, and that the common part is less than e'/n.

Similarly we treat the remaining segments in order, and so get a finite number of the given tiles, the sum of whose lengths differs from that of (a, b) by less than e', covering over the whole of (a, b), and satisfying the condition (iii). This proves the theorem.*

8. The theorems which we have been discussing for a closed interval might equally well have been stated for a closed set; the reasoning requires no sensible alteration, only we have on occasion to prove that certain points belong to the set, *e.g.*, the point X in the proof of the first theorem, and we have to remember that instead of abutting intervals, we have intervals which do not overlap, and whose complementary intervals are free of points of the closed set, that is, are the whole or parts of black intervals of the closed set.

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* The proof originally given of the Tile Theorem is imperfect, owing to the fact that, though a tile may be chipped as much as we please, we must not chip off the point of attachment; in consequence the chipping demanded on p. 68, *loc. cit.*, lines 20, 21, for instance, is not always allowable. This remark refers to both the papers quoted. In fact the proof depended in the first instance on the Heine-Borel theorem, which, without previous preparation of the tiles, is certainly insufficient for the purpose in hand. It is only by the use of the more powerful theorem here given that the result has actually been obtained. In fact, if the closed segment be $(0, 1)$ and the given tiles are such that they can be reduced by means of the Heine-Borel theorem to the two tiles $(-\frac{1}{8}, \frac{7}{8})$ with $\frac{3}{4}$ as point of attachment, and $(\frac{1}{8}, \frac{9}{8})$ with $\frac{1}{4}$ as point of attachment, no chipping will cause these tiles to have the property (iii).

The theorem here given (Tile Theorem for a closed Segment) was originally used by W. H. Young to prove that the upper integral of the sum of two upper semi-continuous functions was the sum of their upper integrals. This was thus proved *before* it was shown that these upper integrals are, in fact, the generalised integrals in this case. The general Tile Theorem, a proof of which is given in the present communication, was applied by the same author ("General Theory of Integration," 1905, *Phil. Trans.* (A), Vol. 204, p. 235) to prove that the upper integral of a function with respect to a measurable set S was the lower bound of the corresponding upper summations. The proof there given seems not to utilise the property (iii), given by the Tile Theorem; this is, however, only due to a small error of calculation at the bottom of p. 235, "less than" being used for "greater than"; the correction will afford no difficulty to the careful reader.

[N.B.—D. Mirimanoff has just shown me a very elegant proof of the Tile Theorem founded on the reasoning in the original paper, and involving an ingenious preparation of the tiles. He is going to publish this.—G. E. C. Y. (Written after completion of paper.)]

It is, however, still more clear if we deduce these theorems from those stated for the closed intervals, using the artifice of associating with each point not belonging to the closed set a set of intervals of the same form as that associated in the enunciation with each point of the closed set, but lying in the black interval containing the point in question.

We thus have the following theorems:—

THEOREM.—*If with each point x of a closed set S we have associated a pair*[*] *of intervals r_x and l_x, such that*

(i) *x is the left-hand end-point of r_x and the right-hand end-point of l_x;*

(ii) *if x' is a point of S internal to l_x, then x is an internal or end-point of r_x;*

then we can find a finite number of the intervals r_x, non-overlapping and containing every point of S as internal or end-point.

COR. 1 (*Generalised Heine-Lusin Theorem*[†]).—*If associated with each point of a closed set S we have all the intervals with x as end-point in a certain neighbourhood of the point x on both sides, then a finite number of these intervals can be found, non-overlapping, and containing all the points of the closed set S as internal or end-points.*

COR. 2 (*The Generalised Heine-Borel Theorem*).—*If corresponding to each point x of a closed set S we are given one or more intervals containing the point x as internal point, then we can find a finite number of these intervals, so that each point of the closed set S is an internal point (not an end-point) of one of these intervals.*

COR. 3 (*Young's First Lemma*[‡]).—*If with each point of a closed set S as left-hand end-point we are given an interval, or several intervals, forming a finite or infinite set, we can find a finite number of these nowhere overlapping, and such that the sub-set of S in the complementary intervals can be enclosed in a finite number of intervals whose content is less than any pre-assigned positive quantity e.*

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[*] In the case of the end-points of the black intervals only one of these intervals need exist, viz., that on the contrary side to the black interval, if it is uniquely determined by the point (as will be the case if S is perfect), and when x is an end-point of two black intervals (*i.e.*, x is an isolated point of S), it is immaterial whether it is l_x or r_x which exists.

[†] N. Lusin, *loc. cit.*

[‡] W. H. Young and Grace Chisholm Young, "On the Existence of a Differential Coefficient," 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 325–335.

COR. 4 (*The Tile Theorem for a Closed Set*).—*Given a set of tiles, which may be chipped as much as we please, and whose points of attachment fill up a closed set S, then we can determine a finite number of the tiles having the following properties :—*

(i) *The length of each tile* d_{P_i} *is less than e ;*

(ii) *each point of S is covered by one or more of the chosen tiles* d_{P_i} *;*

(iii) *the point of attachment of any chosen tile is only covered by that tile, and not by any other of the chosen tiles ;*

(iv) *the sum of the tiles differs from the content of S by less than* e' *;*

here e and e' *are any assigned positive quantities.*

9. The theorems hitherto obtained involve, as has been pointed out, the principle of arbitrary choice at most a finite number of times. The following theorems involve it a countably infinite number of times.

THEOREM (*Of the Equivalent Countable Set of Overlapping Intervals*).*—*Given a set of overlapping intervals, we can determine a countable set from among them having the same internal points, the same external points and the same isolated end-points as the given set.*

The points not internal to the given intervals form a closed set, whose black intervals, considered as completely open intervals, consist precisely of the internal points of the given set of overlapping intervals. Since these black intervals are countable, it is only necessary to prove the theorem in each of them separately, and it will follow that the theorem, as stated, is true.

Let then (a, b) be any one of these black intervals and M its middle point. Corresponding to each point x of the completely open interval (a, b), we define the pair of intervals l_x and r_x as follows :—

l_x *is the smallest interval with* x *as right-hand end-point containing the left-hand end-points of all those of the given intervals which contain* x *as internal point ;*

r_x *is defined in like manner, interchanging left and right.*

Now, if x' is a point of l_x, there is one of the given intervals containing both x and x' as internal points ; therefore x is internal to $r_{x'}$.

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* W. H. Young, "Overlapping Intervals," 1902 (not 1903), *Proc. London Math. Soc.*, Ser. 1, Vol. xxxv, pp. 384–386. Subsequent proofs have been supplied by the author and others. The present proof is a fresh one.

Thus we may apply the Heine-Young theorem for a half-open interval (Cor. 1, § 3 *supra*) to the interval (M, b).

Interchanging left and right, we may apply the same theorem to the half-open interval (a, M), using the intervals l_x instead of r_x to cover it up.

We have therefore the whole interval (a, b) covered up by a countably infinite set of the abutting intervals l_x and r_x. But each of these can be covered by *two* of the given intervals; for instance, if (c, d) is an interval r_x, there is an interval of the given set covering c and reaching so near to d that it overlaps with l_d, and therefore with one of the given intervals containing d. This therefore proves the theorem in so far as the internal points are concerned.

That such a set has also the same isolated end-points, external points and semi-external points as the given set is easily seen. Indeed an end-point of a set of intervals is a point which, without being an internal point of the intervals, is the extremity at least on one side of an interval consisting entirely of such points; thus the identity of internal points enforces also the identity of end-points. If the end-point is an isolated one, it is the common extremity of two completely open intervals, one on each side of it, one at least of which consists entirely of internal points of the intervals, while the other either does the same, or contains none of these internal points; a semi-external point, however, has no such interval on one side of the point, while on the other there is the interval required to define the point to be an end-point. In both these cases, therefore, the identity of internal points renders the defining property invariant.

This proves the theorem.

It should be noticed that the construction here given is such that, *in any closed interval containing only internal points of the set there is only a finite number of the given intervals.*

10. The proof of the Tile Theorem in its general form, which also involves the principle of arbitrary choice a countably infinite number of times, will now be given by means of the Heine-Young theorem.

THEOREM (*The Tile Theorem*).—*Given a set of tiles, each of which may be chipped as much as is convenient, whose points of attachment fill up a measurable set S; we can determine a finite or countably infinite number of the tiles, having the following properties:—*

(i) *the linear dimensions of each of the chosen tiles d_{P_i} are less than e;*

(ii) *each point of S is covered by one or more of the chosen tiles;*

(iii) *the point of attachment P_i of the chosen tile d_{P_i} is not covered by any other of the chosen tiles, but only by its own tile d_{P_i};*

(iv) *the sum of the tiles differs from the content of S by less than e';*

here e and e' are any assigned small positive quantities.

By the definition of the content of a measurable set, we can find a set of non-overlapping intervals of content lying between S and $S+\frac{1}{2}e'$, and having all the points of S as internal points. Since the tiles may be chipped, and their points of attachment are all inside these intervals, we may so chip them that they themselves all lie inside these intervals. Whatever set of the tiles we then choose will certainly have content less than $S+\frac{1}{2}e'$. We shall suppose all the tiles chipped so as to satisfy the condition (i). If we now show how to choose the tiles, so chipped, so as to satisfy the condition (iii), it is evident that we can further chip the tiles so that the sum of their overlapping parts, which contain no point of attachment, is less than $\frac{1}{2}e'$; the sum of the chosen tiles will then differ from their content by less than $\frac{1}{2}e'$. If the chosen tiles then contain the whole set S, their content will lie between S and $S+\frac{1}{2}e'$; and therefore their sum will differ from S by less than e', so that the condition (iv) will be satisfied.

It remains therefore only to show how to choose the tiles so as to satisfy the conditions (ii) and (iii).

We may evidently suppose that the tiles are so chipped that their points of attachment are their middle points. Let q_x denote the right-hand half of d_x. The point of attachment of such a half-tile q_x is then its left-hand end-point, and the half-tiles may be chipped on the right as much as we please, but not on the left at all.

These half-tiles for values of x in the set S fill up a set of non-overlapping intervals, whose internal points are the same as those of the half-tiles, and whose left-hand end-points are either points of S or limiting points of S, while the right-hand end-points are certainly not points of S. Let these intervals be arranged in the usual way[*] in order (A_1, B_1), (A_2, B_2), ...

Let (a_1, b_1) be a closed interval inside (A_1, B_1), which, if A_1 is a point of S, is such that a_1 coincides with A_1; for the moment we shall neglect this latter case, but return to it subsequently. With each point x of the closed interval (a_1, b_1) we can associate a pair of intervals l_x and r_x, where

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[*] So that greater intervals precede smaller, and of two equal intervals that comes first which lies more to the left.

l_x is the least interval with x as right-hand end-point containing all the points of attachment of half-tiles containing x as internal point, and r_x is the least interval with x as left-hand end-point such that $l_x + r_x$ contains all those half-tiles. Then, if x' is a point internal to l_x, there is a half-tile containing both x and x', so that x is internal to $r_{x'}$. We may therefore apply the Heine-Young theorem.

We have thus a finite number of the intervals r_x covering every point of the closed interval (a_1, b_1). Let these be $r_1, r_2, \ldots, r_m$, and the points of division be P_1 or a_1, $P_2, P_3, \ldots, P_m$, and P_{m+1} beyond b_1.

Then, by the definition of r_x, every point internal to it may be the right-hand end-point of a half-tile containing its left-hand end-point x.

There is therefore a half-tile q_1 containing P_m and b_1. If Q_1 be the point of attachment of q_1, Q_1 falls between two of the points of division, say P_i and P_{i+1}, or coincides with one of them, say P_{i+1}, where i is less than m.

We then complete this half-tile by adding on a piece on the left so small as not to reach to P_i. We can then, since $P_i P_{i+1}$ is r_{P_i} and contains Q_1, find a half-tile containing P_i and overlapping with the first chosen tile, but so little as not to contain Q_1. We then treat this tile as we did the first chosen tile, and go on a stage further. After at most m stages therefore we shall have obtained at most m tiles, covering over the closed interval (a_1, b_1), and such that none of their points of attachment lie in the parts where they overlap. Let (a_1', b_1') be the interval thus tiled over, containing (a_1, b_1) and contained in (A_1, B_1).

Bisecting the intervals (A_1, a_1') and (b_1', B_1) at a_2 and b_2 respectively, we can in like manner cover over (a_2, a_1) and (b_1, b_2), previously chipping all the tiles whose points of attachment are not internal to the tiles already chosen so as not to contain any of the points of attachment of those chosen tiles.

We thus add on to the tiles first chosen a finite number of tiles so as to cover the larger interval (a_2, b_2), and still to have the required property that the point of attachment of any chosen tile should not be covered by any other of the chosen tiles.

Thus we proceed stage by stage to cover intervals (a_2, b_2), (a_3, b_3), $\ldots$, each lying inside the next following, and approaching (A_1, B_1) as limiting interval. We have thus defined such a countable set of the given tiles as was required, covering every point of the completely open interval (A_1, B_1), and having the required property.

Here we have assumed that A_1 is not a point of S. In the contrary case the argument still holds, the intervals (A_1, a_1), (a_2, a_1), $\ldots$ disappearing altogether. We only have to remark that, in this case the chosen tile containing A_1 must have A_1 for point of attachment, since there was no half-tile containing points internal to $(A_1 B_1)$ whose point of attachment lay outside (A_1, B_1). We may then choose the left-hand end-point of this tile so as not to be internal to any of the intervals $(A_i B_i)$, for A_1, being a point of S cannot be a right-hand end-point of one of the intervals $(A_i B_i)$, so that there is certainly a sequence of points external to the intervals $(A_i B_i)$, and having A_1 for limiting point. We may then omit from consideration all those of the intervals (A_i, B_i) which lie in this tile. Thus, whereas in the former case, when A_1 is not a point of S, we go on to (A_2, B_2), in this latter case we should go on to that (A_i, B_i) which has the lowest index i among those not already omitted.

Proceeding thus in order through the intervals (A_1, B_i) we obtain a countably infinite set of the given tiles covering every point of S, and having the required property (iii). This proves the theorem since the remaining conditions were already satisfied.

11. Lebesgue* in his proof of the Heine-Borel theorem has introduced a very valuable principle, which is itself in embryo in Heine's proof. Heine, as we have seen, replaces all the *given* intervals with a as left-hand end-point by the smallest interval containing them, which is in his case at the same time the largest of these given intervals. Lebesgue replaces all the intervals with a as left-hand end-point *in which the theorem is true* by the smallest interval containing them. In replacing the actual point b by a hypothetical point X, Heine has passed from the consideration of a closed interval (a, b) about which we cannot yet affirm that the theorem is even approximately true, to a hypothetical interval (a, X), open on the right, which is such that in every sufficiently small interval contained in it the oscillation of the function is less than $6e$, that is to say a certain approximation to the theorem is true. Lebesgue's device consists in taking a hypothetical interval (a, X), the smallest interval containing all the intervals with a as left-hand end-point in which the theorem to be proved is itself true. That such an interval, if it does not reach to b, must be open on the right is evident, since otherwise we could extend it on the right, by adding on one of the given intervals associated with the point X. It remains therefore only to show that such an interval (a, X) cannot be open on the right.

This principle of Lebesgue's enables us to prove theorems which would seem to lie beyond the scope of the methods so far used.

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* H. Lebesgue, *Leçons sur l'Intégration*, 1904, p. 105.

We shall apply this principle first to prove Young's Second Lemma, which includes Young's First Lemma as a special case.

YOUNG'S SECOND LEMMA.*—*If with each point of a closed interval* (a, b) *as left-hand end-point, we are given an interval, or several intervals, forming a finite or infinite set, we can find a finite number of the given intervals, nowhere overlapping, and such that the complementary intervals, or gaps, have lengths which are respectively less than different terms of any chosen monotone descending sequence.*

Let r_a denote the smallest interval with a as left-hand end-point containing all the intervals with a as left-hand end-point in which the theorem is true, r_a taking the place of (a, b) in the enunciation. There must be such an interval r_a, because the least interval containing all the given intervals associated with the point a is such an interval in which the theorem is true. Then, if r_a contains b as internal or end-point, the theorem is true in (a, b). If not, let, if possible, a point X, internal to (a, b), be the right-hand end-point of r_a. Then X determines at least one interval D_X of the given set.† If therefore the theorem is true in (a, X), it is true in the larger interval got by appending D_X to r_a. Since r_a, however, contains all such intervals with a as left-hand end-point, this is impossible. Therefore the interval (a, X), if it exists, must be open on the right, that is to say, the theorem is not true in (a, X).

But we see at once that this is impossible. For if y be a point of (a, X) at a distance from X less than e_1, the theorem is true in (a, y). Therefore taking the monotone descending sequence

$$e_2 > e_3 > \dots,$$

each of whose terms is supposed less than e_1, we can find a finite number of intervals from the given set, non-overlapping and such that the remaining parts of (a, y) are respectively less than different terms of this sequence. Since (y, X) is less than e_1, it follows that these chosen intervals cover over the whole of (a, X) except a finite number of gaps which are respectively less than different terms of the monotone descending sequence

$$e_1 > e_2 > e_3 \dots .$$

Since this is any monotone descending sequence, this proves that we were

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[*] "On Derivates and their Primitive Functions," 1912, *Proc. London Math. Soc.*, Ser. 2, Vol. 12, pp. 210–212.

[†] If we wish to be more definite, we can take the smallest interval containing all the intervals D_X. We can in what follows then append, not D_X, but say, half of D^X.

mistaken in supposing that the theorem was not true in (a, X). Thus, by a *reductio ad absurdum* it is proved that the theorem is true.

12. As before, using the artifice of associating with each point not belonging to a certain closed set an interval, or set of intervals, contained in the corresponding black interval of the closed set, we have the usual generalisation of the preceding Lemma.

YOUNG'S SECOND LEMMA IN ITS GENERALISED FORM.—*If with each point of a closed set S as left-hand end-point, we are given an interval, or several intervals forming a finite or infinite set, we can find a finite number of the given intervals, nowhere overlapping, and such that, in the complementary intervals the points of S can be enclosed in a finite number of extra intervals, not overlapping with one another, or with the chosen intervals, and whose lengths are respectively less than different terms of any chosen monotone descending sequence.*

13. It does not seem that this principle of Lebesgue's will aid us in obtaining a proof of Lebesgue's own Lemma, the enunciation of which is as follows :—

LEBESGUE'S LEMMA.—*If with each point of a closed interval (a, b) as left-hand end-point we are given an interval, or several intervals, forming a finite or infinite set, we can find a countable set of the intervals, non-overlapping, and containing every point of the closed interval (a, b) as internal or left-hand end-point.*

Such a set of intervals is said to form *a Lebesgue chain, stretching from a to b.*

In fact, if (a, X) is, as before, the smallest interval containing all the intervals (a, y) in which the theorem is true, it is perfectly true that if y_1 is a point of (a, X), and y_2 another point of (a, X), between y_1 and X, there is a Lebesgue chain stretching from a to y_1, and another from a to y_2, but it does not follow that these two chains have a common end-point, so that we can replace the first chain by a part of it which is also a part of the second chain. If we could make this assertion, we could find a series of chains, each contained in the next, and having X for the limiting point of their right-hand end-points, from which we could conclude that there was a Lebesgue chain stretching from a to X. As this assertion cannot be made, we cannot draw any such conclusion, and the method seems to fail.

If, however, with Pal,[*] we consider only the special case of the Lemma where there is a definite interval connected with each point, and not a set of intervals, we can prove the missing point, and so apply the argument to obtain the required proof without Cantor's numbers.

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[*] J. Pal, "Beweis des Lebesgue-Young'schen Satzes," 1912, *Rend. di Pal.*, Vol. XXXIII, pp. 352, 353.

In fact, from the definition of a Lebesgue chain it follows that, whatever property we may take as a test, there is always a nearest interval of such a chain to a, having the required property; it is in consequence of this that the intervals of the chain form a well-ordered set. For, if not, we could choose out an infinite succession of intervals of the chain having the property, each lying to the left of the preceding. These would accordingly determine a limiting point Y to the left of all of them and not to the left of a. The interval of the chain to which Y is an internal or left-hand end-point must therefore overlap with some of the intervals of the succession, contrary to the property of the chain that its intervals are non-overlapping, Thus the assumption cannot be true, which proves the statement made at the beginning of this paragraph.

Now let L_1 denote the part (or whole) of the Lebesgue chain from a to y_1 got by omitting any of its intervals to the right of that which contains y_1, and let L_2 denote the second chain, that from a to y_2, where y_2 lies between y_1 and X, as in the argument given above. Then, by what has just been proved, there is a nearest interval of L_1 to a which is not an interval of L_2, or else every interval of L_1 is an interval of L_2, in which latter case L_1 is a part of L_2, which, as we have seen, will suffice for the purposes of proof. Suppose then that this latter is not the case, and let Y be the left-hand end-point of the interval of L_1 nearest to a, not belonging to L_2. Then all the intervals of L_1 to the left of Y are intervals of L_2, and therefore are identical with the part of L_2 to the left of Y, since the intervals so characterised cover over every point of (a, Y), and the intervals of L_2 do not overlap. Hence Y cannot be an internal point of any intervals of L_2 to the left of Y, and must therefore be a left-hand end-point of an interval of L_2 not belonging to L_1. But, by the hypothesis that there is only one of the given intervals associated with the point Y, and this interval is an interval of L_1, which is a contradiction. Therefore this assumption is untenable, and L_1 is a part of L_2.

Thus in the special case when there is only one interval associated with each point, and not a set of intervals, it appears that the principle of Lebesgue will afford us a proof of the Lemma without the use of Cantor's numbers. In the applications this simplified form of the Lemma will often suffice, in fact, it suffices to prove the theorems of Lebesgue on the derivates of continuous functions, owing to the fact that the intervals $(x, x+h)$ associated with a point x, being characterised by an equality of such a form as the following

$$[f(x+h)-f(x)]/h \leqslant e,$$

include, by reason of the continuity of $f(x)$, a largest such interval, which may accordingly be taken to be the unique interval associated with the point x. In the more general theorems in the Theory of Functions of a Real Variable of the type of Lebesgue's theorems, however, we have no such unique interval, and we are bound to have recourse to the general form of Lebesgue's Lemma, or to employ other means.

The only proof* extant of Lebesgue's Lemma is that sketched by him on p. 62 of his *Leçons sur l'Intégration* (1902), and written out at length in "Functions of Bounded Variation," by W. H. Young and Grace Chisholm Young, *Quarterly Journal of Mathematics*, Vol. XLII, p. 73 (1910). It depends on Cantor's numbers of the second class, and involves for the purposes of proof those of still higher classes.

On the other hand, Young's Second Lemma, which has shown itself capable of achieving all that has been effected by Lebesgue's Lemma, and has served to prove fresh results, has here been proved† without the introduction of Cantor's numbers, or even of that of transfinite ordinal types.

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* In our paper "On the Existence of a Differential Coefficient" (1910), *Proc. London Math. Soc.*, Ser. 2, Vol. 10, pp. 332 *seq.*, an attempt was made to show the virtual identity of Lebesgue's Lemma with Young's First Lemma, and, in point of fact, the reasoning there given deducing the latter from the former was correct; the proof of the remaining half of the statement seems to contain a flaw, so that a rigid proof of Lebesgue's Lemma without Cantor's numbers is still a desideratum. Lebesgue's Lemma, as stated above, though it involves the distinction of right and left, does not contain the idea of order, nor do transfinite numbers enter into its enunciation. On the other hand, we have shown that all the results which have been obtained by means of it including all Lebesgue's results, can be obtained without the use of transfinite numbers. Indeed although the flaw in question is unfortunately repeated in the proof of Young's second Lemma without transfinite numbers in the paper "Derivates and their Primitive Functions" (1912), *Proc. London Math. Soc.*, Ser. 2, Vol. 12, p. 212, the proof given in the present paper may be substituted for it, so that the main argument of the paper applies equally whether the use of transfinite numbers be allowed or not. It may be well to point out that the original flaw, which occurs in the second paragraph of p. 333 of Vol. 10, is due to the fact that at each stage intervals will have in general to be omitted, as being too large for the gaps in which we are operating. This requires the intervals at each point to have zero as the lower bound of their lengths, a condition which is not demanded in the enunciation on p. 332. Moreover, even if we add this demand to the enunciation the interval $(c, c+h_c)$, referred to on p. 333, would be certain to drop out at some stage. It is moreover evident that, when there is an unique interval associated with each point, the Lebesgue chain from a to b is unique, whereas the intervals obtained as in Young's Lemma may, in general, be chosen in a variety of ways, with end-points not coinciding with those of the chain. This shows the impossibility of deducing Lebesgue's Lemma from Young's.

† See preceding footnote.

ON THE MODIFICATION OF A TRAIN OF WAVES AS IT ADVANCES INTO SHALLOW WATER

By W. BURNSIDE.

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1. In his memoir "On the Motion of Waves in a Variable Canal of small Breadth and Depth" (*Camb. Phil. Trans.*, 1838), Green shewed that if h is the depth of the canal, the height of a wave travelling along it varies as $h^{-\frac{1}{4}}$ and the length as $h^{\frac{1}{2}}$. Incidentally Green's analysis shews that if the depth varies slowly enough, there is practically no reflection due to varying depth.

If it be assumed that, as a train of waves advances from deep into shallow water, the depth varies so slowly that the effect of reflection at the bottom may be neglected, it is evident that the constancy of the time-period and of the rate of transmission of energy are sufficient data from which to determine the change in the height and the length of the waves. The results thus obtained are markedly different from those in the case of "long" waves.

2. If β is the amplitude (*i.e.*, half the height from trough to crest) of a train of waves of length $2\pi/m$, and time-period $2\pi/n$, in water of depth h, the velocity function is

$$\phi = \frac{\beta n}{m} \operatorname{cosech} mh \cosh m(y-h) \cos(mx-nt),$$

where

$$n^2 = gm \tanh mh. \tag{i}$$

The mean rate of transmission of energy, *i.e.*, the mean value of

$$-\rho \int_0^h \frac{\partial\phi}{\partial t} \frac{\partial\phi}{\partial y}\, dy$$

is

$$\frac{g^2\beta^2\rho}{8n} \frac{2mh+\sinh 2mh}{\cosh^2 mh}.$$

In the absence of reflection, the length and amplitude of the waves in water of depth h is given by the two equations

$$\frac{g^2\beta^2\rho}{8n}\,\frac{2mh+\sinh 2mh}{\cosh^2 mh} = \text{constant},$$

$$n^2-gm\tanh mh = 0,$$

where $2\pi/n$ is the constant time-period.

If a is the amplitude in deep water ($h=\infty$), and L the wave-length in deep water,

$$L = \frac{2\pi g}{n^2},$$

and $$\beta^2\,\frac{2mh+\sinh 2mh}{2\cosh^2 mh} = a^2. \qquad \text{(ii)}$$

Now the function $$\frac{2\theta+\sinh 2\theta}{2\cosh^2\theta}$$

is zero when $\theta=0$, and unity when $\theta=\infty$. It has a single maximum when

$$\theta\tanh\theta = 1.$$

The corresponding value of θ, to two significant figures, is

$$\theta = 1.2,$$

and then $$\frac{2\theta+\sinh 2\theta}{2\cosh^2\theta} = 1.2.$$

When $$mh = 1.2, \qquad \tanh mh = .83,$$

and, from (i), $$\frac{2\pi}{m} = .83L.$$

Hence as the wave-length diminishes with diminishing depth, the amplitude begins by diminishing, and reaches a minimum given by

$$\beta = \frac{a}{\sqrt{(1.2)}} = .91a,$$

when the depth has so far diminished that the wave-length is $.83L$.

After this, with diminishing depth the wave-length steadily diminishes, while the amplitude increases.

The solution of the equation

$$\frac{2\theta+\sinh 2\theta}{2\cosh^2\theta} = 1,$$

other than $\theta = \infty$, is $\theta = .64$,

giving $\tanh\theta = .56$.

Hence, when the wave-length has diminished to $.56L$, the amplitude has increased up to the original amplitude. At this stage, the ratio of the depth to the actual wave-length, viz., $mh/2\pi$, is about .1, and the waves are still far from being "long" waves in the ordinary sense of the term.

When $\beta = 2a$, *i.e.*, when the original amplitude is doubled, $\theta = .125$, and the ratio of the depth to the actual wave-length is .02 so that the motion is practically one of "long" waves. Beyond this point Green's formula is certainly applicable.

3. The interest these results have depends on how far the assumption of no reflection is justified. To test this directly in the hydrodynamical problem would probably be difficult. In the somewhat analogous problem of the transverse vibrations of a tense cord, Lord Rayleigh ("On the Reflection of Vibrations at the Confines of Two Media between which the Transition is Gradual," *Proc. London Math. Soc.*, Ser. 1, Vol. XI, pp. 51–56) has shewn there is sensibly no reflection if the change of density of the cord is sufficiently slow. It can hardly be doubted that a similar result is true in the hydrodynamical problem if the depth changes with sufficient slowness.

ON THE DIFFERENTIATION OF A SURFACE INTEGRAL AT A POINT OF INFINITY

By J. G. LEATHEM.

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1. The present paper is a study of a particular case of differentiation of a surface integral taken over a curved surface. The subject of integration is assumed to depend upon the coordinates of a particular point O in the surface and to have an infinity at O, and the differentiation contemplated corresponds to displacement of O in the surface. The resulting formula is applied to some surface integrals which are important in potential theory.

A Theorem on Surface and Contour Integrals.

2. We begin by proving a lemma with regard to surface integrals which may perhaps be described as a theorem in calculus of variations.

Consider a curved surface S and an area of integration upon it bounded by a curve T; it is assumed that the surface has no singular points in the area of integration. Let f be a function of position on the surface.

Consider an infinitesimal correspondence between points on the surface, namely P' corresponding to P. Denote PP' by δs, and let the correspondence depend upon an infinitesimal parameter $\delta\lambda$ in such a way that for each point P the ratio $\delta s/\delta\lambda$ tends, for $\delta\lambda \to 0$, to a definite value, say $ds/d\lambda$, a function of position of P.

For a selected correspondence of this character let the range of P be the area enclosed by T, and let the corresponding range of P' be the area enclosed by a curve T' on the surface, so that T' is the curve corresponding to T. In general the curve T' lies near to T, and the greater part of the area within T is also within T'.

Let dS be an element of area at P, and dS' the corresponding element of area at P'. If f_P be the value of f at P,

$$f_P\,dS = f_P\,(dS/dS')\,dS',$$

and so
$$\int^{T} f\,dS = \int^{T'} f'\,dS',$$

where the first integral is over the area bounded by T, the second is over the area bounded by T', and
$$f'_{P'} = (f\,dS/dS')_{P}.$$

It is clear, however, that if we express $\int^{T} f\,dS$ in terms of P' instead of P we get the same value provided the boundary is properly chosen. In fact
$$\int^{T} f_P\,dS = \int^{T} f_{P'}\,dS',$$

where the notation implies that P' ranges over the area bounded by T; hence
$$\int^{T} f_P\,dS = \int^{T'} f_{P'}\,dS' - \int_{T}^{T'} f_{P'}\,dS',$$

the second integral on the right being through the strip between T' and T, dS' being reckoned positive or negative according as it is outside or inside the curve T.

Thus, putting $f_P + (df/ds)_P\,\delta s$ for $f_{P'}$, and assuming the properties of f to be such that the closeness of this approximation can be ensured to any arbitrary degree of accuracy by taking $\delta\lambda$ sufficiently small, we have, by equating the two different modified expressions for the original integral,
$$\int^{T'}\left(f_P\frac{dS}{dS'}\right)dS' = \int^{T'}\left\{f_P+\left(\frac{df}{ds}\right)_P\delta s\right\}dS' - \int_{T}^{T'}\left\{f_P+\left(\frac{df}{ds}\right)_P\delta s\right\}dS'.$$

If in this we neglect small quantities of higher order than δs, and note that in the strip between T and T' the element dS' may be put equal to $ds\,\delta s\cos(\delta s, \nu)$, where ν is the outward normal to the contour in the tangent plane to the surface and ds is an element of arc of the contour, we get, to an approximation which can be rendered arbitrarily close,
$$\int^{T'}\left\{f_P\left(1-\frac{dS}{dS'}\right)+\left(\frac{df}{ds}\right)_P\delta s\right\}dS' = \int_{T} f_P\,\delta s\cos(\delta s, \nu)\,ds.$$

Dividing by $\delta\lambda$ and noticing that the degree of approximation is unaltered on the left-hand side by taking the element of area as dS and the boundary as T, we get
$$\int^{T}\left\{f\left(\frac{dS'-dS}{dS\,\delta\lambda}\right)+\left(\frac{df}{ds}\right)\frac{\delta s}{\delta\lambda}\right\}dS = \int_{T} f\frac{\delta s}{\delta\lambda}\cos(\delta s, \nu)\,ds,$$

as an equality which is approximate to an arbitrary degree of closeness depending on the value of $\delta\lambda$.

Proceeding to the limit for $\delta\lambda \to 0$, and denoting $\lim\limits_{\delta\lambda\to 0}\left(\frac{dS'-dS}{dS\,\delta\lambda}\right)$ by θ, we get the result

$$\int^T \left(f\theta + \frac{df}{ds}\frac{ds}{d\lambda}\right) dS = \int_T f\frac{ds}{d\lambda}\cos(\delta s, \nu)\, ds. \tag{1}$$

3. The theorem of the preceding article is a somewhat general relation between a surface integral and a line integral taken round the boundary edge of the surface. Its significance may be illustrated by two special cases.

First, let the surface be a plane surface and let the infinitesimal correspondence be a translation of the whole area of integration, as if rigid, in its own plane. Such a correspondence is given by $\delta\lambda = \delta x$, δs equal and parallel to δx, and $\theta = 0$. The theorem then yields the equality

$$\int \frac{\partial f}{\partial x}\, dS = \int f\cos(x, \nu)\, ds,$$

a familiar result.

Second, let the surface be a curved surface on which there is a set of orthogonal coordinates p, q such that the element of arc is given by

$$ds^2 = P^2dp^2 + Q^2dq^2.$$

Let the correspondence consist in an increase of the p coordinate of every point by the same amount δp. Then θ is $\frac{1}{PQ}\frac{\partial}{\partial p}(PQ)$, and the theorem takes the form

$$\int \frac{1}{PQ}\frac{\partial}{\partial p}(PQf)\, dS = \int fP\cos(p, \nu)\, ds, \tag{2}$$

another well known result.

Differentiation of a Surface Integral.

4. Passing now to the differentiation problem, we consider a surface integral taken over the portion of a curved surface S which is bounded by a curve T, and suppose the subject of integration f to be a function not only of the position of the point P at which the element of area dS is

situated but also of the position of a certain point O in the surface. When necessary this may be emphasised by writing the subject of integration f_{OP}.

We suppose O to be in the region enclosed by T, and if the function f has an infinity at O the area of integration must be separated from O by a surrounding cavity ϵ whose dimensions are made to tend to vanishing.

We want to differentiate, with respect to any vanishing displacement of O in the surface, the integral $\int^T f_{OP}\,dS$, that is

$$\operatorname*{Lim}_{\epsilon \to 0} \int_\epsilon^T f_{OP}\,dS.$$

Let us consider a displacement of O to a neighbouring point O', and denote OO' by $\delta\lambda$.

Let us associate with the passage from O to O' an infinitesimal correspondence of the kind discussed in Art. 2, which makes O' correspond to O, P' to P, T' to T, and gives a cavity ϵ' round O' corresponding to ϵ round O.

The incremental ratio of the integral is

$$\frac{1}{\delta\lambda}\left[\operatorname*{Lim}_{\epsilon' \to 0} \int_{\epsilon'}^T f_{O'P}\,dS - \operatorname*{Lim}_{\epsilon \to 0} \int_\epsilon^T f_{OP}\,dS\right],$$

$$= \frac{1}{\delta\lambda}\left[\operatorname*{Lim}_{\epsilon' \to 0} \int_{\epsilon'}^{T'} f_{O'P'}\,dS' - \operatorname*{Lim}_{\epsilon \to 0} \int_\epsilon^T f_{OP}\,dS - \int_T^{T'} f_{O'P'}\,dS'\right],$$

which, if we denote dS'/dS by χ and bear in mind the infinitesimal correspondence, may be written

$$\frac{1}{\delta\lambda}\left[\operatorname*{Lim}_{\epsilon \to 0} \int_\epsilon^T (f_{O'P'}\chi - f_{OP})\,dS - \int_T^{T'} f_{O'P'}\,dS'\right].$$

In the former of the integrals in this expression it is to be noted that $f_{O'P'}$ is regarded as a function of the position of P; it has an infinity for $P \to O$ because then $P' \to O'$, but it has no infinity at O'. If χ is a function free from peculiarities at O (and we shall assume it continuous at all points in the area bounded by T), then the function $f_{O'P'}\chi$ has an infinity at O of exactly the same character as the infinity of f_{OP} at O. Thus the integrals $\int_\epsilon^T f_{O'P'}\chi\,dS$ and $\int_\epsilon^T f_{OP}\,dS$ are subject to the same conditions as to convergence; if the latter is convergent either absolutely or for some particular mode of vanishing of the cavity ϵ, the former also is convergent either absolutely or for the same mode of vanishing of the cavity as the case may be.

Assuming convergence of the integral to be differentiated, we may accordingly close the cavity in the above formula, and we get that the incremental ratio equals

$$\frac{1}{\delta\lambda}\left[\int^{T}(f_{O'P'}\chi-f_{OP})\,dS-\int_{T}^{T'}f_{O'P'}\,dS'\right]. \tag{3}$$

When the nature of the (P, P') correspondence has been settled as well as the particular arc through O upon which O' is to lie, both $f_{O'P'}$ and χ are functions of the position of P which depend on the parameter OO', *i e.*, $\delta\lambda$. Moreover when O' coincides with O, since the correspondence becomes an identity, $f_{O'P'}\chi$ has the value f_{OP}. Hence by the theorem of mean value, assuming it to be applicable,

$$f_{O'P'}\chi_{O'P'}-f_{OP}=\delta\lambda\left[\frac{D}{d\,\delta\lambda}(f\chi)\right]_{O''},$$

the right-hand side having the value corresponding to some point O'' lying between O and O'. The differentiation here denoted by the symbol D is one in which, for a given P, the points O' and P' (or O'' and P'') move simultaneously, varying as the correspondence varies quantitatively but not qualitatively with the variation of the parameter $\delta\lambda$.

Thus the first term of the incremental ratio is

$$\int^{T}\left(\frac{Df}{d\,\delta\lambda}\,\chi+f\,\frac{D\chi}{d\,\delta\lambda}\right)_{O''}dS. \tag{4}$$

Making now a supposition which must be regarded as provisional pending further study, we assume that the function, say $\phi_{O''}$, which constitutes the subject of integration in this formula is such (i) that $\int\phi_{O}\,dS$ is convergent, and (ii) that we can, by choosing $\delta\lambda$ sufficiently small, make $\int(\phi_{O''}-\phi_{O})\,dS$ less than any arbitrarily assigned small quantity, so that

$$\int\phi_{O''}\,dS\rightarrow\int\phi_{O}\,dS \quad \text{as} \quad O''\rightarrow O.$$

Since $(\chi-1)/\delta\lambda\rightarrow\theta$ for $\delta\lambda\rightarrow 0$, $(D\chi/d\,\delta\lambda)_{O}=\theta$; also $\chi_{O}=1$. Substituting these in ϕ_{O}, we see that our provisional conclusion is

$$\operatorname*{Lim}_{\delta\lambda\rightarrow 0}\frac{1}{\delta\lambda}\int^{T}(f_{O'P'}\chi-f_{OP})\,dS=\int\left(\frac{Df}{d\,\delta\lambda}+f\theta\right)_{O}dS. \tag{5}$$

5. Let us now look more closely into the assumptions provisionally made at the conclusion of the previous article. In this connexion two things are of importance, namely the properties of the (P, P') correspondence, and the nature of the function f.

As regards the correspondence we may postulate that it shall be continuous throughout the whole of the area enclosed by T, so that not only is the ratio of δs to $\delta\lambda$ a continuous function, but also the direction of δs varies continuously. It has already been postulated that when P is at O δs is the same as $\delta\lambda$. Hence the continuity of the correspondence involves that, as P approaches O, δs tends to equality and parallelism with $\delta\lambda$ as to a limit. Further we assume that the correspondence is regular, so that the difference between δs and $\delta\lambda$, whether in magnitude or in direction, tends to smallness of the order of OP as $P \rightarrow O$. Obviously this might not be possible if O were a conical point or point of other singularity on the surface, but we have already excluded such points from consideration.

As regards the nature of f, it seems desirable to restrict the manner of its dependence on the position of O. Denoting the coordinates of O by (x_0, y_0, z_0) and those of P by (x, y, z), we shall suppose that f is the product of two factors of which the first is a function of $(x-x_0, y-y_0, z-z_0)$, while the second is a function of (x, y, z). In fact

$$f = g(x-x_0,\ y-y_0,\ z-z_0)\, h(x, y, z).$$

We shall further suppose that $h(x, y, z)$ is continuous at O.

With these postulates before us, let us consider the integral on the right-hand side of equation (5). The function θ has no infinity or other discontinuity at O, and so $\int (f\theta)_0 dS$ is convergent in the same manner as $\int f dS$. The other part of the subject of integration involves the operator $(D/d\delta\lambda)_0$, which for the sake of brevity we may denote by ϑ_0. In the limit for $P \rightarrow O$, ϑ_0 represents a rate of pure translation in the tangent plane at O corresponding to increase of $\delta\lambda$, $\delta\lambda$ itself being zero since O' is at O, as is implied by the suffix $({}_0)$; let this limit operator of pure translation be denoted by t_0. Since $(x-x_0, y-y_0, z-z_0)$ are unaffected by translation, if k be any function of $(x-x_0, y-y_0, z-z_0)$ which is continuous and so free from infinity at O, $t_0 k = 0$; in fact the continuity and regularity of the correspondence make $D(x-x_0)/d\delta\lambda$, etc., small of the order of OP or r, so that $\vartheta_0 k = r t_0' k$, where t_0' is an operator which does not introduce any infinity at O.

As the function g has an infinity at O it is not safe to take for granted that the operation of ϑ_0 upon it necessarily obeys the same law as its operation upon k. Suppose g to be of the form $kr^{-\mu}$, k being as described above and μ being positive. Then

$$\vartheta_0 kr^{-\mu} = r^{-\mu}\vartheta_0 k \quad -\mu kr^{-\mu-1}\vartheta_0 r$$

$$= r^{-\mu+1}t'_0 k - \mu kr^{-\mu}t'_0 r.$$

Hence in this case $\vartheta_0 g$ has just the same kind of infinity at O as has g itself; a similar argument would generally apply to infinities depending on negative powers of $x-x_0$, etc. The occurrence in f of a negative power of the distance of P from the tangent plane at O, though not formally covered by the above argument, could be included by a slight extension, account being taken of the definite curvature of the surface at O.

Now $$\vartheta_0 f = h\vartheta_0 g + g\vartheta_0 h,$$

and as we have supposed h to be continuous at O this shews that $\vartheta_0 f$ has the same kind of infinity at O as has f. Consequently the integral

$$\int \phi_0\, dS \quad \text{or} \quad \int \left(\frac{Df}{d\,\delta\lambda} + f\theta\right)_0 dS$$

has the same convergence as $\int f dS$.

Passing now to $\int \phi_{O''} dS$, we consider first the effect upon f of the operator $(D/d\delta\lambda)_{O''}$ or $\vartheta_{O''}$, bearing in mind that the value of f, though associated for purpose of integration with dS at P, is determined by the positions of O'' and P''. As regards the effect of the operator on P'' we see that for $P'' \rightarrow O''$ (which involves $P \rightarrow O$) the operator tends to a limit operator $t_{O''}$ which represents a rate of pure translation in the tangent plane to the surface at O''. And since $x_{P''}-x_{O''}$, etc., are unaffected by translation, if k be any function of these arguments which is continuous and so free from infinity for $P'' \rightarrow O''$, (which involves, when the function is associated with dS at P, continuity for $P \rightarrow O$), $t_{O''}k = 0$; in fact the continuity and regularity of the correspondence make $\vartheta_{O''}(x_{P''}-x_{O''})$, etc., small of the order of $O''P''$ or r'', *i.e.*, of the order of r, so that

$$\vartheta_{O''} k = rt'_{O''} k$$

where $t'_{O''}$ is an operator which does not introduce any infinity for $P'' \rightarrow O''$.

The function $g_{O''}$ has an infinity for $P'' \rightarrow O''$ and this may be treated by precisely the same reasoning as has been applied to g_O, r'' taking the place of r in the formulæ, and it being borne in mind that r and r'' are of the same order of smallness and tend simultaneously to zero. The conclusion is that $\vartheta_{O''}g_{O''}$ and $g_{O''}$ have, when regarded as functions of the position of P'' the same kind of infinity at O'', and when regarded as functions localised at P the same kind of infinity at O. Consequently $\vartheta_{O''}f_{O''}$ when localised at P has the same kind of infinity at O as $f_{O''}$ or f_O.

The function χ, being continuous, does not introduce any complication into $\phi_{O''}$ as compared with ϕ_O, and so we conclude that

$$\int \phi_{O''}\,dS \quad \text{or} \quad \int \left(\frac{Df}{d\,\delta\lambda}\,\chi + f\,\frac{D\chi}{d\,\delta\lambda}\right)_{O''} dS$$

has the same convergence as $\int f dS$. This conclusion has been arrived at without reference to the particular values of OO'' or OO', and is not invalidated by any changes in these values.

Now the assumptions with regard to $\phi_{O''}$ made at the end of the last article were important for the following reason. If at all points in the area of integration ϕ is a continuous function of $\delta\lambda$ it is generally possible, by taking $\delta\lambda$ sufficiently small, to make $|\phi_{O''} - \phi_O|$ less than an arbitrarily assigned small quantity κ, and therefore also $\left|\int (\phi_{O''} - \phi_O)\,dS\right|$ less than an arbitrarily assigned small quantity κ'. If, however, after a value of $\delta\lambda$ has been fixed, P in the course of integration has to approach a point of infinity, there is a danger that the function $\phi_{O''} - \phi_O$ may become so great as to invalidate an inequality of the type

$$\left|\int (\phi_{O''} - \phi_O)\,dS\right| < \kappa'.$$

But the assumptions and reasoning of the present article have provided against this contingency, for we have shewn that if $\int f dS$ is convergent then both $\int \phi_{O''}dS$ and $\int \phi_O dS$ are convergent, so that necessarily $\int (\phi_{O''} - \phi_O)\,dS$ is convergent. This convergence is independent of $\delta\lambda$ and cannot be modified or nullified by any arbitrary diminution of $\delta\lambda$. Hence the inequality stands, and the formula (5) is justified, provided the transformation and the function f comply with the restrictions set out at the beginning of the present article.

6. The second term of the incremental ratio as set out in formula (3) is a surface integral over the strip between T and T', which strip tends to vanishing as $\delta\lambda \to 0$. Taking the element of area as extending across the strip, so that it is ultimately a parallelogram having for base the element of arc ds of T and for altitude $\delta s \cos(\delta s, \nu)$ where ν is the outward normal to T, we see that

$$\int_T^{T'} f_{O'P'}\, dS' \to \int_T f_{OP}\;\delta s\;\cos(\delta s, \nu)\, ds$$

$$\to \delta\lambda \int_T f_{OP} \frac{ds}{d\lambda} \cos(\delta s, \nu)\, ds.$$

Putting together the two terms of the incremental ratio, dividing by $\delta\lambda$, and proceeding to the limit for $\delta\lambda \to 0$, we get

$$\frac{d}{d\lambda}\int^T f_{OP}\, dS = \int^T (\vartheta_O f_O + f_O\,\theta)\, dS - \int_T f \frac{ds}{d\lambda} \cos(\delta s, \nu)\, ds. \qquad (6)$$

Now let us apply the theorem of Art. 2 to the function f in the area bounded externally by T and internally by a cavity η surrounding the point O. We get

$$\int_T f \frac{ds}{d\lambda} \cos(\delta s, \nu)\, ds - \int_\eta f \frac{ds}{d\lambda} \cos(\delta s, \nu)\, ds = \int_\eta^T \left(\frac{df}{ds}\frac{ds}{d\lambda} + f\theta\right) dS, \qquad (7)$$

where, for the curve η, ν is outward.

Eliminating the line-integral round T from these two equations (6) and (7), and passing to the limit for $\eta \to 0$, we find

$$\frac{d}{d\lambda}\int^T f\, dS = \operatorname*{Lim}_{\eta \to 0} \int_\eta^T \left(\vartheta f - \frac{df}{ds}\frac{ds}{d\lambda}\right) dS - \operatorname*{Lim}_{\eta \to 0} \int f \frac{ds}{d\lambda} \cos(\delta s, \nu)\, ds. \qquad (8)$$

It is to be noted that ϑf is a differentiation of f corresponding to a simultaneous transfer of O to O' and P to P', while $\frac{df}{ds}\frac{ds}{d\lambda}$ is a differentiation corresponding to transfer of P to P' only. Hence $\vartheta f - \frac{df}{ds}\frac{ds}{d\lambda}$ is the differentiation which corresponds to displacement of O to O' leaving P undisturbed. Call this $\partial f/\partial\lambda$. Also we note that, by the continuity of the transformation, $ds/d\lambda \to 1$ for points on η as $\eta \to 0$. So our final formula for the differentiation is

$$\frac{d}{d\lambda}\int^T f\, dS = \operatorname*{Lim}_{\eta \to 0} \left\{ \int_\eta^T \frac{\partial f}{\partial\lambda}\, dS - \int_\eta f \cos(\delta\lambda, \nu)\, ds \right\}. \qquad (9)$$

This is the general result, but we must not fail to note that in passing from equations (6) and (7) to equation (8) it has been tacitly assumed that $\int (\vartheta f + f\theta)\, dS$ may be replaced by $\lim\limits_{\eta \to 0} \int_\eta (\vartheta f + f\theta)\, dS$. This is clearly legitimate without restriction upon the form of the cavity η if $\int f\, dS$, and therefore also $\int (\vartheta f + f\theta)\, dS$ are absolutely convergent. But if $\int f\, dS$ is semi-convergent, so that it can only be rendered definite by specifying a particular kind of vanishing cavity ϵ, the argument of Art. 5 implies that the same cavity must be associated with $\int (\vartheta f + f\theta)\, dS$; consequently in this case it is necessary in formula (9) to make η the same as ϵ.

Illustrations in Potential Theory.

7. A simple illustration of the use of the differentiation formula which has been obtained above is afforded by the potential and force integrals of a surface concentration of gravitating matter of surface-density σ. We put $f = \sigma r^{-1}$, noting that σ is a function of the position of P only while r^{-1} is a function of the coordinates of P relative to O. The potential V is $\int \sigma r^{-1} dS$, an absolutely convergent integral, and we differentiate for a displacement of O typified by $d\lambda$. The result is

$$\frac{dV}{d\lambda} = \lim_{\eta \to 0} \left\{ \int_\eta^T \sigma \frac{\partial}{\partial \lambda}\left(\frac{1}{r}\right) dS - \int_\eta \frac{\sigma}{r} \cos(\delta\lambda, \nu)\, ds \right\}.$$

Denoting the force-component at O, when surrounded by the cavity η, by $F(\eta)$, we see that our result is

$$\frac{dV}{d\lambda} = \lim_{\eta \to 0} \left\{ F(\eta) - \int_\eta \frac{\sigma}{r} \cos(\delta\lambda, \nu)\, ds \right\}. \qquad (10)$$

The force-integral is semi-convergent, and the difference between the potential gradient $dV/d\lambda$ and the force-component $\operatorname{Lim} F(\eta)$ is

$$-\lim_{\eta \to 0} \int \sigma r^{-1} \cos(\delta\lambda, \nu)\, ds,$$

which is generally the same as

$$-\sigma_0 \lim_{\eta \to 0} \int r^{-1} \cos(\delta\lambda, \nu)\, ds.$$

The value of this difference clearly depends upon the form of the vanishing cavity η. It is zero if the cavity vanishes as a circle with O as centre, or as any curve having symmetry about the line through O perpendicular to $d\lambda$ in the tangent plane.

8. A second illustration is afforded by the potential integral of a double sheet of strength τ. The main facts as to the convergence and discontinuity of this integral are well known, but before proceeding to the differentiation formula it seems worth while to give a fresh discussion of these, in the hope that the method employed may be found an interesting alternative to the methods in use elsewhere.

The cosines of the normal on the positive side of the double sheet at a point P, (x, y, z), being denoted by (l, m, n), the potential at a point O, not necessarily in the sheet, is

$$V = \int \tau \left(l \frac{\partial}{\partial x} + m \frac{\partial}{\partial y} + n \frac{\partial}{\partial z} \right) \frac{1}{r}\, dS,$$

r being OP. Now $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r}$, considered as a vector localised at P, is a solenoidal vector and therefore is the curl of some other vector or family of vectors. Such a vector is easily specified when we think of grad $(1/r)$ as corresponding hydrodynamically to symmetrical flow from the point O. We take O as the origin of a set of spherical polar coordinates (r, θ, ϕ), and we try a vector of magnitude Ω perpendicular to the plane of the angle θ, in the direction of ϕ increasing. The flow through the circle ($r =$ const., $\theta =$ const.) is $-2\pi(1-\cos\theta)$, and this must equal the line-integral of Ω, that is $2\pi r \sin\theta\,\Omega$; so

$$\Omega = -\frac{1}{r}\,\frac{1-\cos\theta}{\sin\theta} = -\frac{1}{r}\tan\tfrac{1}{2}\theta. \tag{11}$$

Having guessed this vector Ω it is easy to verify that it really has the property

$$\operatorname{curl}\Omega = \operatorname{grad}(1/r). \tag{12}$$

We note that Ω has an infinity for $\theta = \pi$, independent of r, as well as the infinity $r = 0$.

Considering a region of the surface S bounded by a curve T and not containing any infinity of Ω, assuming the existence of orthogonal curvilinear coordinates p, q as in Art. 3, and indicating the components of Ω tangential to the curve T and the curves of increase of p and q by suffixes t, p, q respectively, we apply the second theorem of Art. 3, and so get

$$\int_T \tau\Omega_t ds = \int \tau \left\{ -\Omega_p \cos(q, \nu) + \Omega_q \cos(p, \nu) \right\} ds$$

$$= \int \frac{1}{PQ} \left\{ \frac{\partial}{\partial p}(\tau Q\Omega_q) - \frac{\partial}{\partial q}(\tau P\Omega_p) \right\} dS$$

$$= \int \tau(\text{curl } \Omega)_N \, dS + \int \left(\Omega_q \frac{1}{P}\frac{\partial \tau}{\partial p} - \Omega_p \frac{1}{Q}\frac{\partial \tau}{\partial q} \right) dS,$$

where the suffix N refers to the normal to the surface. The first integral of the last formula equals $\int \tau(\text{grad } 1/r)_N \, dS$, and so is the potential at O of the part of the double sheet bounded by T. The second has for subject of integration the normal component of the vector product $[\text{grad } \tau, \Omega]$.

It will be convenient to regard the boundary T as made up of two curves, an outer edge which we shall henceforth call T and an inner boundary ϵ surrounding any point at which Ω has an infinity. Our formula now takes the form

$$V = \int_T \tau\Omega_t \, ds - \int_\epsilon \tau\Omega_t \, ds - \int_\epsilon^T [\text{grad } \tau, \Omega]_N \, dS, \tag{13}$$

the two line-integrations being taken in the same sense, namely that corresponding to increase of ϕ.

If O is not in the surface the only infinities of Ω occur at points where the negative direction of the axis of spherical polar coordinates chosen for the origin O cuts the surface, for at these points $\theta = \pi$. Let Q be such a point, ϵ the cavity round it, and P a point in the region of integration near to Q. At P it is clear that Ω is great of the order of $1/QP$, so that the surface integral converges absolutely as $\epsilon \to 0$. Also, from the original definition of Ω, $\int_\epsilon \Omega_t ds$ equals all of the symmetrical flux from O except that part which passes through ϵ, and so tends to -4π whatever be the manner of vanishing of ϵ; consequently

$$\int_\epsilon \tau\Omega_t ds \to \tau_Q \int \Omega_t ds \to -4\pi\tau_Q,$$

τ_Q being reckoned positive when in the sense from Q to O. Thus, if O is not in the surface

$$V = 4\pi\Sigma\tau_Q + \int_T \tau\Omega_t ds - \int^T [\text{grad } \tau, \Omega]_N \, dS. \tag{14}$$

If O is in the surface (at an ordinary point of the surface), $\theta = \frac{1}{2}\pi$, and Ω tends to infinity of the order $1/r$. The surface integral is therefore absolutely convergent. The integral $\int_\epsilon \Omega_t ds$ tends to equal half the total flux from O, that is -2π, whatever the shape of ϵ, and so

$$\int_\epsilon \tau\Omega\, ds \to -2\pi\tau_0.$$

Thus V has a definite value, wherever O may be, but has a discontinuity at the surface. For, as O approaches the surface in a direction making an acute angle with the negative sense of the axis of spherical polar coordinates for O, it gets nearer to a point Q which supplies a term $4\pi\tau_Q$ in V. When O is at Q this term disappears, being replaced by $2\pi\tau_Q$: and when O crosses the surface the term $2\pi\tau_Q$ disappears also; and during these changes all the other parts of V vary continuously.

9. Passing now to the differentiation of V, we notice that f or $\tau(l\partial/\partial x + m\partial/\partial y + n\partial/\partial z)(1/r)$, though not as it stands the product of two factors of the g and h types, is however the sum of three terms each of which separately is such a product. Hence the theorem of Art. 6 is applicable and we have, using the notation of Art. 7,

$$\frac{dV}{d\lambda} = \operatorname*{Lim}_{\eta\to 0} \left\{ F(\eta) - \int_\eta \tau \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) \cos(\delta\lambda, \nu)\, ds \right\}. \quad (15)$$

The force-integral is semi-convergent, and the potential gradient $dV/d\lambda$ differs from the force by

$$-\operatorname*{Lim}_{\eta\to 0} \int_\eta \tau\Sigma \left(l\frac{\partial}{\partial x}\right)\left(\frac{1}{r}\right) \cos(\delta\lambda, \nu)\, ds,$$

which is the same as

$$-\tau_0 \int_\eta \Sigma \left(l\frac{\partial}{\partial x}\right)\left(\frac{1}{r}\right) \cos(\delta\lambda, \nu)\, ds,$$

or, if we put ζ for the distance of a point P from the tangent plane at O,

$$\tau_0 \int_\eta \zeta r^{-2} \cos(\delta\lambda, \nu)\, ds. \quad (16)$$

Projecting on the tangent plane at O, taking as axes the principal directions of curvature corresponding to principal radii ρ_1 and ρ_2, putting $(\cos\theta, \sin\theta)$ for the cosines of OP, (L, M) for the cosines of $\delta\lambda$, and (l', m') for the cosines of the normal to the projection η' of η, the difference between potential gradient and force takes the form

$$\tfrac{1}{2}\tau_0 \int_\eta \left(\frac{\cos^2\theta}{\rho_1} + \frac{\sin^2\theta}{\rho_2}\right)(Ll' + Mm')\, ds. \quad (17)$$

This is different for different vanishing forms of η. If η' be circular, with O as centre, $l' = \cos\theta$, $m' = \sin\theta$, and the line integral has zero limit.

THEOREMS RELATING TO FUNCTIONS DEFINED IMPLICITLY, WITH APPLICATIONS TO THE CALCULUS OF VARIATIONS

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IT is well known that, if $F(u, v)$ be a function of two real variables, which, together with its partial differential coefficients, is continuous in a neighbourhood of the point $u = \alpha$, $v = \beta$, and if $\partial F/\partial u$ is not zero at this point, then for any sufficiently small positive number δ, another positive number ϵ can be so determined, that there exists one and only one value of u, such that $|u-\alpha| < \delta$, corresponding to each value of v such that $|v-\beta| < \epsilon$, so that these corresponding values of u and v satisfy the equation $F(u, v) = 0$. Moreover this value of u is expressible in the form $\phi(v)$, where $\phi(v)$ has a continuous differential coefficient in the neighbourhood $|v-\beta| < \epsilon$, of β. This theorem* was established by Cauchy for the case of a function that is analytic in the neighbourhood of the point (α, β), and was freed from this restriction by Dini.

The theorem has been extended to the case in which u is replaced by n variables $u_1, u_2, \ldots, u_n$, and v by m variables $v_1, v_2, \ldots, v_m$; the function $F(u, v)$ being replaced by n functions

$$F_r(u_1, u_2, \ldots, u_n;\ v_1, v_2, \ldots, v_m) \quad (r = 1, 2, 3, \ldots, n).$$

In this case the condition

$$\frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u_1, u_2, \ldots, u_n)} \neq 0,$$

at the point
$$u_1 = \alpha_1,\ u_2 = \alpha_2,\ \ldots,\ u_n = \alpha_n;$$
$$v_1 = \beta_1,\ v_2 = \beta_2,\ \ldots,\ v_m = \beta_m,$$

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* For proofs of the theorem see W. H. Young's tract "The Fundamental Theorems of the Differential Calculus," or Osgood's "Lehrbuch der Funktionentheorie," Vol. I, pp. 47–65.

corresponds to the condition $\partial F/\partial u \neq 0$, for the case $n=1$, $m=1$. Two numbers δ, ϵ can then be so determined that for each set of values of $v_1, v_2, \ldots, v_m$ such that

$$|v_s-\beta_s| < \epsilon \quad (s=1, 2, \ldots, m),$$

there exists one and only one set of values of $u_1, u_2, \ldots, u_n$, such that

$$|u_r-\alpha_r| < \delta \quad (r=1, 2, \ldots, n),$$

so that these corresponding sets of values satisfy all the equations $F_r=0$. These values of $u_1, u_2, \ldots, u_n$ are expressible as functions of $v_1, v_2, \ldots, v_m$, which have continuous partial differential coefficients with respect to all the variables. This theorem was also established by Dini.

There are two directions in which this general theorem can be extended, and in which such extensions are requisite with a view to applications in the Calculus of Variations, connected with proofs of the existence of Weierstrassian fields of extremals.

In the first place, it may be supposed that the functions F_r involve, besides the two sets of variables u, v, a set of parameters $\xi_1, \xi_2, \ldots, \xi_p$, where the values of these parameters are such that the point $(\xi_1, \xi_2, \ldots, \xi_p)$ belongs to some closed domain in the p-dimensional (ξ) space. It is then possible to shew that, subject to certain conditions, one and the same set of values of δ and ϵ will be applicable whatever the values of $\xi_1, \xi_2, \ldots, \xi_p$ may be, so long as the corresponding point is in the given closed domain.

Secondly, it may be supposed that the point $(\alpha_1, \alpha_2, \ldots, \alpha_n; \beta_1, \beta_2, \ldots, \beta_n)$ is no longer fixed, but may be any point of a certain closed domain in the $n+m$ dimensional space of these variables. In this case also, it can be shewn that, subject to certain conditions, the numbers δ, ϵ can be so fixed that the theorem remains valid for all the different positions of the points (α, β).

Lastly, both these modes of extension may be combined, and a very general theorem, investigated in the present communication, is obtained, which applies to the case when the functions F_r contain p parameters, and when the point (α, β) is variable in the manner prescribed.

Extensions of the kind indicated have been developed by Bolza,[*] and by Hadamard,[†] and have been applied by them in the Calculus of Variations.

In the present paper, these extensions have been treated in a manner which leads to greater generality in the results, and in some respects to simplification in the proofs.

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[*] See his *Vorlesungen über Variationsrechnung*, pp. 154–168.

[†] See his *Leçons sur le Calcul des Variations*, Vol. I, pp. 497–502.

Properties of Sets of Points in a Manifold.

1. Let $(a_1, a_2, \ldots, a_n)$ be a point in an n-dimensional manifold, and let h be a fixed positive number; a point $(x_1, x_2, \ldots, x_n)$ of the manifold is said to be in the *incomplete neighbourhood* (h) of $(a_1, a_2, \ldots, a_n)$, if

$$|x_1-a_1|, \quad |x_2-a_2|, \quad \ldots, \quad |x_n-a_n|$$

are all less than h; and it is said to be in the *complete neighbourhood* (h) if

$$|x_1-a_1|, \quad |x_2-a_2|, \quad \ldots, \quad |x_n-a_n|$$

are all $\leqslant h$. The set of all points in an incomplete, or in a complete, neighbourhood (h) is said to form the incomplete or the complete, neighbourhood (h) of the point. Such a set may also be spoken of as forming an open or closed cell (h) with centre at $(a_1, a_2, \ldots, a_n)$.

If G be a given set of points in the n-dimensional manifold, the set consisting of all those points each of which is in the incomplete neighbourhood (h) of at least one point of G is said to form the incomplete neighbourhood (h) of the set G.

Similarly, the set of all those points each of which is in the complete neighbourhood (h) of at least one point of G is said to form the complete neighbourhood (h) of the set G.

Every point of the incomplete neighbourhood (h), denoted by H, of a given set G, is an interior point of H. For any point P of the set H is in the interior of a cell, all the interior points of which belong to H; hence a neighbourbood $(h', < h)$ of P can be determined so that all its points belong to H, and thus P is an interior point of H.

(*a*) *The complete neighbourhood* (h), *denoted by* $\bar{H}$, *of a limited set* G, *is a closed set, if the set* G *is a closed set.*

For, if $Q_1, Q_2, Q_3, \ldots$ be a sequence of points of $\bar{H}$ which converges to the limiting point Q, and if $(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$ be the point Q_m, and $(x_1, x_2, \ldots, x_n)$ the point Q, there exists a point $P_m\,(\xi_1^{(m)}, \xi_2^{(m)}, \ldots, \xi_n^{(m)})$ of G, such that $|x_r^{(m)}-\xi_r^m| \leqslant h$ for all the values $1, 2, 3, \ldots, n$ of r. The set of points $\{P_m\}$ has at least one limiting point $P\,(\xi_1, \xi_2, \ldots, \xi_n)$, which is necessarily a point of G, since G is closed. A set $m_1, m_2, \ldots, m_s, \ldots$ of increasing integers can be so chosen that the sequence $P_{m_1}, P_{m_2}, \ldots, P_{m_s}, \ldots$ converges to P as its sole limiting point.

Since $$x_r^{(m_s)}-h \leqslant \xi_r^{(m_s)} \leqslant x_r^{(m_s)}+h,$$

and since this holds for every value of s, we have in the limit

$$x_r - h \leqslant \xi_r \leqslant x_r + h;$$

and since this holds for all the values of r, it follows that $Q(x_1, x_2, \ldots, x_n)$ is in the complete neighbourhood (h) of the point $P(\xi_1, \ldots, \xi_n)$ which belongs to G. It follows that the set $\bar{H}$ is closed.

It is well known that if G_1, G_2 be two closed sets of points which have no point in common, the distance of a pair of points

$$(x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}), \qquad (x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}),$$

defined by $$\left\{ \sum_{s=1}^{s=n} (x_s^{(1)} - x_s^{(2)})^2 \right\}^{\frac{1}{2}},$$

has a minimum $d\,(>0)$, for all possible pairs of points of the two sets, and that this minimum d is attained by at least one pair of points. The following theorem follows immediately:—

(*b*) *If G_1, G_2 be two limited closed sets of points which have no point in common, a neighbourhood of G_1 can be determined which contains no points of G_2.*

Let d be the minimum distance of pairs of points of G_1 and G_2. The complete, or the incomplete, neighbourhood (h), where $h < d/\sqrt{n}$, of G_1 contains no points of G_2; for the distance of a point of G_2 from any point of G_1 being $\geqslant d$, such a point of G_2 cannot be in the neighbourhood (h) of any point of G_1.

The following theorem can be immediately deduced:—

(*c*) *If* the limited closed set G is in the interior of a set H, a neighbourhood (h) of G, complete or incomplete, can be so determined that it is interior to H.*

The points of G have a maximum distance ρ from the point $(0, 0, \ldots, 0)$. Choose a number $\rho' > \rho$, and let H_1 be the part of H that consists of points at a distance not exceeding ρ' from the point $(0, 0, \ldots, 0)$. It is clear that G is interior to H_1.

Let L be the set of points on the boundary of H_1; it is necessarily a closed set. A neighbourhood (h) of G can be determined which, whether complete or incomplete, contains no points of L. Every point of this neighbourhood (h) is an interior point of H_1, and therefore of H. For let

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* This theorem and Theorem (*d*) have been established by Bolza in a different manner. *Loc. cit.*, pp. 155–158.

Q be such a point; then it is in the neighbourhood (h) of some point P belonging to G. If the neighbourhood (h) of P does not consist entirely of interior points of H_1, it must contain one or more points of the boundary of H_1, *i.e.*, of the set L; and this is not the case; therefore Q is an interior point of H_1.

(*d*) *If* $f(x_1, x_2, \ldots, x_n)$ *is a function that is continuous in a given set* H, *and is either* (1) *positive, or* (2) *different from zero, at every point of a limited closed set* G *that consists of interior points of* H, *then a neighbourhood, either complete or incomplete, of* G *can be so determined that the function has the same property* (1) *or* (2), *in this neighbourhood as in* G.

Let a closed neighbourhood K of G be determined, that consists of interior points of H. This set K is closed. Those points of K (if any), for which, in case (1), $f \leqslant 0$, or, in case (2), $f = 0$, form a closed set k, as follows from the continuity of the function in K. The two closed sets G, k have no point in common, and therefore a neighbourhood of G can be so determined as to contain no points of k; it can also be determined so as to be interior to K, and therefore to H. In this neighbourhood the condition (1) or (2) is satisfied.

(*e*) *If* $f(x_1, x_2, \ldots, x_n)$ *is a function that is continuous in a given set* K, *and if it have the value zero at all points of a limited closed set* G *interior to* K, *a neighbourhood of* G *can be determined, interior to* K *such that at every point of that neighbourhood* $|f(x_1, x_2, \ldots, x_n)|$ *is less than an assigned positive number* ϵ.

Let a complete neighbourhood H of G be determined, which is interior to K. The points of H (if any) at which $|f(x_1, x_2, \ldots, x_n)| \geqslant \epsilon$ form a closed set L, which has no points in common with G. Determine a neighbourhood of G which contains no points of L and is interior to H; at every point of this neighbourhood the condition $|f(x_1, x_2, \ldots, x_n)| < \epsilon$ is satisfied, and it is interior to H and therefore to K.

General Theorem relating to Implicit Functions.

2. *Let the* p *variables* $\xi_1, \xi_2, \ldots, \xi_p$ *have the systems of values corresponding to points in a* p*-dimensional closed domain* X_p, *and let the* $n+m$ *variables* $u_1, u_2, \ldots, u_n$; $v_1, v_2, \ldots, v_m$ *have the systems of values corresponding to points in an* $n+m$*-dimensional closed domain* D_{n+m}; *the* $n+m+p$ *variables having accordingly the systems of values corresponding*

to points in an $(n+m+p)$*-dimensional domain* D_{n+m+p}. *Let it be further assumed that all the points of* D_{n+m+p} *are interior points of a domain* D'_{n+m+p} *for which n functions*

$$F_r(u_1, u_2, \ldots, u_n;\ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p) \quad (r = 1, 2, 3, \ldots, n)$$

are defined, and which satisfy the following conditions :—

(1) *The functions* F_r *are all continuous in* D'_{n+m+p}, *and have all the* $n(n+m+p)$ *partial differential coefficients* $\partial F/\partial u$, $\partial F/\partial v$, $\partial F/\partial \xi$ *continuous in that domain.*

(2) *At a fixed point* $(\alpha_1, \alpha_2, \ldots, \alpha_n;\ \beta_1, \beta_2, \ldots, \beta_m)$ *interior to* D_{n+m}, *the conditions* $F_1 = 0$, $F_2 = 0$, ..., $F_n = 0$ *are all satisfied, for all points* $(\xi_1, \xi_2, \ldots, \xi_p)$ *in* X_p.

(3) *The Jacobian* $\dfrac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u_1, u_2, \ldots, u_n)}$ *is different from zero at the point* $(\alpha_1, \alpha_2, \ldots, \alpha_n;\ \beta_1, \beta_2, \ldots, \beta_m)$ *of* D_{n+m}, *for all points* $(\xi_1, \xi_2, \ldots, \xi_p)$ *of* X_p.

It then follows that a neighbourhood (δ) *of the point* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, *and a neighbourhood* (ϵ) *of the point* $(\beta_1, \beta_2, \ldots, \beta_m)$ *can be so determined that, for each point* $(\xi_1, \xi_2, \ldots, \xi_p)$ *in* X_p, *there exists a uniquely determined point* $(u_1, u_2, \ldots, u_n)$ *interior to the neighbourhood* (δ) *of* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *corresponding to each point* $(v_1, v_2, \ldots, v_m)$ *interior to the neighbourhood* (ϵ) *of* $(\beta_1, \beta_2, \ldots, \beta_m)$, *such that the equations*

$$F_r(u_1, u_2, \ldots, u_n;\ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p) = 0 \quad (r = 1, 2, \ldots, n)$$

are all satisfied. The values of the variables u corresponding to given values of the variables v depend in general on the values of $\xi_1, \xi_2, \ldots, \xi_p$.

Moreover, if these values of $u_1, u_2, \ldots, u_n$ *be denoted by*

$$u_r = \phi_r(v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p) \quad (r = 1, 2, \ldots, n),$$

the functions ϕ_r *are all continuous and have continuous partial differential coefficients with respect to the* $m+p$ *variables* v, ξ. *For* δ *any sufficiently small number can be chosen, and then* ϵ *can be chosen.*

First, the theorem will be proved in the case $n = 1$.

In this case we have a single function $F(u;\ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p)$ that has the value zero when $u = \alpha$, $v_1 = \beta_1$, $v_2 = \beta_2$, ..., $v_n = \beta_m$, whatever point $(\xi_1, \xi_2, \ldots, \xi_p)$ may be of X_p. Moreover, for the same system of values $\alpha, \beta_1, \ldots, \beta_m, \xi_1, \xi_2, \ldots, \xi_p$, the Jacobian $\partial F/\partial u$ is always different from zero, and since it is continuous in X_p it is everywhere of the same sign; let it be assumed that it is positive. In accordance with the

Theorem (*d*) of § 1, a neighbourhood (h) of the set of points in D_{1+m+p} given by $u=\alpha$, $v_1=\beta_1$, $v_2=\beta_2$, ..., $v_m=\beta_m$, $(\xi_1, \xi_2, \ldots, \xi_p)$ in X_p, can be determined such that, for all points in that neighbourhood, $\partial F/\partial u$ is positive. Thus for

$$\alpha-h<u<\alpha+h, \quad \beta_s-h<v_s<\beta+h \quad (s=1, 2, \ldots, m),$$

$(\xi_1, \xi_2, \ldots, \xi_p)$ in X_p, $\partial F/\partial u$ is positive. If $v_1, v_2, \ldots, v_m$ have the values $\beta_1, \beta_2, \ldots, \beta_m$, since $\partial F/\partial u>0$, as u is increased from $\alpha-h$ to $\alpha+h$, F is increased, and therefore, since it is zero when $u=\alpha$, it is negative when $\alpha-h \leqslant u<\alpha$, and it is positive when $\alpha<u \leqslant \alpha+h$. This holds for every point of X_p.

A neighbourhood of the $m+p$-dimensional set of points $(\beta_1, \beta_2, \ldots, \beta_m; \xi_1, \xi_2, \ldots, \xi_p)$ at which, for $u=\alpha-h$, F is negative, can be so determined that F is negative in that neighbourhood. Thus, for a properly chosen positive number k_1 (which can be taken not to exceed h),

$$F(\alpha-h;\ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p)<0,$$

for

$$\beta_s-k_1<v_s<\beta_s+k_1 \quad (s=1, 2, \ldots, m).$$

Similarly, a positive number k_2 (not exceeding h) can be so chosen that

$$F(\alpha+h;\ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p)>0,$$

for

$$\beta_s-k_2<v_s<\beta_s+k_2 \quad (s=1, 2, \ldots, m).$$

If k be the smaller of the two numbers k_1, k_2, we see that, for

$$\beta_s-k<v_s<\beta_s+k \quad (s=1, 2, \ldots, m),$$

$F(u;\ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p)$ is negative when $u=\alpha-h$, and is positive when $u=\alpha+h$; whatever point $(\xi_1, \xi_2, \ldots, \xi_p)$ may be of X_p. Since $\partial F/\partial u>0$, for all points such that

$$\alpha-h<u<\alpha+h, \quad \beta_s-k<v_s<\beta_s+k \quad (s=1, 2, \ldots, m),$$

$(\xi_1, \xi_2, \ldots, \xi_p)$ in X_p, we see that, for each point of X_p, there is one point u, and only one, such that $|u-\alpha|<h$, at which F vanishes, for each set of values of $v_1, v_2, \ldots, v_m$.

It has now been shewn that, for h fixed and sufficiently small, there is one value of u, and only one, such that $|u-\alpha|<h$, corresponding to each set of values of $v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p$ that satisfy the conditions

$$|v_s-\beta_s|<k \quad (s=1, 2, \ldots, m),$$

$(\xi_1, \xi_2, \ldots, \xi_p)$ in X_p, which is such that

$$F(u;\ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p)=0.$$

It should be remarked that when h and k have been chosen, we can then choose any smaller value of h, provided k be if necessary suitably diminished.

In accordance with the Theorem (c) of § 1, a complete and closed neighbourhood (η) of D_{1+m+p} can be so determined as to be interior to D'_{1+m+p}.

The difference ΔF of the values of the function F at any two points

$$(u\,;\ v_1,\, v_2,\, \ldots,\, v_m\,;\ \xi_1,\, \xi_2,\, \ldots,\, \xi_p),$$

$$(u+\Delta u\,;\ v_1+\Delta v_1,\, \ldots,\, v_m+\Delta v_m\,;\ \xi_1+\Delta\xi_1,\, \ldots,\, \xi_p+\Delta\xi_p),$$

of the domain D_{1+m+p}, and such that

$$|\Delta u|,\ |\Delta v_1|,\ \ldots,\ |\Delta v_m|,\ |\Delta\xi_1|,\ \ldots,\ |\Delta\xi_p|,$$

are all $\leqslant \eta$, is given by

$$\Delta F = \left(\frac{\partial F}{\partial u}+\rho\right)\Delta u + \sum_{s=1}^{s=m}\left(\frac{\partial F}{\partial v_s}+\sigma_s\right)\Delta v_s + \sum_{r=1}^{r=p}\left(\frac{\partial F}{\partial \xi_r}+\tau_r\right)\Delta\xi_r,$$

where all the expressions ρ, σ_s, τ_r denote continuous functions that converge to zero as Δu, Δv_1, Δv_2, ..., Δv_m, $\Delta\xi_1$, $\Delta\xi_2$, ..., $\Delta\xi_p$ do so, uniformly in D_{1+m+p}.

When $$u = \phi\,(v_1,\, v_2,\, \ldots,\, v_m,\ \xi_1,\, \xi_2,\, \ldots,\, \xi_p),$$

$$u+\Delta u = \phi(v_1+\Delta v_1,\, v_2+\Delta v_2,\, \ldots,\, \xi_1+\Delta\xi_1,\, \ldots,\, \xi_p+\Delta\xi_p),$$

we have $$\Delta F = 0.$$

Thus, if $\Delta\xi_1$, $\Delta\xi_2$, ..., $\Delta\xi_p$ all vanish, and Δv_1, ..., Δv_{s-1}, Δv_{s+1}, ..., Δv_m also vanish, we have

$$\frac{\Delta u}{\Delta v_s} = -\frac{\dfrac{\partial F}{\partial v_s}+\sigma_s}{\dfrac{\partial F}{\partial u}+\rho},$$

which converges uniformly to $$-\frac{\dfrac{\partial F}{\partial v_s}}{\dfrac{\partial F}{\partial u}},$$

since σ_s, ρ converge uniformly to zero.

Thus $$\frac{\partial\phi}{\partial v_s} = -\frac{\partial F}{\partial v_s}\Big/\frac{\partial F}{\partial u}\,;$$

and in a similar manner we see that

$$\frac{\partial \phi}{\partial \xi_r} = -\frac{\partial F}{\partial \xi_r}\Big/\frac{\partial F}{\partial u}.$$

The theorem has now been established in the case $n = 1$.

It will next be proved that, if the theorem holds for $n = \nu-1$, it also holds for $n = \nu$.

As before, if $$\frac{\partial(F_1, F_2, \ldots, F_\nu)}{\partial(u_1, u_2, \ldots, u_\nu)} \gtrless 0,$$

when $$u_1 = a_1,\ u_2 = a_2,\ \ldots,\ u_\nu = a_\nu,$$
$$v_1 = \beta_1,\ v_2 = \beta_2,\ \ldots,\ v_m = \beta_m,$$

for all points $(\xi_1, \xi_2, \ldots, \xi_p)$ of X_p, there exists a positive number h, such that the same inequality holds, provided

$$|u_r - a_r| < h \quad (r = 1, 2, \ldots, \nu), \qquad |v_s - \beta_s| < h \quad (s = 1, 2, \ldots, m),$$

$(\xi_1, \xi_2, \ldots, \xi_p)$ in X_p.

At a point at which the Jacobian is not zero it is impossible that all the differential coefficients $\frac{\partial F_1}{\partial u_1}, \frac{\partial F_1}{\partial u_2}, \ldots, \frac{\partial F_1}{\partial u_\nu}$ can vanish. Choosing a number $h' < h$, at each point of the closed domain E given by

$$|u_r - a_r| \leqslant h' \quad (r = 1, 2, \ldots, \nu), \qquad |v_s - \beta_s| \leqslant h' \quad (s = 1, 2, \ldots, m),$$

$(\xi_1, \xi_2, \ldots, \xi_p)$ in X_p, one at least of the above partial differential coefficients is different from zero. For any point of E a complete neighbourhood of the point can be determined in the whole of which one and the same differential coefficient is different from zero. Such a neighbourhood exists for each point of the closed set E. By employing the Heine-Borel theorem, we see that a finite number of the cells that constitute these neighbourhoods can be so selected that each point of E is interior to at least one of the cells of this finite set. We obtain in this manner a division of the domain E into a finite number of parts, such that in each part one of the differential coefficients $\frac{\partial F_1}{\partial u_1}, \frac{\partial F_1}{\partial u_2}, \ldots, \frac{\partial F_1}{\partial u_\nu}$ is different from zero in the whole of that part. Let us assume that, in one of these parts, $\partial F_1/\partial u_1 > 0$, and that this part contains the points

$$(a_1, a_2, \ldots, a_\nu;\ \beta_1, \beta_2, \ldots, \beta_m;\ \xi_1, \xi_2, \ldots, \xi_p)$$

in its interior, provided that $(\xi_1, \ldots, \xi_p)$ belongs to a certain closed part of K_p which we may denote by K_p^0.

We now consider

$$F_1(u_1, u_2, \ldots, u_\nu;\ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p)$$

as a function of $u_1, v_1, v_2, \ldots, v_m$, and of the $p+\nu-1$ parameters $\xi_1, \xi_2, \ldots, \xi_p$, $u_2, u_3, \ldots, u_\nu$. Applying now the result that has been established for the case $n=1$, we see that positive numbers h_1, $k_1 (\leqslant h')$ can be so determined that one and only one value of u_1 such that $|u_1-a_1| < h_1$ exists, corresponding to each set of values of $v_1, v_2, \ldots, v_m$; $u_2, u_3, \ldots, u_\nu$; $\xi_1, \xi_2, \ldots, \xi_p$, such that

$$|v_s-\beta_s| < k_1 \quad (s = 1, 2, \ldots, m),$$

and that $(u_2, u_3, \ldots, u_\nu;\ \xi_1, \xi_2, \ldots, \xi_p)$ is in a certain closed domain of $\nu-1+p$ dimensions. This value of u is given by

$$u_1 = \psi_1(u_2, u_3, \ldots, u_\nu;\ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p),$$

where ψ_1 has continuous partial differential coefficients with respect to all the variables contained in it. Thus

$$\frac{\partial \psi_1}{\partial u_r} = -\frac{\partial F_1}{\partial u_r}\bigg/\frac{\partial F_1}{\partial u_1} \quad (r = 2, 3, \ldots, \nu),$$

$$\frac{\partial \psi_1}{\partial v_s} = -\frac{\partial F_1}{\partial v_s}\bigg/\frac{\partial F_1}{\partial u_1} \quad (s = 1, 2, \ldots, m),$$

$$\frac{\partial \psi_1}{\partial \xi_q} = -\frac{\partial F_1}{\partial \xi_q}\bigg/\frac{\partial F_1}{\partial u_1} \quad (q = 1, 2, 3, \ldots, p).$$

On substitution of this value of u_1 in the equations

$$F_2 = 0,\ F_3 = 0,\ \ldots,\ F_\nu = 0;$$

we obtain a set of $\nu-1$ equations which may be denoted by

$$\Phi_r(u_2, u_3, \ldots, u_\nu;\ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p) = 0,$$

where $r = 2, 3, \ldots, \nu$. These hold when $|v_s-\beta_s| < k_1$, and $(u_2, u_3, \ldots, u_\nu;\ \xi_1, \xi_2, \ldots, \xi_p)$ is in the specified closed domain.

Assuming the theorem to hold for $n = \nu-1$, we see that there exist values of $u_2, u_3, \ldots, u_\nu$, given by equations

$$u_r = \phi_r(v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p) \quad (r = 2, 3, \ldots, \nu);$$

and that these values are unique for every set of values of $v_1, v_2, \ldots, v_m$; $\xi_1, \xi_2, \ldots, \xi_p$, such that

$$|v_s-\beta_s| < k_2 \quad (s = 1, 2, \ldots, m),$$

and such that $(\xi_1, \xi_2, \ldots, \xi_p)$ is in X_p^0; where k_2 is some fixed positive number. The values of $u_2, u_3, \ldots, u_\nu$ are such that $|u_r - a_r| < h_3$, some fixed positive number; this must be so small that the points $(u_2, u_3, \ldots, u_\nu; \xi_1, \xi_2, \ldots, \xi_p)$ are all in the specified closed domain. This holds subject to the condition that

$$\frac{\partial(\Phi_2, \Phi_3, \ldots, \Phi_\nu)}{\partial(u_2, u_3, \ldots, u_\nu)} \neq 0,$$

at the points $(\alpha_2, \alpha_3, \ldots, \alpha_\nu; \beta_1, \beta_2, \ldots, \beta_m; \xi_1, \xi_2, \ldots, \xi_p)$, $(\xi_1, \xi_2, \ldots, \xi_p)$ in X_p^0.

Now $$\frac{\partial\Phi_r}{\partial u_s} = \frac{\partial F_r}{\partial u_s} + \frac{\partial F_r}{\partial u_1}\left(-\frac{\partial F_1}{\partial u_s}\bigg/\frac{\partial F_1}{\partial u_1}\right).$$

On substituting the values of the partial differential coefficients in the Jacobian, we obtain, after a slight transformation of the determinant,

$$\frac{\partial(F_1, F_2, \ldots, F_\nu)}{\partial(u_1, u_2, \ldots, u_\nu)}\bigg/\frac{\partial F_1}{\partial u_1};$$

and thus the condition is that

$$\frac{\partial(F_1, F_2, \ldots, F_\nu)}{\partial(u_1, u_2, \ldots, u_\nu)} \neq 0,$$

at the points $(\alpha_1, \alpha_2, \ldots, \alpha_\nu; \beta_1, \beta_2, \ldots, \beta_m; \xi_1, \xi_2, \ldots, \xi_p)$ where $(\xi_1, \xi_2, \ldots, \xi_p)$ is in X_p^0.

It has now been shewn that there is a unique value of u_1 such that $|u_1 - \alpha_1| < h_1$ for each set of values of $u_2, u_3, \ldots, u_\nu; v_1, v_2, \ldots, v_m; \xi_1, \xi_2, \ldots, \xi_p$ such that $(u_2, u_3, \ldots, u_\nu; \xi_1, \xi_2, \ldots, \xi_p)$ is in a certain closed domain which includes a set of points for which

$$u_2 = \alpha_2, \quad u_3 = \alpha_3, \quad \ldots, \quad u_\nu = \alpha_\nu,$$

and that $$|v_s - \beta_s| < k_1.$$

It has also been shewn that a unique set of values of $u_2, u_3, \ldots, u_\nu$ such that

$$|u_r - \alpha_r| < h_2 \quad (r = 2, 3, \ldots, \nu)$$

can then be determined for each set of values of $v_1, v_2, \ldots, v_m; \xi_1, \xi_2, \ldots, \xi_p$, such that $|v_s - \beta_s| < k_2$, $(\xi_1, \xi_2, \ldots, \xi_p)$ in K_p^0. We can choose any value of h_1 or of h_2 smaller than the value fixed, then k_1 or k_2 will in general have to be diminished. By altering the value of one of the numbers h_1, h_2, we may make them have equal values, say $\bar{h}$, and for $\bar{k}$ we then take the smaller of the corresponding values k_1, k_2. It now appears that there exists a unique set of values of $u_1, u_2, \ldots, u_\nu$, such that

$$|u_r - \alpha_r| < \bar{h} \quad (r = 1, 2, \ldots, \nu),$$

corresponding to each point $(v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p)$, such that

$$|v_s - \beta_s| < \bar{k},$$

and that $(\xi_1, \xi_2, \ldots, \xi_p)$ is in X_p^0.

As the whole set X_p can be divided into a finite number of such sets as X_p^0, corresponding to each of which one of the differential coefficients $\partial F_1/\partial u_r$ is different from zero for the whole of that part, the above reasoning may be applied to each such part X_p^0.

We have in each case a pair of positive numbers $\bar{h}$, $\bar{k}$ determined as above; moreover we may take instead of $\bar{h}$ any number less than $\bar{h}$; $\bar{k}$ being correspondingly diminished. If we take for δ the smallest of the numbers $\bar{h}$, it is clear that a corresponding number ϵ may be determined so that δ and ϵ can be used instead of $\bar{h}$, $\bar{k}$ for all the parts of X_p^0. It has thus been shewn that the theorem holds in the case $n = \nu$ if it is assumed to hold in the case $n = \nu - 1$. It has been proved to hold in the case $n = 1$, and therefore it holds for all values of n.

Some remarks may be made as to the latitude of choice of the two numbers δ, ϵ corresponding to the given point $(\alpha_1, \alpha_2, \ldots, \alpha_n;\ \beta_1, \beta_2, \ldots, \beta_m)$. If δ have a fixed value, ϵ may have any value between 0 and a certain maximum number. If ϵ have a sufficiently small fixed value, there exist two fixed numbers $M(\delta_\epsilon)$, $m(\delta_\epsilon)$ such that δ may have any value such that

$$M(\delta_\epsilon) \geqslant \delta \geqslant m(\delta_\epsilon).$$

The number $M(\delta_\epsilon)$ is determined by the condition that there are either (1) one or more points $(u_1, u_2, \ldots;\ v_1, v_2, \ldots)$ on the boundary of the cell defined by

$$|u_r - \alpha_r| \leqslant M(\delta_\epsilon), \quad |v_s - \beta_s| < \epsilon,$$

which belong to the boundary of D'_{n+m}, or (2) that in this cell, including the boundary, there is more than one set of values of $u_1, u_2, \ldots, u_n$ corresponding to a single set of values of $v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p$, such that $|v_s - \beta_s| < \epsilon$, $(\xi_1, \xi_2, \ldots)$ in X_p. It is clear that, if $\delta \leqslant M(\delta_\epsilon)$, there are no two points $(u_1, u_2, \ldots)$ corresponding to one set of values of $(v_1, v_2, \ldots, v_n;\ \xi_1, \xi_2, \ldots, \xi_p)$, such that $|v_s - \beta_s| < \epsilon$, ξ in X_p, in the interior of the neighbourhood (δ) of $(u_1, u_2, \ldots, u_n)$.

The number $m(\delta_\epsilon)$ is determined by the condition that one or more points corresponding to points $(v_1, v_2, \ldots;\ \xi_1, \xi_2, \ldots, \xi_p)$, such that $|v_s - \beta_s| < \epsilon$, ξ in X_p, lie on the boundary of the neighbourhood $m(\delta_\epsilon)$ of the point $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

It is clear that if δ has a suitable value corresponding to ϵ, it is also suitable when ϵ has a value $\epsilon' < \epsilon$; thus

$$M(\delta_{\epsilon'}) \geqslant M(\delta_\epsilon), \quad \text{and} \quad m(\delta_{\epsilon'}) \leqslant m(\delta_\epsilon).$$

As the number ϵ increases, $M(\delta_\epsilon)$ diminishes or remains constant, and $m(\delta_\epsilon)$ increases or remains constant, until ϵ becomes so large that

$$M(\delta_\epsilon) = m(\delta_\epsilon),$$

or until a boundary is reached.

3. It is easy to see that the general theorem may be modified so as to apply to the establishment of the existence of unique values of $u_1, u_2, \ldots, u_n$, which satisfy the n equations

$$F_r(u_1, u_2, \ldots, u_n;\ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p) = f_r(\xi_1, \xi_2, \ldots, \xi_p),$$

where the functions $f_r(\xi_1, \xi_2, \ldots, \xi_p)$ satisfy the conditions that they are continuous in and have continuous partial differential coefficients in a domain which contains in its interior the closed domain X_p; the functions F_r are to satisfy the same conditions of continuity and of possession of continuous partial differential coefficients as before. We have, in fact, only to consider the n functions $F_r - f_r$ instead of F_r. It is then, as before, assumed that $F_r - f_r = 0$, when $u_r = a_r$, $v_s = \beta_s$, for all values of $(\xi_1, \xi_2, \ldots, \xi_p)$ in X_p.

A very important case of the general theorem is that in which $m = n$, and in which F_r is of the form

$$\phi_r(u_1, u_2, \ldots, u_n;\ \xi_1, \xi_2, \ldots, \xi_p) - v_r.$$

We have then an extension of the theorem of inversion.

The case of the general theorem which arises when $\xi_1, \xi_2, \ldots, \xi_p$ are absent is the well known theorem of Dini relating to implicit functions.*

If we have equations of the form

$$Y_r = \chi_r(u_1, u_2, \ldots, u_n;\ \xi_1, \xi_2, \ldots, \xi_p) - \chi_r(a_1, a_2, \ldots, a_n;\ \xi_1, \xi_2, \ldots, \xi_p),$$

where
$$y_r = \chi_r(u_1, u_2, \ldots, u_n;\ \xi_1, \xi_2, \ldots, \xi_n),$$

we see that there is for each set of values of $(\xi_1, \xi_2, \ldots, \xi_p)$ in X_p a single set of values of $\{u_r\}$ such that $|u_r - a_r| < \delta$, corresponding to each set of values of Y_r, such that $|Y_r| < \epsilon$, *i.e.*, corresponding to each set of values of y_r, such that y_r is between the two values

$$\chi_r(a_1, a_2, \ldots, a_n;\ \xi_1, \xi_2, \ldots, \xi_p) \pm \epsilon.$$

In particular there exists no set of values of $\{u_r\}$ such that

$$|u_r - a_r| < \delta,$$

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* See Jordan's *Cours d'Analyse*, Vol. I, §§ 91, 92; also Osgood's *Lehrbuch der Funktionentheorie*, Vol. I, pp. 47–57.

except the set (a_r) which is such that

$$\chi_r(u_1, u_2, \ldots, u_n;\ \xi_1, \xi_2, \ldots, \xi_p) = \chi_r(a_1, a_2, \ldots, a_n;\ \xi_1, \xi_2, \ldots, \xi_n),$$

for all the values of r; and this holds for each set of values of $\xi_1, \xi_2, \ldots, \xi_p$ in the set X_p.

The Existence of a Field of Extremals.

4. This last result is of importance in relation to the theorem of the existence of a field of extremals, in the functional problem of the Calculus of Variations.

Let us consider the case $p = 1$; and write x for ξ_1. A set of extremals for a variational problem in which there are n dependent variables $y_1, y_2, \ldots, y_n$, is given by a set of equations

$$y_r = \chi_r(u_1, u_2, \ldots, u_n;\ x) \quad (r = 1, 2, 3, \ldots, n);$$

where $u_1, u_2, \ldots, u_n$ are n parameters.

We assume that in an interval (X_1, X_2) of x including the interval (x_1, x_2) in its interior, the condition

$$\frac{\partial(\chi_1, \chi_2, \ldots, \chi_n)}{\partial(u_1, u_2, \ldots, u_n)} \neq 0,$$

is satisfied along the arc of the particular extremal for which

$$u_1 = \overset{0}{u}_1, \quad u_2 = \overset{0}{u}_2, \quad \ldots, \quad u_n = \overset{0}{u}_n,$$

joining the points for which x has the values x_1, x_2.

We see then that, corresponding to each set of values of $\{y_r\}$, such that y_r lies between

$$\chi_r(\overset{0}{u}_1, \overset{0}{u}_2, \ldots, \overset{0}{u}_n, x) + \epsilon \quad \text{and} \quad \chi_r(\overset{0}{u}_1, \overset{0}{u}_2, \ldots, \overset{0}{u}_n, x) - \epsilon,$$

there exists one set and only one of the parameters $\{u_r\}$, such that

$$| u_r - u_r^0 | < \delta,$$

and so as to satisfy the equations

$$y_r = \chi_r(u_1, u_2, \ldots, u_n, x)$$

for all values of x in the interval (x_1, x_2).

Thus the arc of the extremal $(\overset{0}{u}_1, \overset{0}{u}_2, \ldots, \overset{0}{u}_n)$ joining the points corresponding to $x = x_1$, and $x = x_2$ is surrounded by a neighbourhood given by the sets of values

$$y_r = \chi_r(u_1^0, u_2^0, \ldots, u_n^0, x) \pm \epsilon \quad (r = 1, 2, 3, \ldots, n),$$

such that through each point in this neighbourhood there passes one and only one of the extremals of the given set such that

$$| u_r - u_r^0 | < \delta \quad (r = 1, 2, 3, \ldots, n).$$

Moreover the values of $u_1, u_2, \ldots, u_n$ so determined as functions of $y_1, y_2, \ldots, y_n, x$, have continuous partial differential coefficients with respect to these variables. Thus the field exists, in accordance with the definition of a Weierstrassian field of extremals.

The Existence of a Quasi-Field of Extremals passing through a Fixed Point.

5. In the case in which all the extremals pass through a fixed point for which $x = x_1$, the above proof of the existence of a field fails, because the Jacobian $\dfrac{\partial(\chi_1, \chi_2, \ldots, \chi_n)}{\partial(u_1^0, u_2^0, \ldots, u_n^0)}$ is zero when $x = x_1$. We assume that the Jacobian does not vanish when $x_1 < x \leqslant x_2$.

Let
$$V_r = \frac{\chi_r(u_1, u_2, \ldots, u_n, x) - \chi_r(u_1^0, u_2^0, \ldots, u_n^0, x)}{x - x_1},$$

when $x \neq x_1$. It will be shewn that V_r has a definite limit when x converges to x_1, for each set of values of $u_1, u_2, \ldots, u_n$. The value of V_r at a point $(u_1, u_2, \ldots, u_n, x_1)$ will be defined to be equal to this limit.

When $x \neq x_1$, we have

$$V_r = \frac{\chi_r(u_1, u_2, \ldots, u_n, x) - \chi_r(u_1. u_2, \ldots, u_n, x_1)}{x - x_1}$$
$$- \frac{\chi_r(u_1^0, u_2^0, \ldots, u_n^0, x) - \chi_r(u_1^0, u_2^0, \ldots, u_n^0, x_1)}{x - x_1},$$

since
$$\chi_r(u_1, u_2, \ldots, u_n, x_1) = \chi_r(u_1^0, u_2^0, \ldots, u_n^0, x_1).$$

It follows that

$$V_r = \frac{\partial}{\partial x'} \chi_r(u_1, u_2, \ldots, u_n, x') - \frac{\partial}{\partial x''} \chi_r(u_1^0, u_2^0, \ldots, u_n^0, x''),$$

where x' and x'' both lie between x_1 and x.

It is now clear that V_r tends to the limit

$$\frac{\partial}{\partial x}\chi_r(u_1, u_2, \ldots, u_n, x_1) - \frac{\partial}{\partial x}\chi_r(\overset{0}{u}_1, \overset{0}{u}_2, \ldots, u_n, x_1),$$

which is taken as the value of V_r at $(u_1, u_2, \ldots, u_n, x_1)$.

To shew that $\partial V_r/\partial x$ exists and is continuous at $x = x_1$, we have

$$V_r(x) = \frac{\partial}{\partial x}\chi_r(u_1, u_2, \ldots, u_n, x_1) + \tfrac{1}{2}(x - x_1)\frac{\partial^2}{\partial x^2}\chi_r(u_1, u_2, \ldots, u_n, \xi)$$

$$- \frac{\partial}{\partial x}\chi_r(\overset{0}{u}_1, \overset{0}{u}_2, \ldots, \overset{0}{u}_n, x_1) - \tfrac{1}{2}(x - x_1)\frac{\partial^2}{\partial x^2}\chi_r(\overset{0}{u}_1, \overset{0}{u}_2, \ldots, \overset{0}{u}_n, \xi'),$$

where ξ and ξ' lie between x_1 and x. The existence and continuity of the second differential coefficients is here assumed.

We have now

$$\lim_{x \sim x_1}\frac{V_r(x) - V_r(x_1)}{x - x_1} = \tfrac{1}{2}\frac{\partial^2}{\partial x^2}\chi_r(u_1, u_2, \ldots, u_n, x_1) - \tfrac{1}{2}\frac{\partial^2}{\partial x^2}\chi_r(\overset{0}{u}_1, \overset{0}{u}_2, \ldots, \overset{0}{u}_n, x_1).$$

Thus $\partial V_r/\partial x$, when $x = x_1$, exists.

When $x \neq x_1$,

$$\frac{\partial V_r}{\partial x} = \frac{1}{x - x_1}\left\{\frac{\partial}{\partial x}\chi_r(u_1, u_2, \ldots, u_n, x) - \frac{\partial}{\partial x}\chi_r(\overset{0}{u}_1, \overset{0}{u}_2, \ldots, \overset{0}{u}_n, x)\right\}$$

$$- \frac{1}{(x - x_1)^2}\left\{\chi_r(u_1, u_2, \ldots, u_n, x) - \chi_r(\overset{0}{u}_1, \overset{0}{u}_2, \ldots, \overset{0}{u}_n, x)\right\}$$

$$= \tfrac{1}{2}\frac{\partial^2}{\partial x^2}\chi_r(u_1, u_2, \ldots, u_n, \xi) - \tfrac{1}{2}\frac{\partial^2}{\partial x^2}\chi_r(\overset{0}{u}_1, \overset{0}{u}_2, \ldots, \overset{0}{u}_n, \xi')$$

$$- \frac{\partial^2}{\partial x^2}\chi_r(u_1, u_2, \ldots, u_n, \xi''') + \frac{\partial^2}{\partial x^2}\chi_r(\overset{0}{u}_1, \overset{0}{u}_2, \ldots, \overset{0}{u}_n, \xi^{\text{iv}}),$$

where ξ, ξ', ξ''', ξ^{iv} are between x_1 and x. This value of $\partial V_r/\partial x$ clearly converges, as x tends to x_1, to the value of $\partial V_r/\partial x$ at x_1. The value of $\dfrac{\partial V_r}{\partial u_s}$ is $\dfrac{\partial \chi_r}{\partial u_s}\Big/(x - x_1)$, when $x \neq x_1$; we thus have

$$\lim_{x \sim x_1}\frac{\partial V_r}{\partial u_s} = \frac{\partial^2 \chi_r}{\partial x\, \partial u_s}\ (x = x_1),$$

it being assumed that this partial differential coefficient exists and is continuous. Since at $x = x_1$,

$$V_r = \frac{\partial}{\partial x}\chi_r(u_1, u_2, \ldots, u_n, x_1) - \frac{\partial}{\partial x}\chi_r(\overset{0}{u_1}, \overset{0}{u_2}, \ldots, \overset{0}{u_n}, x_1),$$

the value of $\dfrac{\partial V_r}{\partial u_s}$ is $\dfrac{\partial^2\chi_r}{\partial x\,\partial u_s}$; hence $\dfrac{\partial V_r}{\partial u_s}$ is continuous at $(u_1, u_2, \ldots, u_n, x_1)$.

The Jacobian of $V_1, V_2, \ldots, V_n$ with respect to $u_1, u_2, \ldots, u_n$ is $\dfrac{1}{x-x_1}$ times the Jacobian of $\chi_1, \chi_2, \ldots, \chi_n$, when $x \neq x_1$, and is therefore different from zero, for $u = u_1^0$, $u = u_2^0$, $\ldots$, $u_n = u_n^0$.

When $x = x_1$, this Jacobian of $V_1, V_2, \ldots, V_n$ is

$$\frac{\partial\left(\dfrac{\partial\chi_1}{\partial x}, \dfrac{\partial\chi_2}{\partial x}, \ldots, \dfrac{\partial\chi_n}{\partial x}\right)}{\partial(u_1\, u_2, \ldots, u_n)};$$

it will be shewn that this does not vanish at the points $(u_1^0, u_2^0, \ldots, u_n^0, x_1)$.

It has now been shewn that, subject to the assumptions made, the conditions of application of the theorem of § 3 to the functions $V_1, V_2, \ldots, V_n$ are all satisfied.

In order to shew that the Jacobian of $\partial\chi_1/\partial x$, $\partial\chi_2/\partial x$, $\ldots$, $\partial\chi_n/\partial x$, with respect to $u_1, u_2, \ldots, u_n$ does not vanish when $x = x_1$, $u_1 = u_1^0$, $u_2 = u_2^0$, $\ldots$, we select a family of the extremals obtained by letting $u_1, u_2, \ldots, u_n$ be functions $\mu_1(a)$, $\mu_2(a)$, $\ldots$, $\mu_n(a)$ of one parameter and which has the value a^0, when $u_1 = u_1^0$, $u_2 = u_2^0$, $\ldots$, these functions having continuous differential coefficients. The extremals of this family are thus given by n equations

$$y_r = F_r(a, x).$$

It is known that when the problem is a regular one, the n functions $\partial F_r/\partial a$ for $a = a_0$ satisfy a system of n linear ordinary differential equations of the second order with respect to x; the analogue of Jacobi's equation in this case $n = 1$. It is impossible that all the n functions $\left(\dfrac{\partial F_r}{\partial a}\right)_{a=a_0}$ and all the n functions $\left(\dfrac{\partial}{\partial x}\dfrac{\partial F_r}{\partial a}\right)_{a=a_0}$ should vanish at the one point $x = x_1$, as follows from a known property of such a system of linear ordinary equations.

Now

$$\left(\frac{\partial}{\partial x}\frac{\partial F_r}{\partial a}\right)_{a=a_0} = \left(\frac{\partial}{\partial u_1}\frac{\partial\chi_r}{\partial x}\right)_{a=a_0}\mu_1'(a_0) + \left(\frac{\partial}{\partial u_2}\frac{\partial\chi_r}{\partial x}\right)_{a=a_0}\mu_2'(a_0) + \ldots$$
$$+ \left(\frac{\partial}{\partial u_n}\frac{\partial\chi_r}{\partial x}\right)_{a=a_0}\mu_n'(a_0),$$

where $r = 1, 2, 3, \ldots$.

Since the functions $\mu(a_0)$ cannot be so determined that, at $x = x_1$, all the functions $\left(\frac{\partial}{\partial x}\frac{\partial F_r}{\partial a}\right)_{a=a_0}$ vanish, it follows that the Jacobian

$$\frac{\partial\left(\frac{\partial \chi_1}{\partial x}, \frac{\partial \chi_2}{\partial x}, \ldots, \frac{\partial \chi_n}{\partial x}\right)}{\partial(u_1, u_2, \ldots, u_n)} \neq 0,$$

when $x = x_1$, and $u_1 = u_1^0$, $u_2 = u_2^0$, ... Thus the condition assumed above is actually satisfied in the case of the extremals corresponding to an integral

$$\int_{x_1}^{x_2} F(x, y_1, y_2, \ldots, y_n; y_1', y_2', \ldots, y_n')\,dx,$$

when the extremals are regular.

Now, by applying the theorem of § 3 to the functions V_r, which have been shewn to satisfy all the conditions for the validity of the theorem, we see that, corresponding to each set of values of $V_1, V_2, \ldots, V_r$, such that $|V_r| < \epsilon$, there is for each value of x such that $x_1 \leqslant x \leqslant x_2$, a set of values of $u_1, u_2, \ldots, u_n$, and only one such set, which is such that $|u_r - u_r^0| < \delta$, for $r = 1, 2, \ldots, n$.

Since the functions V_r are continuous, and since they all vanish for all values of x in (x_1, x_2), when $u_1 = u_1^0$, $u_2 = u_2^0$, ..., $u_n = u_n^0$; a number λ can be so determined that $|V_r| < \epsilon$, for all values of $u_1, u_2, \ldots, u_n$, such that $|u_r - u_r^0| < \lambda$, and for all values of x in the interval (x_1, x_2). This follows from Theorem (e) of § 1.

Let $\bar{\delta}$ be the smaller of the two numbers δ and λ, and consider the space filled by the curves

$$y_1 = f_1(u_1, u_2, \ldots, u_n, x),\ y_2 = f_2(u_1, u_2, \ldots, u_n, x),\ \ldots,\ y_n = f_n(u_1, u_2, \ldots, u_n, x)$$

determined by the conditions $|u_r - u_r^0| < \bar{\delta}$, for $r = 1, 2, 3, \ldots, x_1 < x \leqslant x_2$. In the interior of this space, we have at any point $(y_1, y_2, \ldots, y_n, x)$, the conditions $|V_r| < \epsilon$ all fulfilled, and there is accordingly one, and only one, set of values of $\{u_r\}$, such that $|u_r - u_r^0| < \bar{\delta}$.

It has thus been shewn that the arc of the extremal $(u_1^0, u_2^0, \ldots, u_n^0)$ joining the two points for which $x = x_1$ and $x = x_2$ can be surrounded by the family of extremals for which $|u_r - u_r^0| < \bar{\delta}$, in such wise that in the space filled by these extremals, there goes through each point (with the exception of the point x_1) one and only one of the extremals of the family. The set of values of $u_1, u_2, \ldots, u_n$ corresponding to each point $(x, y_1, y_2, \ldots, y_n)$, where $x_1 < x \leqslant x_2$, is such that $u_1, u_2, \ldots, u_n$ are functions of $x, y_1, y_2, \ldots, y_n$ that have continuous partial differential coefficients. The existence of a quasi-field (*uneigentliches Feld*) of extremals surrounding the given arc of an extremal has therefore been established.

The theorem proved may be stated as follows :—

Let a family of curves in space of $n+1$ *dimensions of which the coordinates are* $x,\ y_1,\ y_2,\ \ldots,\ y_n$, *be denoted by*

$$y_1 = \chi_1(u_1,\ u_2,\ \ldots,\ u_n,\ x),\quad y_2 = \chi_2(u_1,\ u_2,\ \ldots,\ u_n,\ x),\quad \ldots,$$
$$y_n = \chi_n(u_1,\ u_2,\ \ldots,\ u_n,\ x)\ ;$$

the functions $\chi_1,\ \chi_2,\ \ldots$ *satisfying the following conditions :—*

(1) *The functions* χ_r *are defined in a domain* D'_{n+1} *of* $(u_1,\ u_2,\ \ldots,\ x)$, *where* $(u_1,\ u_2,\ \ldots,\ u_n)$ *is a point of a domain* D'_n, *and* x *is in the interval* $(X_1,\ X_2)$ *of* x.

This domain D'_{n+1} *contains in its interior a closed domain* D_{n+1} *consisting of a closed domain* D_n *of* $(u_1,\ u_2,\ \ldots,\ u_n)$, *and of the interval* $(x_1,\ x_2)$ *of* x, *where* $X_1 < x_1 < x_2 < X_2$.

(2) *The partial differential coefficients* $\dfrac{\partial\chi_r}{\partial x},\ \dfrac{\partial\chi_r}{\partial u_s},\ \dfrac{\partial^2\chi_r}{\partial x^2},\ \dfrac{\partial^2\chi_r}{\partial u_s\partial x}$ *for all the values of* r *and* s, *exist and are continuous in* D'_{n+1}.

(3) *All the curves passes through a fixed point* $(x_1,\ y_1^0,\ \ldots,\ y_n^0)$, *the same for all values of* $u_1,\ u_2,\ \ldots,\ u_n$.

(4) *The Jacobian* $$\frac{\partial(\chi_1,\ \chi_2,\ \ldots,\ \chi_n)}{\partial(u_1,\ u_2,\ \ldots,\ u_n)} \neq 0,$$

when $$u_1 = u_1^0,\quad u_2 = u_2^0,\quad \ldots,\quad u_n = u_n^0,$$

where $(u_1^0,\ u_2^0,\ \ldots,\ u_n^0)$ *is a fixed point in the interior of* D_n, *provided* $x_1 < x \leqslant x_2$. *Thus the Jacobian is different from zero along a fixed arc except at the end for which* $x = x_1$.

(5) *The Jacobian* $$\frac{\partial\left(\dfrac{\partial\chi_1}{\partial x},\ \dfrac{\partial\chi_2}{\partial x},\ \ldots,\ \dfrac{\partial\chi_n}{\partial x}\right)}{\partial(u_1,\ u_2,\ \ldots,\ u_n)} \neq 0,$$

when $$u_1 = u_1^0,\quad u_2 = u_2^0,\quad \ldots,\quad u_n = u_n^0,$$

$x = x_1$, *i.e., at the end of the arc of the fixed curve for which* $x = x_1$.

It then follows that in the space filled by those curves

$$y_r = f_r(u_1,\ u_2,\ \ldots,\ u_n,\ x) \quad (r = 1,\ 2,\ 3,\ \ldots,\ n),$$

for which $$|u_r - u_r^0| < \bar{\delta} \qquad (r = 1,\ 2,\ 3,\ \ldots,\ n)\ ;$$

where $\bar{\delta}$ *is a properly chosen positive number, through any point* $(x, y_1, y_2, \ldots, y_n)$ *where* $x_1 < x \leqslant x_2$, *there passes one curve and only one of this set ; and if* $u_1, u_2, \ldots, u_r$ *correspond to that curve, and be regarded as functions of* $(x, y_1, y_2, \ldots, y_n)$, *the functions have continuous differential coefficients with respect to* $(x, y_1, y_2, \ldots, y_n)$. *All the conditions are actually satisfied in the case of the extremals through* P_1 *corresponding to the integral*

$$\int_{x_1}^{x_2} F(x, y_1, y_2, \ldots, y_n ;\ y_1', y_2', \ldots, y_n')\, dx,$$

for an arc joining two fixed points P_1, P_2, *in case the arc is a regular arc of an extremal for which Jacobi's condition is satisfied.*

Correspondence between Two Spaces.

6. If a point P be given by the values of $n+m$ coordinates $u_1, u_2, \ldots, u_n;$ $v_1, v_2, \ldots, v_m$, it is said to be a point in the $(n+m)$-dimensional (u, v) space. If Q be the point $(u_1, u_2, \ldots, u_n)$ in the n-dimensional (u) space, and R the point $(v_1, v_2, \ldots, v_m)$ in the m-dimensional (v) space, the points Q, R are said to be the *projections* of P in the (u) and (v) spaces respectively.

A set of points $G(u, v)$ in the (u, v) space has for projections $G(u)$, $G(v)$, sets of points in the (u) and (v) spaces respectively.

The projections of any point P of $G(u, v)$ in the (u) and (v) spaces are points Q, R, such that Q belongs to $G(u)$, and R to $G(v)$.

It is easily seen that, if the set $G(u, v)$ is closed and limited, so also are $G(u)$ and $G(v)$. Moreover, if P is an interior point of $G(u, v)$, Q and R are interior points of $G(u)$ and $G(v)$ respectively.

It is not, in general, the case that any point Q of $G(u)$, and any point R of $G(v)$ are such that Q, R are the projections of a point P of $G(u, v)$.

If $G_h(u, v)$ is a neighbourhood (h) of $G(u, v)$, its projections $G_h(u)$, $G_h(v)$ are neighbourhoods (h) of $G(u)$, $G(v)$ respectively.

For a point p of $G_h(u, v)$ is in the neighbourhood (h) of some point P of $G(u, v)$, of which the projections are Q, R. It is clear that the projections q, r of p are in the neighbourhoods (h) of Q and R respectively. Also, if q be any point in the neighbourhood (h) of any point Q of $G(u)$, and Q is the projection of P, then q is the projection of points in the neighbourhood (h) of P which belong to $G_h(u, v)$.

Consider a system of functions $F_r(u_1, u_2, \ldots, u_n ;\ v_1, v_2, \ldots, v_m)$, where $r = 1, 2, \ldots, n$. To any point of the (v) space there corresponds a finite or an infinite set of points (which may be a null set) of the (u) space, defined by the relations $F_1 = 0$, $F_2 = 0$, $\ldots$, $F_n = 0$.

This set may be called the image (F) of the point of the (v) space in the (u) space. To a given set of points in the (v) space there corresponds an image (F) in the (u) space, which consists of the images of all the points of the given set. That part of the image which is in a given domain of the (u) space may be spoken of as the image (F) in the domain of the (u) space.

7. *Let the variables* $\xi_1, \xi_2, \ldots, \xi_p$ *have the systems of values corresponding to points in a closed limited domain* X_p *of the* (ξ) *space, and let the* $n+m$ *variables* $u_1, u_2, \ldots, u_n;\ v_1, v_2, \ldots, v_m$ *have the system of values corresponding to points of a closed limited domain* $D(u, v)$ *in the* $(n+m)$*-dimensional* (u, v) *space; the* $n+m+p$ *variables having accordingly the systems of values corresponding to points in a limited closed domain* $D(u, v, \xi)$ *in the* (u, v, ξ) *space. Let it be further assumed that* $D(u, v, \xi)$ *is interior to a domain* $D'(u, v, \xi)$ *for which* n *functions*

$$F_r(u_1, u_2, \ldots, u_n;\ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p)$$

are defined, and for which the functions are continuous and such that all their partial differential coefficients with respect to the $n+m+p$ *variables exist and are continuous. Let* $S(u, v)$ *be a limited and closed set in the interior of* $D(u, v)$, *with reference to which the functions* F_r *satisfy the following conditions:—*

(1) *At every point of* $S(u, v)$ *all the functions* $F_1, F_2, \ldots, F_n$ *have the value zero, whatever point* $(\xi_1, \xi_2, \ldots, \xi_p)$ *may be of* X_p.

(2) *The Jacobian* $$\frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u_1, u_2, \ldots, u_n)} \neq 0,$$

at every point of $S(u, v)$, *and for every point* $(\xi_1, \ldots, \xi_p)$ *of* X_p.

(3) *In* $S(u, v)$ *no two points have the same projection in the* (v) *space; and thus a point of* $S(v)$, *the projection of* $S(u, v)$ *in the* (v) *space, has as its image* (F), *in* $S(u)$, *the projection of* $S(u, v)$ *in the* (u) *space, only one point.*

Subject to these conditions, two positive numbers δ, ϵ *can be so determined that any point interior to* $S_\epsilon(v)$, *the neighbourhood* (ϵ) *of* $S(v)$, *has, for each set of values of* $(\xi_1, \xi_2, \ldots, \xi_p)$ *in* X_p, *one and only one point of its image* (F) *in the* (u) *space interior to* $S_\delta(u)$, *the neighbourhood* (δ) *of* $S(u)$.

Moreover, these values of $(u_1, u_2, \ldots, u_n)$ *are given by a set of equations*

$$u_r = \phi_r(v_1, v_2, \ldots, v_n;\ \xi_1, \xi_2, \ldots, \xi_p) \quad (r = 1, 2, \ldots, n),$$

where ϕ_r *and all its partial differential coefficients with respect to the* $m+p$ *variables are continuous in the domain which consists of* $(v_1, v_2, \ldots, v_m)$ *in the interior of* $S_\epsilon(v)$, *and* $(\xi_1, \xi_2, \ldots, \xi_p)$ *in* X_p.

When $S(u, v)$ consists of a single point, the theorem reduces to that established in § 2. The present theorem asserts that one pair of values of δ, ϵ can be determined which apply to all the points of $S(u, v)$. The number δ can first be taken to have any value sufficiently small, and then ϵ can be chosen.

It will first be proved that a number h can be so determined that any interior point of $S_h(v)$, the neighbourhood (h) of $S(v)$, has not more than one point of its image (F) in the (u) space in the interior of $S_h(u)$, the neighbourhood (h) of $S(u)$, for each point $(\xi_1, \xi_2, \ldots, \xi_p)$ of X_p.

In the first place, h_1 may be so fixed that, in the complete neighbourhood (h_1) of that set of points (u, v, ξ) such that (u, v) belongs to $S(u, v)$, and (ξ) to X_p, the Jacobian $\dfrac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u_1, u_2, \ldots, u_n)}$ is nowhere zero. Thus, if $S_{h_1}(u, v)$ denote the complete neighbourhood (h_1) of $S(u, v)$, the Jacobian does not vanish in that neighbourhood, whatever value $(\xi_1, \xi_2, \ldots, \xi_p)$ may have in X_p. We can also suppose $S_{h_1}(u, v)$ to be interior to $D(u, v)$.

Let
$$(u_1, u_2, \ldots, u_n;\ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p),$$
$$(u_1+\Delta u_1,\ u_2+\Delta u_2,\ \ldots,\ u_n+\Delta u_n;\ \ v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p)$$
be two points of the domain formed by $S_{h_1}(u, v)$ and X_p, for both of which all the functions F_r have the value zero. We have then, in accordance with the hypotheses made in the statement of the theorem, a system of equations

$$O = \left(\frac{\partial F_r}{\partial u_1}+\rho_{r1}\right)\Delta u_1+\left(\frac{\partial F_r}{\partial u_2}+\rho_{r2}\right)\Delta u_2+\ldots+\left(\frac{\partial F_r}{\partial u_n}+\rho_{rn}\right)\Delta u_n$$
$$(r = 1, 2, \ldots, n),$$

provided $|\Delta u_1|$, $|\Delta u_2|$, ..., $|\Delta u_n|$ are all less than some fixed positive number η; where $\rho_{r1}, \rho_{r2}, \ldots, \rho_{rn}$ converge to zero as $\Delta u_1, \Delta u_2, \ldots, \Delta u_n$ do so; and uniformly for all values of $u_1, u_2, \ldots, u_n$; $v_1, v_2, \ldots, v_m$; $\xi_1, \xi_2, \ldots, \xi_p$ of the domain made up of $S_{h_1}(u, v)$ and X_p.

If none of the expressions $\Delta u_1, \Delta u_2, \ldots, \Delta u_n$ are zero, we can eliminate them from these relations, and we thereby obtain an eliminant which may be written in the form

$$\frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u_1, u_2, \ldots, u_n)} = \theta(\Delta u_1, \Delta u_2, \ldots, \Delta u_n),$$

where the expression on the right-hand side converges to zero as $\Delta u_1, \Delta u_2, \ldots, \Delta u_n$ do so; and uniformly in the domain made up of $S_{h_1}(u, v)$ and X_p.

Since the absolute value of the Jacobian must have a minimum in the closed domain, that cannot be zero, it follows that the numbers Δu must be such that $\theta(\Delta u_1, \Delta u_2, \ldots, \Delta u_n)$ exceeds in absolute value some fixed positive number; and therefore $\sum\limits_{r=1}^{r=n} (\Delta u_r)^2$ must exceed some fixed positive number.

In case some of the numbers Δu vanish, say $\Delta u_{s+1}, \Delta u_{s+2}, \ldots, \Delta u_n$ all vanish, we then see that all the determinants formed by s rows of

$$\begin{array}{llll} \dfrac{\partial F_1}{\partial u_1}+\rho_{11}, & \dfrac{\partial F_1}{\partial u_2}+\rho_{12}, & \ldots, & \dfrac{\partial F_1}{\partial u_s}+\rho_{1s}, \\ \dfrac{\partial F_2}{\partial u_2}+\rho_{21}, & \dfrac{\partial F_2}{\partial u_2}+\rho_{22}, & \ldots, & \dfrac{\partial F_2}{\partial u_s}+\rho_{2s}, \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{\partial F_n}{\partial u_n}+\rho_{n1}, & \dfrac{\partial F_n}{\partial u_2}+\rho_{n2}, & \ldots, & \dfrac{\partial F_n}{\partial u_s}+\rho_{ns}, \end{array}$$

must vanish. It then follows easily that the same determinant vanishes as before, and thus that we have an equation

$$\frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u_1, u_2, \ldots, u_n)} = \theta(\Delta u_1, \Delta u_2, \ldots, \Delta u_n);$$

from which, as before, we infer that $\sum\limits_{r=1}^{r=s} (\Delta u_r)^2$ must exceed some fixed positive number.

Considering every possible such case, we see that, if

$$(u_1, u_2, \ldots, u_n;\ v_1, v_2, \ldots, v_m),$$

$$(u_1+\Delta u_1,\ u_2+\Delta u_2,\ \ldots,\ u_n+\Delta u_n:\ v_1, v_2, \ldots, v_m),$$

are any two points of $S_{h_1}(u, v)$ at both of which the equations $F_r = 0$ are all satisfied for the same set of values of $(\xi_1, \xi_2, \ldots, \xi_p)$ in X_p, then $\sum\limits_{r=1}^{r=n} (\Delta u_r)^2$ must exceed some fixed positive number, for all such pairs of points of $S_{h_1}(u, v)$ and for all points of X_p. Thus, if one point of $S_{h_1}(v)$ have two points in its image contained in $S_{h_1}(u)$, for any set of values of ξ in X_p, the distance between the two points of the image must exceed some fixed number independent of the particular point of $S_{h_1}(v)$ and of the particular point of X_p.

Let L_{h_1} denote the set of all those points P of $S_{h_1}(u, v)$ which are such that there is another point Q (also belonging to the set), so that all the functions F_r vanish both at P and Q, the two points having the same projection in the (v) space, and the values of $\xi_1, \xi_2, \ldots, \xi_p$ corresponding to some point R of the set X_p. It will be shewn that the set L_{h_1} is closed. For let $P_1, P_2, \ldots, P_n, \ldots$ be a convergent sequence of such points P, and let $Q_1, Q_2, \ldots, Q_n, \ldots$ be the positions of Q that corresponds to $P_1, P_2, \ldots, P_n, \ldots$; and let $R_1, R_2, \ldots, R_n, \ldots$ be the positions of the point R of X_p that correspond to the pairs $(P_1, Q_1), (P_2, Q_2), \ldots, (P_n, Q_n), \ldots$. A sequence of increasing integers $n_1, n_2, n_3, \ldots$ can be so chosen that $Q_{n_1}, Q_{n_2}, Q_{n_3}, \ldots$ converge to a single limiting point q; then $P_{n_1}, P_{n_2}, P_{n_3}, \ldots$ converge to the same limiting point p as do $P_1, P_2, P_3, \ldots$. Out of the sequence $n_1, n_2, n_3, \ldots$ another increasing sequence $n'_1, n'_2, n'_3, \ldots$ of integers can be so chosen that the points $R_{n'_1}, R_{n'_2}, R_{n'_3}, \ldots$ converge to a single limiting point r. The three sequences

$$P_{n'_1}, P_{n'_2}, P_{n'_3}, \ldots; \quad Q_{n'_1}, Q_{n'_2}, Q_{n'_3}, \ldots; \quad R_{n'_1}, R_{n'_2}, R_{n'_3}, \ldots,$$

are all convergent, their limiting points being p, q, r respectively.

The continuity of the functions F_r has as consequence that these functions vanish both at p and q, when $\xi_1, \xi_2, \ldots, \xi_p$ have the values represented by the point r of X_p, and that p, q have the same projections in the (v) space. It is impossible that p and q should be coincident; for if they were so, there would exist in the arbitrarily small neighbourhood of the point pairs of points of the set L_{h_1}, such that the distance between the projections in the (u) plane have arbitrarily small values; and this is impossible, since that distance has been shewn to be greater than some fixed finite number. Therefore p and q are distinct points, and they therefore belong to the set L_{h_1}, which is consequently closed.

The two points p, q cannot both belong to $S(u, v)$, since that set contains no two points that have the same projection in the (v) space. Let $L_{h_1}^{(1)}$ be the set of those points of L_{h_1} that belong to $S(u, v)$; then since $L_{h_1}^{(1)}$ is the set of points common to two closed sets, it is itself a closed set. The set of points q such that p belongs to $L_{h_1}^{(1)}$ can be shewn, as before, by considering a convergent sequence of different positions of q, to be a closed set M; and M can have no points in common with $S(u, v)$. A complete neighbourhood (h_2) of $S(u, v)$ can now be so determined that it contains in its interior no points of M. Let L_{h_2} be the set of all those points P of the complete neighbourhood (h_2) of $S(u, v)$ which are such that another point q of the same neighbourhood exists so that the functions F_r all vanish at p and q for the same set of values of $\xi_1, \xi_2, \ldots$, and so that p and q have the same projections in the (v) space. As before,

L_{h_2} is a closed set, and it has no points in common with $S(u, v)$; therefore a neighbourhood (h) of $S(u, v)$ can be determined which contains in its interior no point of L_{h_2}. This neighbourhood (h) of $S(u, v)$ is the neighbourhood the existence of which was to be established.

The neighbourhood $S_h(v)$ of $S(v)$ is such that, to any point v of it, there corresponds not more than one point interior to $S_h(u)$ that is in correspondence (F) with it for any one set of values of $\xi_1, \xi_2, \ldots, \xi_p$. Here $S_h(u)$, $S_h(v)$ denote the projections of $S_h(u, v)$ in the (u) and (v) spaces respectively.

In accordance with the theorem of § 2, for each fixed point of $S(u, v)$ two positive numbers δ, ϵ can be fixed, so that to each interior point of $S_\epsilon(v)$ the neighbourhood (ϵ) of $S(v)$, and for each point of X_p, there corresponds one and only one interior point of $S_\delta(u)$, the neighbourhood (δ) of $S(u)$.

It will now be shewn that it is possible so to fix δ and ϵ that these same values are applicable to every point of $S(u, v)$. Consider any point β of $S(v)$.

If ϵ be taken $\leqslant h$, and sufficiently small, there will be one point, and only one, interior to the neighbourhood (h) of α the point of $S(u)$ that corresponds to β, corresponding to each particular point interior to the neighbourhood (ϵ) of β, for any particular point of X_p. For h has been so fixed that there cannot be more than one such point, and if ϵ be small enough such a point must necessarily exist. Let $\bar{\epsilon}_\beta$ denote the upper limit of all possible such values of ϵ; $\bar{\epsilon}_\beta$ being in no case taken to be $> h$. Then to each point interior to the neighbourhood $(\bar{\epsilon}_\beta)$ of β there corresponds, for each point of X_p, one and only one point interior to the neighbourhood (h) of α; the value of $\bar{\epsilon}_\beta$ depending in general on the particular point β of $S(v)$. Consider the cells which consist of the closed neighbourhoods $(\bar{\epsilon}_\beta)$ of all the points β of $S(v)$. This set of cells contains every point of $S(v)$ in the interior of at least one of the cells. Consequently, employing the Heine-Borel theorem, a finite number of these cells can be selected which contain all the points of $S(v)$ in the interior of at least one of the cells of the finite set. Let these cells be the neighbourhoods $(\bar{\epsilon}_{\beta_1}), (\bar{\epsilon}_{\beta_2}), \ldots, (\bar{\epsilon}_{\beta_t})$ of the points $\beta_1, \beta_2, \ldots, \beta_t$ of $S(v)$. The totality of all the points of these cells forms a closed set that contains $S(v)$ in its interior. Therefore a neighbourhood (ϵ) of $S(v)$ can be so determined as to be interior to this composite closed set. Every point interior to this neighbourhood (ϵ) of $S(v)$ is interior to at least one of the neighbourhoods $(\bar{\epsilon}_{\beta_1}), (\bar{\epsilon}_{\beta_2}), \ldots, (\bar{\epsilon}_{\beta_t})$ of the points $\beta_1, \beta_2, \ldots, \beta_t$. Consider a point interior to $(\bar{\epsilon}_{\beta_s})$; to such a point there corresponds a point in the neighbourhood (h) of the point α_s of $S(u)$ that corresponds to β_s. This point is interior to the neighbourhood (h) of $S(u)$; and no other point in that neighbourhood can correspond to the same point of the (v)-space for the same set of values of $\xi_1, \xi_2, \ldots, \xi_p$. The value of ϵ thus found is the one required; for the required value of δ we may take h.

The values of $u_1, u_2, \ldots, u_n$ for the interior point of $S_h(u)$ that is in correspondence (F) with a point interior to $S_\epsilon(v)$ for a set of values $(\xi_1, \xi_2, \ldots, \xi_p)$ belonging to X_p, must be given by a set of equations

$$u_r = \phi_r(v_1, v_2, \ldots, v_m;\ \xi_1, \xi_2, \ldots, \xi_p) \quad (r = 1, 2, \ldots, n);$$

where ϕ_r and all its partial differential coefficients are continuous in the interior of each of the cells that constitute the neighbourhoods

$$(\bar{\epsilon}_{\beta_1}),\ (\bar{\epsilon}_{\beta_2}),\ \ldots,\ (\bar{\epsilon}_{\beta_t})$$

of the points $\beta_1, \beta_2, \ldots, \beta_t$. It follows that ϕ_r and its partial differential coefficients are continuous in the neighbourhood (ϵ) of $S(v)$.

8. The special case of the theorem of § 7 which arises when $n = m$, and the functions F_r are all of the form $f_r(u_1, u_2, \ldots, u_n;\ \xi_1, \xi_2, \ldots, \xi_p) - v_r$ will now be considered.

In this case there is a single point $(v_1, v_2, \ldots, v_n)$, for each point of X_p, corresponding to each set of values of $(u_1, u_2, \ldots, u_n)$.

It has been shewn that, if the two positive numbers δ, ϵ are properly chosen, corresponding to each point interior to $S_\epsilon(v)$, and for each point of X_p there is one and only one point interior to $S_\delta(u)$, such that

$$f_r(u_1, u_2, \ldots, u_n;\ \xi_1, \xi_2, \ldots, \xi_p) = v_r \quad (r = 1, 2, \ldots, n).$$

Let us consider those points of the complete neighbourhood $S_\delta(u)$ of $S(u)$, of which the corresponding points in the (v) space are not interior to $S_\epsilon(v)$ for some or all of the values of (ξ) in X_p. These points form a set $H_\delta(u)$ which will be shewn to be a closed set. Let $P_1, P_2, \ldots, P_n, \ldots$ be a sequence of points of $H_\delta(u)$ which converges to a limiting point P; this point is certainly in the complete neighbourhood $S_\delta(u)$. Let $R_1, R_2, \ldots, R_n, \ldots$ be a set of points in X_p, such that the points corresponding to $P_1, P_2, \ldots, P_n, \ldots$ in the (v) space, are $Q_1, Q_2, \ldots, Q_n, \ldots$, none of which are interior to $S_\epsilon(v)$, when the sets of values of (ξ) in the correspondence are those represented by $R_1, R_2, \ldots, R_n, \ldots$ respectively. A sequence of increasing integers $n_1, n_2, \ldots$ can be defined so that $R_{n_1}, R_{n_2}, R_{n_3}, \ldots$ converge to a single limiting point R in X_p. Then $P_{n_1}, P_{n_2}, \ldots, P_{n_m}, \ldots$ converge to P; and $Q_{n_1}, Q_{n_2}, \ldots, Q_{n_m}, \ldots$ converge to some point Q, as is seen by considering the fact that the functions f_r

which determine Q_{n_m} when P_{n_m}, R_{n_m} are given, are all continuous. It is clear that the point Q cannot be interior to $S_\epsilon(v)$, since none of the points $Q_{n_1}, Q_{n_2}, \ldots$ are so. It then follows that P belongs to the set $H_\delta(u)$, which is therefore closed. Since $H_\delta(u)$ has no point in common with $S(u)$, the number $\delta' (\leqslant \delta)$ can be so chosen that no points of $H_\delta(u)$ are interior to $S_{\delta'}(u)$, the neighbourhood (δ') of $S(u)$. The set $H_{\delta'}(u)$ has the property that a point (v) which corresponds to (u) for any value of (ξ) is interior to $S_\epsilon(v)$.

The points corresponding to the interior points of $H_{\delta'}(u)$ form a set $A_\xi(v)$ dependent on the point (ξ) of X_p, such that $A_\xi(v)$ is interior to $S_\epsilon(u)$. For a given point (ξ) the two sets $A_\xi(v)$, $S_{\delta'}(u)$ are in (1, 1) correspondence.

Let P be a point of $S_{\delta'}(u)$, and Q the corresponding point of $A_\xi(v)$, for the point (ξ) of X_p. Two positive numbers δ'', ϵ'' can be so determined that to every point in the neighbourhood (ϵ'') of Q there corresponds a point in the neighbourhood (δ'') of P. The number δ'' can be chosen so small that this latter neighbourhood is interior to $S_{\delta'}(u)$, since P is an interior point of $S_{\delta'}(u)$. It follows that to every point of the neighbourhood (ϵ'') of Q there corresponds a point interior to $S_{\delta'}(u)$; therefore Q is an interior point of $A_\xi(v)$.

The following theorem* has now been established :—

Let the p variables $\xi_1, \xi_2, \ldots, \xi_p$ have the systems of values corresponding to points in a closed limited domain X_p of the (ξ) space. Let $u_1, u_2, \ldots, u_n$ be a set of variables having the values corresponding to points in a limited closed domain $D(u)$; the $n+p$ variables having therefore the values corresponding to points in a limited closed domain $D(u, \xi)$.

Let it further be assumed that $D(u, \xi)$ is interior to a domain $D'(u, \xi)$ for which n functions

$$F_r(u_1, u_2, \ldots, u_n;\ \xi_1, \xi_2, \ldots, \xi_p) \quad (r = 1, 2, \ldots, n)$$

are defined, and in which the functions and all their partial differential coefficients with respect to the $n+p$ variables are continuous.

Let the variables $v_1, v_2, \ldots, v_n$ be defined by the n equations

$$v_r = F_r(u_1, u_2, \ldots, u_n;\ \xi_1, \xi_2, \ldots, \xi_p),$$

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* The special case of this theorem which arises when $\xi_1, \xi_2, \ldots, \xi_p$ are absent has been established otherwise by Bolza, and applied by him to the existence theorem of a Weierstrassian field.

and let $S(u)$ be a closed set of points interior to $D(u)$, for which the following conditions are satisfied :—

(1) *For all the points of $S(u)$, all the functions F_r become independent of the point $(\xi_1, \xi_2, \ldots, \xi_n)$ of X_p. Moreover the equations $F_r = v_r$ define a (1, 1) correspondence between $S(u)$ and the set $S(v)$ in the (v) space.*

(2) *The Jacobian $\dfrac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u_1, u_2, \ldots, u_n)} \neq 0$, at every point of $S(u)$, for every point of X_p.*

Subject to these conditions, two positive numbers δ, ϵ can be so determined that, to any set of values of (v) in $S_\epsilon(v)$ there corresponds one and only one set of values of (u) in $S_\delta(u)$ for each point of X_p.

These values of u are given by equations

$$u_r = \phi_r(v_1, v_2, \ldots, v_n;\ \xi_1, \xi_2, \ldots, \xi_p) \quad (r = 1, 2, \ldots, n),$$

where ϕ_r and all its partial differential coefficients with respect to $v_1, v_2, \ldots, v_n;\ \xi_1, \xi_2, \ldots, \xi_p$ are continuous in the domain consisting of $S_\epsilon(v)$ and X_p.

Moreover, an incomplete neighbourhood (δ') of $S(u)$ can be so determined that, for each point of X_p, there is a (1, 1) correspondence between $S_{\delta'}(u)$ and its image in the (v) plane; and every point of that image is an interior point of the image.

The Fields of Extremals in the Neighbourhoods of Points in a Plane.

9. The general theorem of § 8 may be applied to prove an important property of the extremals in a regular parametric problem of the Calculus of Variations.

It will be assumed that in a domain X_2^0 of the plane of (x_1, y_1) the Weierstrassian function $F_1(x_1, y_1, \cos\alpha, \sin\alpha) \neq 0$ for all values of α.

The extremal that passes through a point (x_1, y_1) and is there inclined at an angle α to the x-axis, has for its equations

$$x = X(s, \alpha;\ x_1, y_1),$$

$$y = Y(s, \alpha;\ x_1, y_1),$$

where s denotes the length of the arc of the extremal measured from the point (x_1, y_1). The functions X, Y have continuous differential coefficients with respect to s, α, x_1, y_1 for all values of α, when (x_1, y_1) is in a closed and limited domain X_2 interior to X_2^0. Also a number h can be so determined that the extremals are all regular in the arcs given by $|s| \leqslant h$, throughout the domain X_2.

Let

$$v_1 = [\{X(s, a;\ x_1, y_1) - x_1\}^2 + \{Y(s, a;\ x_1, y_1) - y_1\}^2]^{\frac{1}{2}},$$

$$v_2 = \tan^{-1} \frac{Y(s, a;\ x_1, y_1) - y_1}{X(s, a;\ x_1, y_1) - x_1},$$

be equations connecting the variables v_1, v_2 with the variables s, a; the variables x_1, y_1 being regarded as parameters.

These relations can be so interpreted that, corresponding to each pair of values of s and a, there is a unique pair of values of v_1 and v_2. For this purpose we assume that the positive or the negative value of v_1 is taken, according as s has a positive or a negative value.

Since $\quad X(0, a;\ x_1, y_1) = x_1, \quad Y(0, a;\ x_1, y_1) = y_1,$

we have $\quad X(s, a;\ x_1, y_1) - x_1 = s \dfrac{\partial}{\partial s} X(\theta s, a;\ x_1, y_1),$

$$Y(s, a;\ x_1, y_1) - y_1 = s \frac{\partial}{\partial s} Y(\theta' s, a;\ x_1, y_1),$$

where $0 < \theta < 1$, $0 < \theta' < 1$; thus

$$v_1 = s \left[\left\{ \frac{\partial}{\partial s} X(\theta s, a;\ x_1, y_1) \right\}^2 + \left\{ \frac{\partial}{\partial s} Y(\theta' s, a;\ x_1, y_1) \right\}^2 \right]^{\frac{1}{2}},$$

and the positive value of the square root is to be taken, whichever sign s may have.

The value to be assigned to v_2 may be taken as that of the unique function

$$a + \int_0^s \frac{(X - x_1) \dfrac{\partial Y}{\partial s} - (Y - y_1) \dfrac{\partial X}{\partial s}}{(X - x_1)^2 + (Y - y_1)^2}\, ds.$$

In order to apply the theorem of § 8, the coordinates there denoted by u_1, u_2 must be taken to be s, a, and ξ_1, ξ_2 must be taken to denote x_1, y_1.

The domain for which $X(s, a;\ x_1, y_1)$, $Y(s, a;\ x_1, y_1)$ has continuous partial differential coefficients is given by $0 \leqslant |s| \leqslant h$, $-\infty < a < \infty$, (x_1, y_1) in X_2.

We take the closed set $S(u)$ to be given by $s = 0$, $-\pi \leqslant a \leqslant \pi$; and the closed set $S(v)$ by $v_1 = 0$, $-\pi \leqslant v_2 \leqslant \pi$. The closed set $S(u, v)$ is given by $t = 0$, $-\pi \leqslant a \leqslant \pi$, $v_1 = 0$, $v_2 = a$; and this is interior to the domain $0 \leqslant s \leqslant h$, $-\infty < a < \infty$, $-\infty < v_1 < \infty$, $-\infty < v_2 < \infty$.

It is necessary to verify that the differential coefficients

$$\frac{\partial v_1}{\partial s},\ \frac{\partial v_1}{\partial a},\ \frac{\partial v_1}{\partial x_1},\ \frac{\partial v_1}{\partial y_1}\quad \text{and} \quad \frac{\partial v_2}{\partial s},\ \frac{\partial v_2}{\partial a},\ \frac{\partial v_2}{\partial x_1},\ \frac{\partial v_2}{\partial y_1}$$

are all continuous functions of $(s, a;\ x_1, y_1)$; the only difficulty in doing this arises in connection with $s = 0$.

If we employ the expressions

$$X - x_1 = s\,\frac{\partial X(0)}{\partial s} + \tfrac{1}{2}s^2\,\frac{\partial^2 X}{\partial s^2},$$

where, in $\partial^2 X/\partial s^2$ the argument has some value between 0 and s, with a similar expression for $Y - y_1$; and if we use

$$\frac{\partial X(s)}{\partial s} = \frac{\partial X(0)}{\partial s} + s\,\frac{\partial^2 X}{\partial s^2},$$

where the argument in $\partial^2 X/\partial s^2$ is between 0 and s, we find that $\partial v_1/\partial s$ is continuous for $s = 0$, the differential coefficient converging to the value 1 as $s \sim 0$, whatever values a, x_1, y_1 may have. Also $\partial v_2/\partial s$ will be found to converge to the value

$$\tfrac{1}{2}\left\{\frac{\partial X(0)}{\partial s}\,\frac{\partial^2 Y_0}{\partial s^2} - \frac{\partial Y(0)}{\partial s}\,\frac{\partial^2 X(0)}{\partial s^2}\right\},$$

which has a finite and limited value in the case of regular extremals. It will be found that $\partial v_2/\partial a$ converges to the value 1 as $s \sim 0$, and this is the value for $s = 0$.

As $\dfrac{\partial X(0)}{\partial x_1} = 1$, $\dfrac{\partial X(0)}{\partial y_1} = 0$, $\dfrac{\partial Y(0)}{\partial x_1} = 0$, $\dfrac{\partial Y(0)}{\partial y_1} = 1$,

it can easily be seen that $\dfrac{\partial v_1}{\partial x_1}, \dfrac{\partial v_1}{\partial y_1}, \dfrac{\partial v_2}{\partial x_1}, \dfrac{\partial v_3}{\partial y_1}$ converge to zero, as $s \sim 0$, and are thus continuous for $s = 0$.

The Jacobian $\dfrac{\partial(v_1, v_2)}{\partial(s, a)}$ has the value 1, in $S(u, v)$, and thus the conditions for the applicability of the theorem of § 8 are all satisfied.

There can consequently be found two fixed numbers δ, ϵ, such that there exists one and only one pair of values of s, a, such that $|s| < \delta$, $-\pi-\delta < a < \pi+\delta$, corresponding to each set of values of v_1 and v_2, such that $|v_1| < \epsilon$, $-\pi-\epsilon < a < \pi+a$: or such that $|v_1| < \epsilon$, $-\pi < a \leqslant \pi$, for each set of values of x_1, y_1 in the domain X_2. Instead of a value of a, such that $-\pi-\delta < a < \pi+\delta$, we can always employ one such that $-\pi < a \leqslant \pi$, as the functions X, Y are periodic in a, of period 2π.

Now, let
$$v_1 = \sqrt{(x-x_1)^2+(y-y_1)^2},$$
$$v_2 = \tan^{-1}\frac{y-y_1}{x-x_1},$$
where that value of the inverse trigonometrical function is taken that satisfies the condition $-\pi < v_2 \leqslant \pi$, and the sign of v_1 is taken to be positive; thus for each set of values of x, y, x_1, y_1 there is a unique pair of values of v_1, v_2. It follows that for any set of values of x, y, x_1, y_1, such that $(x-x_1)^2+(y-y_1)^2 < \epsilon^2$, there exists a unique pair of values of a and s, such that $-\pi < a \leqslant \pi$, $0 < s < \delta$.

The following theorem has been established:—

In the regular Weierstrassian problem for $\int F(x, y, x', y')\,dt$, *two numbers* δ, ϵ *can be so determined that any pair of points in a closed domain of regularity whose distance from one another is less than* ϵ *can be joined by one and only one extremal of length less than* δ.

The advantage of the use of the theorem of § 8, in proving this known theorem is that we are not confined to the consideration of the neighbourhood of a single point (x_1, y_1), but that the theorem is established at once for all such points in the closed domain of regularity.

There is no difficulty in principle in extending this theorem to the case in which there is a larger number of variables.

MATHEMATICAL RESEARCH

(*Presidential Address.*)

By A. E. H. LOVE.

[Read November 12th, 1914.]

MR. PRESIDENT, LADIES AND GENTLEMEN :—

We are met together at a serious time. We are all so much moved by the terrible suffering and misery which the War causes, so distressed by the woeful destruction of ancient cities and historic monuments by which it has been marked, so shocked by the failure of civilised opinion to restrain the predatory ambitions of the leaders of a nation supposed to be friendly, that we find it difficult to concentrate our attention on ordinary affairs, and may even have entertained doubts as to the appropriateness of holding our Annual General Meeting. But I think that we may be reassured on this point. While, at the present crisis, it is the simple duty of every man of suitable age and sound physique to place his services at the disposal of the military authorities, it is equally the duty of other citizens to meet whatever happens with courage and calmness; and the best means to this end is to maintain the normal course of civil life with as little disturbance as possible. For this reason scientific societies, as distinguished from their younger and more robust members, should, as far as may be, act in the same way as in time of peace. Among scientific societies, mathematical societies occupy a special position owing to the nature of the trust confided to them. Such a society is a trustee for a treasure which has accumulated through the ages, a treasure which war cannot destroy and even barbarism can but temporarily dim, a treasure which has been described by my predecessor in the Chair as "the main heritage of man, his little beacon of light amidst the solitudes and the darknesses of infinite space." That heritage is mathematics. The process by which it has been won is research. To maintain and improve it is the purpose for which our Society exists. I would therefore crave your indulgence while I set before you some considerations bearing upon various aspects of mathematical research.

The chief matter which I propose to discuss is a question which must be especially interesting to the Council of the Society. It is this: What is it that constitutes value or importance in mathematical research? To put it another way: Wherein lies the difference between valuable research and laborious trifling? Or, again, it may be asked: How is valuable research to be distinguished from the construction of examination questions or from mathematical recreations? The distinction is recognized to exist. It is recognized that some kinds of work are better worth doing than others. We are told, for example, concerning Ritz that he had a very true sense of the relative importance of problems. But I do not remember to have come across any general discussion of the question. It is easy, however, to note certain qualities by which valuable research work is characterized.

One of these qualities is novelty. Prof. Baker, in his Address to Section A of the British Association at Birmingham, has emphasized the aspect of mathematics as a creative art. In any valuable research this element of creation or novelty cannot be absent. The new thing, or the created thing, may be a new idea, or a new method, or a new result, or a new proof of a known result. Mathematics is primarily an affair of ideas much more than of formulæ or calculations or technique. But it seems to be necessarily true that a new idea, to gain acceptance, must be developed into a method and fruitful of results, so that a difficulty may arise in judging of the value of a piece of work which purports to contain a new idea. As examples of new ideas I would mention the idea of incommensurables, which is said to have arisen from a contemplation of the solution of the problem of dividing a segment in extreme and mean ratio; the idea of the comparison of evanescent increments, virtually present in Napier's invention of logarithms, and subsequently developed into the Differential Calculus; the idea of a group, first noted by Ruffini in connexion wlth the substitutions which leave a rational function unchanged, and afterwards so much developed, especially by Galois in algebra and Lie in geometry. These examples suggest that new ideas in mathematics grow from small beginnings, and that a moment comes when some man of genius seizes their essential import and makes them part of our intellectual heritage. Many examples might be given of valuable researches which have been regarded by their authors as introducing or establishing new methods. It may suffice to recall the fact that Leibniz' invention of the Differential Calculus was published in a short paper of less than seven pages as a method for finding maxima and minima and the tangents of curves. It is unnecessary to give instances of writings which contain new results or new proofs of known results, though it may be permitted to refer to a curious example of a famous theorem for which a new proof had to be found because the original proof had disappeared: I mean Pascal's theorem of the inscribed hexagon. Pascal's investigations, which were never completely published, seem to have been founded upon the neglected work of Desargues on perspective, but the theorem is not contained in that work. Our proofs are due to much later writers, such as Brianchon, who was the first to develop systematically the theory of anharmonic ratio, although the fundamental result that such ratios are unaltered by projection was known to Desargues. In this fragment of history we have the emergence of a new mathematical idea—anharmonic ratio, the development of a new geometrical method founded on this idea, a new and striking result, a new and elegant proof. It seems that the progress of mathematics needs many kinds of work. A worker who introduces a new idea may be compared with the exploring prospector who discovers that there is gold in a country; one who invents a new method, with the mechanician who devises the processes and perfects the tools by which the gold can be extracted; one who obtain new results, with the miner who extracts the gold; one who obtains new proofs of known results, with the metallurgist who refines the gold and uses it for making beautiful objects. Few of us can hope to play the part of the prospector or the mechanician, but their efforts would be fruitless without the work of the miner and the metallurgist.

After this all-important character of novelty, or artistic creation, we may note other qualities which valuable research must possess. One of these, which is not very easy to define, I propose to call "relevancy." A piece of work, to be valuable, must be a branch of the tree of knowledge; it must stand in a proper relation to the state of mathematical knowledge existing at the time when it is produced. If it is isolated or has no such relation, it is irrelevant. A proposition may be new and true and difficult to prove and yet it may be irrelevant. Let me give an example. Prof. Hobson, in his little book on the *Squaring of the Circle*, has given us a series of most interesting surveys of the state of knowledge in regard to this problem existing at various periods. Anyone who should now spend time on developing new series for calculating approximate values for π, after the fashion of Gregory's series for the inverse tangent, or Newton's series for the inverse sine, would be doing work that might have been valuable in the seventeenth century but would be irrelevant now. This example is rather extreme, but the quality of relevancy or irrelevancy attaches in greater or less degree to all original work in mathematics.

Another quality which characterizes valuable original research may be named "definiteness." A piece of research work should aim at giving a definite answer to a definite question. For example, the most famous work of Galois aimed at answering the question: What algebraic equations can be solved by means of radicals? We observe in regard to this question that when asked it was supremely relevant. In the first half of the sixteenth century more than one Italian mathematician, Cardan being the best remembered, had found how to reduce the solution of the cubic equation to that of a quadratic equation, and Ferrari had found how to reduce the solution of a biquadratic equation to that of a cubic equation. At a later time, in the early part of the eighteenth century, Euler had found a different way of reducing the problem of solving the biquadratic equation to that of solving a cubic equation, and had entertained the idea that a similar reduction must be possible for an equation of any degree to one of the next lower degree. The development of this idea had been undertaken by Lagrange, with the result that the suggested process had been found to fail definitely for the quintic; and Abel had finally proved that the general quintic cannot be solved by means of radicals. The precision which had been given to the question by the work of Abel would have been unattainable in the time of Cardan, and thus Galois was able to propose a perfectly definite question. He also found a perfectly definite answer. In regard to this quality of definiteness it is of interest to bear in mind the influence that has been exerted upon the progress of mathematics by problems. We shall find that many valuable researches have arisen in the effort to obtain the solution of some definite problem. As an example, I would cite the Problem of Three Bodies, especially as illustrating the way in which a problem becomes transformed. In its first form it is the theory of the motion of the moon around the earth, as disturbed by the attraction of the sun, a problem which can be solved, with sufficient approximation for practical purposes, by various methods. In the more general form, in which it is the problem of n bodies, it includes the problem of planetary theory, also solvable for practical purposes by various methods of successive approximation. But the theoretical interest of the mathematical problem remains undiminished after the practically useful approximations are completed; and it has given rise to numerous interesting questions in regard to the general differential equations of analytical dynamics and the theory of differential equations. We know that it has now been proved that the equations of the problem do not admit any other integral of the same type as the known integrals, which express the constancy of the energy, the linear momentum, and the moment of momentum; but the question of completing the integration remains unsolved. This question has perhaps not yet been asked in the definite form that it must receive before any advance can be made. It is sometimes true that a question properly asked is half answered.

We have noted three qualities as characteristic of valuable research in mathematics: novelty, relevancy, definiteness. I would add to this catalogue a fourth: generality. The best work is never parochial, it is never restricted to a narrow outlook. The quality of generality may seem to be opposed to the quality of definiteness, but generality must not be confused with vagueness. As an example of a piece of work which shows conspicuously the mark of generality, I would cite Gauss' famous memoir on the hypergeometric series. At the time when this was published it could be said of it that it included the theory of almost all the functions which up to that time had been investigated by analysts. But there is nothing indefinite or vague about Gauss' work. As another example of generality, I would cite the shifting of the theory of elliptic functions from the relatively narrow basis of the Jacobian theory, with its modulus, its three related functions, and its appalling array of quasi-trigonometrical relations, to the comparatively simple but much more comprehensive foundation afforded by the theory of the doubly periodic functions. These examples lead us to think that generality is the mark of good theories, while definiteness attaches rather to problems, but it must not be forgotten that theories have their roots in problems and bear their fruits in the solution of problems. Is there then such a thing as excessive generality? A story is told concerning a certain variety of roses which were in great demand among the makers of bouquets, not only on account of their beauty, but especially because the stalks were very long and stiff. The growers took steps to increase the length of the stalk, and were very successful, producing blooms with stalks as much as seven feet long. But unfortunately as the stalk lengthened the bloom dwindled, indeed most of the very long stalks bore no flowers at all. It may be treading on dangerous ground to suggest that there is such a thing as excessive generality, though even so convinced an analyst as Picard is not without misgivings on the subject. Yet it can sometimes be wished that writers who develop general theories at great length would pause to enquire how far they are available for the solution of special problems. Let me give an example. The theory of ordinary linear differential equations is very highly developed, but applications to particular equations are beset by a difficulty which the theory leaves untouched. I do not undervalue the interest or importance of the existing theory; everyone who has studied it must have been impressed by it. To proceed to the difficulty, let us suppose, for example, that we have before us a linear differential equation of the second order with rational algebraic coefficients and known singular points. We can form, and solve, the indicial equations relating to the finite singularities, and obtain sequence equations to determine the coefficients of the series which represent regular integrals in the corresponding neighbourhoods; but we cannot usually write down the coefficient of the n-th term of one of our series, unless it should happen that the sequence equation is a linear difference equation of the first order, or one of the second order of the type that can be solved by means of relations between hypergeometric series. When the sequence equation has not one of these characters, if it is an equation of the second order, an equation of three terms, we may obtain a practically sufficient solution by the use of continued fractions, as Laplace did in the theory of the tides; but if it is of an order higher than the second, say an equation of four terms, even this road is blocked. Now it seems to me that much, if not all, of the theory of linear differential equations is developed on the understanding that the coefficients of the series which represent the integrals in the neighbourhoods of the singularities have been effectively obtained, whereas it is only the sequence equations that are really obtained. This is not the occasion to do more than indicate the existence of this gap in the theory, but the example suggests that the value of a piece of research work is increased if the quality of definiteness is combined with the quality of generality, just as the quality of relevancy should be coordinated with that of novelty.

Let us turn for a few moments from this general discussion of the qualities of valuable research in mathematics to consider the distinctive features of research in mathematical physics, including, of course, such subjects as mechanics and hydrodynamics, as well as such subjects as electricity and thermodynamics. It would be taking too narrow a view to contend that mathematical physics and pure mathematics are identical, although there is a constant tendency for mathematical physics to be absorbed in pure mathematics. Thus Mr. Bertrand Russell claims dynamics as a branch of pure mathematics, and the German Encyclopædia of mathematics classifies the theory of potential under differential equations. On the other hand, it is known that geometry had originally an empirical foundation, and it may be suspected that the same is true even of arithmetic. We can trace the passage of geometry from the experimental to the abstract stage; the beginnings of arithmetic are hidden in the times before the dawn of history. We might then enter a counterclaim to the effect that all mathematics is mathematical physics. But it is well understood that this view also would be too narrow. Mathematics is not concerned with the process of passing from empirical data to an abstract theory. This is the primary concern of mathematical physics. Mathematics is not much occupied with the numerical tabulation of the results that can be obtained, not at all with the comparison of such tabulated results with the results of experimental measurements: but a piece of valuable research work in mathematical physics may be composed largely of such matter. Mathematics is not so much occupied with apparently isolated special problems as mathematical physics necessarily is. I say "apparently" isolated, because in valuable research work, however isolated a problem may appear, its solution is desired for the sake of the light which it is expected to throw upon some question of greater generality. In the attempt to solve such problems there arises in mathematical physics a kind of research work which would be trivial in pure mathematics—the kind of work in which we take some piece of pure mathematical theory and use it to obtain the solutions of the special physical problems which it is competent to solve. This kind of work has been humorously described in the phrase "Given the solution it is required to find the problem." Kirchhoff's solutions of problems in discontinuous fluid motion illustrate the process perfectly. Such work can be of great value, as indeed Kirchhoff's was, when it really helps us to understand some natural phenomena; it is ridiculous to do it for its own sake. I would say then that, in addition to the qualities of novelty, relevancy, definiteness, and generality, there is necessary to valuable research work in mathematical physics another quality which may be described as "realism" or adherence to fact. There is work to be done in showing how new observations fall under existing theories, how existing theories must be modified in order that new observations may fall under them, how previously existing, or newly proposed, theories may be tested by deducing analytically the results that follow from them, and comparing these results with laboratory experiments or general experience. Facts make their appearance at the beginning and also at the end. Even the argument by which a question is solved may have a physical interpretation, as happens, for example, when the notion of an "image" is introduced; and this is the ideal of mathematical physics—to conduct the analysis in such terms and by such processes that the argument may be couched in physical language with the minimum use of uninterpreted auxiliary quantities or relations devoid of physical significance. Often it happens, however, that a new result, obtained at first by a more abstract method, is afterwards proved by methods approaching more closely to this ideal. For example, Poisson solved the problem of induction, for a conducting sphere under the influence of a point charge, by means of harmonic analysis, years before electric images were thought of.

The quality of realism may attach to valuable research work in mathematical physics in very various degrees. A theory, or the solution of a problem, may conduce directly to the increase of material well-being, as Maxwell's theory of electrodynamics contained the germ of wireless telegraphy, or it may satisfy intellectual curiosity, as Darwin's theory of tidal friction throws light upon the past history of the earth and moon. There is a certain mental satisfaction in knowing that "Sodium is in the sun," although sodium in the earth can alone be used in chemical industries. The thirst for knowledge is not confined to the immediately practical. Such matters as the origin and past history of the solar system are legitimate objects of curiosity even if an advance of knowledge concerning them is not likely to lead to improvements in the art of navigation. An investigation may be of value even though it may have but a remote bearing upon the rational scheme under which a wide range of facts are, as Prof. Pearson would say, "resumed." Reference has already been made to the Problem of Three Bodies. Apart from approximate solutions, such as constitute the lunar theory, its bearing upon theoretical astronomy may be conceded to be remote. In regard to this problem we may observe that one of the numerous cosmogonies, which have been put forward without proof, makes a great point of the capture theory of comets. The sun by itself cannot capture a comet. Can the solar system capture a comet? Could even a simpler system, consisting of a central sun attended by a single planet, do it? The answer to this question would be found in a particular solution of the Problem of Three Bodies.

A rather difficult question sometimes arises as to the validity of approximations made by workers in mathematical physics. For example, the whole theory of elasticity is founded upon the assumption that quantities of an order higher than the first in the components of strain may be disregarded. The effect of such assumptions underlying a theory is to make the general equations by which the theory is expressed more tractable. Even when the approximation, which is involved in such a reduction of the general equations as is here referred to, has been made, the solution of the equations appropriate to some special problem may be impossible of attainment with existing analytical resources. This happens, for example, in the problem of calculating the distribution of stress in a masonry dam. In such cases it is necessary to simplify the problem by disregarding circumstances which may be assumed to be of little consequence. For instance, in the problem of the dam, the material is regarded as homogeneous, and the system is replaced by a two-dimensional one, while for the actual boundary there is substituted a more regular geometrical shape. Even when a problem has been simplified to the utmost, in such ways as this, it may remain defiant; and it may be necessary to have recourse to some numerical method of approximate solution. Much of the work of the mathematical physicist may take the form of devising, and applying, suitable methods of approximation. A trained physical instinct is the only sure guide to the selection of the circumstances to be disregarded. One of the most remarkable examples of the success of approximate methods specially devised with a view to a particular class of questions is to be found in Fresnel's theory of diffraction. Fresnel did not know the differential equations which govern the propagation of light *in vacuo;* his geometrical notions of the process of wave-transmission took their place. He did not know, nor do we as precisely as we could wish, how the presence of matter in its course would affect the passage of a wave; but he made an assumption which has been found by experience to be approximately true, the assumption that the light that comes to an aperture in obstructing matter is alone concerned in the construction of the effects observable beyond the aperture. He could not even sum exactly the series constructed by the aid of this assumption, but he divined the principle by which the important terms can be distinguished from the comparatively unimportant residuum. Similar problems arise in the theory of sound. Here the analytical conditions are more precisely known, while the physical interest is much smaller. Yet the problems remain, for the most part, unsolved. Lord Rayleigh remarks: "Although the general character of the phenomena is well understood, and therefore no very startling discoveries are to be expected, the exact theoretical solution of a few of the simpler problems, which the subject presents, would be interesting." The only exact theoretical solution we have is that, first obtained by Sommerfeld, for the diffraction of plane waves at a straight edge. There seems to have been a certain element of good fortune in the discovery of this solution, as efforts to extend the method to other problems have so far proved fruitless. The theory of diffraction is not the only department in which the exact theoretical solution of a few simple problems would be welcome, even though the general character of the phenomena is well understood; and we may note here that it may not be necessary to aim at any startling discoveries in physics in order to do valuable research work in mathematical physics.

An examination of the solitary known theoretically exact solution of the problem of diffraction prompts the remark that a mathematical physicist cannot know too much mathematics. And this remark brings me to the equipment of an investigator. Some essential elements of this equipment seem to be skill, knowledge, curiosity. In order to produce valuable research work, either in pure mathematics or in mathematical physics, the worker must be an accomplished mathematician. It is not enough to have native ability. It is necessary to be ready with the various artifices by which mathematical work is effected. More than this, it is necessary to have wide and deep knowledge of the mathematics that has been invented, even of the most recent work that has not yet found a place in the current textbooks. It has been said that the first essential for research is knowledge, especially a minute knowledge of the work that has been done in the subject proposed for investigation. Evidently the first thing to be done by anyone who would advance knowledge is to determine the position of the front. But it is equally necessary not to be oppressed by the vastness or the intricacy of that which is known if progress is to be made into the unknown. The effort to obtain knowledge must not be overwhelming; the worker must retain his freshness. He must still be able to ask himself questions, and to wish to know the answers. The attitude of a mind which is always seeking to know the answers to questions which are relevant, definite, and general, is what I mean by "curiosity" as an element in the equipment of an investigator. It cannot be acquired, though it may be encouraged. Skill can be acquired by developing native ability through constant practice, but the native ability must be forthcoming. Knowledge is constantly being increased. Fortunately the means for acquiring it are also constantly being improved. Text-books and treatises include always later additions to knowledge. Encyclopædias, such as the *Encyclopädie der Mathematischen Wissenschaften*, enable the would-be investigator to get rapidly to the place where, at the time when one of its articles was written, new work in the subject could be attempted. Indexes, such as the *Jahrbuch über die Fortschritte der Mathematik*, the *Revue Semestrielle*, or the *International Catalogue of Scientific Literature*, put it in his power to ascertain what new work has been done since the date of the Encyclopædia article. Reprinted collections of the writings of great investigators enable him to become acquainted with the best of the older work at first hand; and all the indexes and encyclopædias that have ever been compiled cannot take the place of the great original memoirs. They are the corner-stones of the edifice.

And here I would plead for more attention to the history of mathematics. Towards elucidating this history Great Britain has not done very much; the study of it receives here but little encouragement. Yet it seems to me to be extremely desirable, if not actually indispensable, for entering into the heritage that has been bequeathed to us, and for seeking to enhance its value. It may be hoped that the celebration this year of the tercentenary of Napier's invention of logarithms, and the series of excellent monographs on the work of the ancient Greek mathematicians, which we owe to Sir Thomas Heath, may induce us, as a nation, to take more interest in the history of our science.

Finally, to revert to the aspect of mathematics as a creative art, I would urge that an essential element in the equipment of an investigator is a literary education, or, if you prefer it, a training in the means of expression. It is necessary to be articulate, but more than this is desirable. It is desirable to be mathematically articulate, to be able to express mathematical ideas in such a way that they can be comprehended easily by those who have the requisite training. Some great work is marred by obscurity. This charge has been brought against even so great an originator as Abel. Others, such as Laplace, are models of lucidity. There is such a thing as style in mathematics, and it is worth cultivating. The mathematician is an artist; and every artist, we have been told by Mr. Bernard Shaw, must grow his own style out of himself. But there are points of style to which it is desirable to attend, such as clearness, arrangement, rigour, avoidance of haste, conciseness, notation. It is desirable to say exactly what one means, neither less nor more. It is desirable to introduce new ideas, or new relations, one at a time, so that each one seems to arise naturally just at the place where it makes its appearance in a piece of written work. No trouble is too great to secure rigour, if it can be secured. We have all heard how Newton kept back the publication of the work, which was ultimately embodied in the *Principia*, until he had obtained a conclusive proof that spheres attract as if their masses were condensed at their centres. If absolute rigour has occasionally to be sacrificed, it should be made perfectly clear at what points it is absent. A memoir should not bear marks of hurry; the argument should be developed in a straightforward fashion from the premisses to the conclusion. On the other hand, it should not waste time, as, for instance, by undue restriction of conditions in the main argument, with the object of excluding exceptional cases, or by overloading the main argument with details; it should always be possible to distinguish the wood from the trees. The choice of notation is not to be despised; it may make all the difference to the ease with which a piece of work can be assimilated, or a new idea applied to new questions. It may be necessary to rewrite a memoir more than once or twice if these advantages are to be secured. It is worth while.

ON THE DISTRIBUTION OF THE SET OF POINTS $(\lambda_n\theta)$

By R. H. FOWLER.

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1. In a paper on "Some Problems of Diophantine Approximation", recently published,[*] Messrs. G. H. Hardy and J. E. Littlewood have investigated in detail the distribution of the set of points $(\lambda_n\theta)$ [$(\lambda_n\theta)$ denotes the fractional part of $\lambda_n\theta$] in the particular case in which

$$\lambda_n = a^n,$$

where a is an integer greater than 1, and n takes the values $1, 2, \ldots, \rightarrow \infty$. The authors point out that this is equivalent to studying the distribution of the digits in the expression of θ as a decimal in the scale of a, and it is this view of the problem which is the more interesting in this particular case, and on which they concentrate their attention. In the following paper some of these results are extended, by means of an adaptation of Messrs. Hardy and Littlewood's arguments, to the set of points resulting from any sequence λ_n which satisfies the inequalities

$$(1) \qquad \lambda_n/\lambda_{n-1} \geqslant \beta^{n^{-1+\zeta}} \quad (n \geqslant n_0),$$

where ζ is any number > 0, and $\beta > 1$. It is easily verified that these inequalities imply that

$$(2) \qquad \lambda_n > \exp(Hn^{\zeta}) \quad (n \geqslant n_1),$$

where H is a constant. Of course the relation (2) does not imply the relations (1); but, to give some idea of the range of applicability of the results that we shall obtain, we may say, speaking roughly, that *our theorems apply to sufficiently regular sequences which increase faster than*

$$\exp(n^{\zeta})$$

for some positive value of ζ.

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[*] *Acta Mathematica*, Vol. 37, pp. 155-190.

In the particular case discussed by Messrs. Hardy and Littlewood, there are, as we have mentioned, two ways of regarding the results which are both of great interest. In the more general case, though the discussion proceeds by the construction of a quasi-decimal expression for θ and the determination of the distribution of the integers employed (corresponding to the digits in a decimal), this aspect of the problem is no longer fundamental, and the results must be exhibited from the other point of view.

Messrs. Hardy and Littlewood prove the following theorems:—

THEOREM 1·48.*—*It is almost always true that, when a number θ is expressed in any scale of notation, the number of occurrences of any digit, or any combination of digits, is asymptotically equal to the average number that might be expected.*

THEOREM 1·481.*—*It is almost always true that the deviation from the average in the first n places of decimals, is not of order exceeding* $\sqrt{n \log (n)}$.

THEOREM 1·482.*—*It is almost always true that the deviation, in both directions, is sometimes of order exceeding* $\sqrt{n}$.

THEOREM 1·483.*—*The number Δ_n, of the first n numbers $(a^{\nu}\theta)$ which fall inside an interval of length δ included in the interval* $(0, 1)$ *is almost always asymptotically equal to δn.*

In these theorems, a statement is said to be almost always true when it is true for all θ's between 0 and 1, with the exception of a set of measure zero.

Theorem 1·483 is, of course, practically the same as Theorem 1·48, regarded from the other point of view. Messrs. Hardy and Littlewood suggest that it may be possible to give corresponding forms to Theorems 1·481, 1·482. It is, however, very doubtful if this can be done.

In the case of the more general sequences considered in this paper, I shall prove theorems which are to a certain extent analogous to Theorem 1·481, and strictly analogous to Theorem 1·48. They differ from strict analogues of Theorem 1·481 in that there seems to be no way of proving—at any rate by Messrs. Hardy and Littlewood's methods—that the error term is of an order so small as $\sqrt{n \log (n)}$. I have not succeeded in proving any analogue of Theorem 1·482 for these general sequences.

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* These theorems are so numbered in the paper referred to, p. 190.

2. We shall start by considering sequences for which

$$\lambda_n/\lambda_{n-1} \geqslant \beta^n \quad (n \geqslant n_0),$$

where β is a constant > 1; we can then extend the theorems so obtained to less restricted sequences by another argument. There will be no loss of generality in supposing that $n_0 = 2$, and $\lambda_1 > 1$.

Let G be any integer conveniently chosen. G will eventually be made large, but for the present it is to be regarded as fixed. Let

$$n = n_1 + i,$$

and consider the sequence $(\lambda_i\theta)$ which is formed by omitting the first n_1 terms of the old sequence $(\lambda_n\theta)$. If n_1 be so chosen that

$$\beta^{n_1} \geqslant G,$$

i.e., if $$n_1 \geqslant \log(G)/\log(\beta),$$

the new sequence will satisfy the inequalities

$$\lambda_i/\lambda_{i-1} > G\beta^i,$$

for all values of $$i \geqslant 2.$$

Let a be the greatest integer contained in $\sqrt{G}$. Then the λ's in our new sequence $(\lambda_i\theta)$ satisfy the inequalities

$$\lambda_1 < a\lambda_1 < \lambda_2 < a\lambda_2 \ldots < \lambda_i < a\lambda_i < \lambda_{i+1} < \ldots.$$

By means of this increasing sequence of numbers, we can build up a quasi-decimal expression for θ which will enable us to make use of Messrs. Hardy and Littlewood's arguments, and so to obtain the desired result. The possibility of the required expansion is established by the Lemma of the next section.

3. Let $[x]$ denote *the integer next less than x*, so that when x is not an integer,

$$x > [x] > x-1,$$

and when x is an integer,

$$[x] = x-1.$$

Let us also write

(3) $$[x]+1 = x+(x)_f;$$

$(x)_f$ will satisfy the inequalities

$$1 > (x)_f \geqslant 0.$$

Let K denote the set of expressions of the form

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\}, \tag{4}$$

where ϕ_i takes any one of the values 0, 1, 2, ..., $[\lambda_i/a\lambda_{i-1}]$,* χ_i any one of the values 0, 1, 2, ..., $a-1$, and x_i is that number satisfying

$$1 > x_i \geqslant 0,$$

which is such that $$\lambda_i \sum_{\varepsilon=1}^{i-1} \frac{1}{\lambda_s} \left\{ \phi_s + \frac{\chi_s}{a} + x_s \right\} + x_i$$

is an integer. Referring to the equation (3) we see that

$$x_i = \left(\lambda_i \sum_{s=1}^{i-1} \frac{1}{\lambda_s} \left\{ \phi_s + \frac{\chi_s}{a} + x_s \right\} \right)_f. \tag{5}$$

For given values of $\phi_1 \ldots \phi_{i-1}$, $\chi_1 \ldots \chi_{i-1}$, the values of $x_1 \ldots x_i$ are uniquely determined and can be calculated in order, beginning with x_1, by means of the equation (5). A change in the values of ϕ_i and χ_i will therefore not affect the values of $x_1 \ldots x_i$, though, in general, the values of x_{i+1}, x_{i+2}, ... will all be changed. It may be observed that $x_1 = 0$.

Any expression of the form (4), in which all the ϕ's and χ's after a certain stage are zero, will be spoken of as a terminating expression, though, in general, of course, such an expression will contain the non-terminating convergent series

$$\sum_{i=1}^{\infty} x_i/\lambda_i.$$

It is easily seen that the series (4) will in all cases converge, and so will always represent a definite positive number.

Let $_0x_i$ be the value assumed by x_i when all the ϕ's and all the χ's have their greatest permissible values. Let Λ denote the sum of the convergent series

$$1 + \frac{(\lambda_1)_f}{\lambda_1} + \sum_{i=2}^{\infty} \frac{1}{\lambda_i} \left\{ \left(\frac{\lambda_i}{a\lambda_{i-1}} \right)_f + {}_0x_i \right\}. \tag{6}$$

Then $$\Lambda \geqslant 1.$$

We shall prove the following Lemma :—

LEMMA I.—*Every member of K represents a definite number between* 0 *and* Λ *inclusive.*

Conversely every number between 0 *and* Λ *inclusive is represented by at least one member of K.*

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[*] $\lambda_i/a\lambda_{i-1}$, when $i = 1$, is always to be taken to represent λ_1.

It is an obvious consequence of Lemma I that K is a set of measure Λ.

We have already pointed out that any member of K denotes a definite number θ. Consider the change produced in any member k of K by adding 1 to any ϕ or χ which has not its maximum value, to χ_s for example. We have

$$k < \sum_{i=1}^{s} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\} + \sum_{i=s+1}^{\infty} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + 1 \right\},$$

and $$k' > \sum_{i=1}^{s} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\} + \frac{1}{a\lambda_s} + \sum_{i=s+1}^{\infty} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} \right\},$$

where the ϕ's, χ's, and x's are the same in both expressions. Therefore

$$k' - k > 1/a\lambda_s - \sum_{i=s+1}^{\infty} (1/\lambda_i).$$

But it is easily seen that

$$\sum_{i=s+1}^{\infty} (1/\lambda_i) < (1/\lambda_{s+1}) \sum_{r=0}^{\infty} \beta^{-r(s+1)} = \beta^{s+1}/\lambda_{s+1}\{\beta^{s+1}-1\}.$$

Now $$\beta^{s+1}/\{\beta^{s+1}-1\} < \beta/\{\beta-1\},$$

and $$\lambda_{s+1} > \sqrt{G}\, a\lambda_s,$$

so that, provided G be greater than a certain constant depending only on β,

$$k' - k > 0.$$

Therefore any member of K is increased by increasing any one of its constituent integers. The first part of the Lemma is an immediate consequence of this, for it follows that every member of K is less than the number obtained by giving their maximum values to *all* the ϕ's and χ's, and it is easily verified that the number so formed is Λ as defined by (6).

To prove the converse, suppose that θ is any number $< \Lambda$. Since the sum of the series (4) is increased by increasing any of its constituent integers, and since the sum is Λ when all these integers have their maximum values, it must be possible to determine the first q ϕ's and the first q χ's in such a way that

$$\sum_{i=1}^{q} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\} + \sum_{i=q+1}^{\infty} \frac{x_i}{\lambda_i} \leqslant \theta < \sum_{i=1}^{q} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\}$$
$$+ \sum_{i=q+1}^{\infty} \frac{1}{\lambda_i} \left\{ \left[\frac{\lambda_i}{a\lambda_{i-1}}\right] + 1 - \frac{1}{a} + x_i' \right\}.$$

The first q terms on each side agree exactly, so that the first $q+1$ x's are the same on each side; there is, however, no reason to suppose that the later values of x_i will be the same on both sides. This fact is indicated by writing x'_i for x_i in the later terms on the right. This choice of the ϕ's and χ's may be effected by giving to each ϕ and to each χ in turn the greatest possible value that leaves the expression

$$\sum_{i=1}^{q} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\} + \sum_{i=q+1}^{\infty} \frac{x_i}{\lambda_i}$$

not greater than θ at each stage of the process, which can be continued indefinitely if no case of equality ever occurs. If a case of equality does turn up, we have a terminating expression of the form (4) for θ. If no such case turns up, we construct thus a definite convergent series which is such that the difference between the sum of its first q terms and θ tends to 0 as $q \to \infty$. It therefore represents the number θ, and the truth of the Lemma is established.

4. The set of terminating expressions included in K is enumerable, and therefore is a set of measure zero. We shall in future consider only the set K' of non-terminating expressions included in K, which consequently is a set of measure Λ. It should also perhaps be pointed out that we have not proved that more than one member of the set K' cannot represent the same number θ. It can, in fact, be shown that more than one member of K' can represent the same θ (at least for some values of θ) except when all the members $\lambda_i/a\lambda_{i-1}$ are integers. Consequently the relation between the set K' and the set of numbers θ between 0 and Λ, which cannot have a terminating expression, is a many-one, not a one-one relation. This fact, however, need not concern us further, for the following reason. For we proceed to prove theorems about almost all members of K', and the fact that the relation may be many-one does not prevent us from deducing the corresponding theorems about almost all the θ's.

5. All members of K', whose first ν ϕ's and first ν χ's have assigned values, can be enclosed in an interval whose length is certainly not greater than I_ν, where

$$I_\nu = \sum_{i=\nu+1}^{\infty} \frac{1}{\lambda_i} \left\{ \left[\frac{\lambda_i}{a\lambda_{i-1}}\right] + 2 - \frac{1}{a} \right\}.$$

On using the equation (3), we find that

$$I_\nu = \frac{1}{a\lambda_\nu} + \sum_{i=\nu+1}^{\infty} \frac{1}{\lambda_i} \left\{ 1 + \left(\frac{\lambda_i}{a\lambda_{i-1}}\right)_f \right\}. \tag{7}$$

Among the terminating expressions (in number a^ν),

$$\sum_{i=1}^{\nu} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\} + \sum_{i=\nu+1}^{\infty} \frac{x_i}{\lambda_i},$$

in which the ϕ's are assigned and the χ's are not, there will be $p(\nu, m)$ in which χ_i takes the particular value b for exactly m values of i, where

$$(8) \qquad p(\nu, m) = \frac{\nu!}{m! \, \{\nu - m\}!} \{a-1\}^{\nu-m}.*$$

All such members of K' can be enclosed in intervals of total length $< p(\nu, m) \, I_\nu$. Consequently *all* members of K' whose first ν χ's contain exactly m b's can be enclosed in intervals whose total length L_ν satisfies the inequality

$$(9) \qquad L_\nu \leqslant \prod_{i=1}^{\nu} \{1 + [\lambda_i / a\lambda_{i-1}]\} \, I_\nu \, p(\nu, m).$$

On using the equations (3) and (7), this becomes

$$L_\nu \leqslant \left[\prod_{i=1}^{\nu} \left\{ 1 + \frac{a\lambda_{i-1}}{\lambda_i} \left(\frac{\lambda_i}{a\lambda_{i-1}} \right)_f \right\} \right]$$
$$\times \left[1 + a\lambda_\nu \sum_{i=\nu+1}^{\infty} \frac{1}{\lambda_i} \left\{ 1 + \left(\frac{\lambda_i}{a\lambda_{i-1}} \right)_f \right\} \right] a^{-\nu} p(\nu, m).$$

Now we have assumed that

$$\lambda_i / \lambda_{i-1} > G\beta^i;$$

and therefore

$$(10) \qquad L_\nu < \left\{ 1 + 2\beta^{-\nu} \sum_{r=0}^{\infty} \beta^{-r(\nu+1)} \right\} \prod_{r=1}^{\infty} \{1 + \beta^{-r}\} \, a^{-\nu} p(\nu, m),$$
$$< M a^{-\nu} p(\nu, m),$$

where M depends only on β.

6. We can now treat the set K' in exactly the same way as Messrs. Hardy and Littlewood treat the set of irrational numbers expressed as decimals in the scale of a. We shall need, however, to revise the Lemmas on which the discussion is based, since it will be necessary to let $a \to \infty$.

Let $$\mu(\nu) = m - \nu / a,$$

so that $\mu(\nu)$ is the excess of the number of b's above the average. We proceed to prove that—

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* *Loc. cit.*, 1·441.

Lemma II.—*If* $f(\nu)$ *is any function of* ν *that* $\rightarrow \infty$ *with* ν, *and satisfies the inequalities*

$$1 < f(\nu) < \nu/a - 1,$$

then, for all values of ν,

$$a^{-\nu} \Big\{ \sum_{|\mu(\nu)| \geqslant f(\nu)} p(\nu, m) \Big\} < M \sqrt{\{a\nu\}} \exp \{ -\tfrac{1}{4} a [f(\nu)]^2/\nu \},$$

where M *is a constant independent of* a, ν, *and* $f(\nu)$.*

Let us consider values of $\mu(\nu)$, such that

$$\mu(\nu) \geqslant f(\nu),$$

where $f(\nu)$ is positive, so that

$$m \geqslant \nu/a + f(\nu).$$

Then

$$\begin{aligned} p(\nu, m+1)/p(\nu, m) &= \{\nu - m\} / \{a-1\} \{m+1\} \\ &\leqslant \{\nu \{a-1\} - af(\nu)\} / \{a-1\} \{\nu + af(\nu)\} \\ &< 1. \end{aligned}$$

It follows that, for all values of $\mu(\nu) \geqslant f(\nu)$,

$$\begin{aligned} p(\nu, m) &< p[\nu, \{\nu/a + f(\nu)\}] \\ &< \frac{\nu!}{\{\nu/a + f(\nu)\}! \, \{\nu - \nu/a - f(\nu)\}!} \{a-1\}^{\nu - \nu/a - f(\nu)}. \end{aligned}$$

Now it follows from Stirling's theorem that, if $s \geqslant 1$,

$$K \exp \{(s+\tfrac{1}{2}) \log(s) - s\} > s! > k \exp \{(s+\tfrac{1}{2}) \log(s) - s\},$$

where K and k are independent of s. Provided, therefore, that

$$\nu \geqslant 1, \quad \nu/a + f(\nu) \geqslant 1, \quad \nu - \nu/a - f(\nu) \geqslant 1,$$

we have

$$a^{-\nu} p(\nu, m) < M \nu^{-\frac{1}{2}} \exp \{ F[\nu, a, f(\nu)] \},$$

where M is independent of ν, a, and $f(\nu)$, and $F[\nu, a, f(\nu)]$ is a function which can easily be reduced to the form

$$\begin{aligned} F[\nu, a, f(\nu)] = - \{\nu/a\} \{ (1 + a\zeta) \log(1 + a\zeta) & \\ + (a - 1 - a\zeta) \log [1 - a\zeta/(a-1)] \} & \\ - \tfrac{1}{2} \log \{ (1/a + \zeta)(1 - 1/a - \zeta) \}, & \end{aligned}$$

---

* This Lemma is merely an adaptation of Lemma 1·444, *loc. cit.*, p. 185.

where $$\zeta = f(\nu)/\nu.$$

The coefficient of $-\nu/a$ in this expression may be written in the form

$$A = (1+x)\log(1+x)+(a-1-x)\log[1-x/(a-1)],$$

where $$x = a\zeta.$$

On differentiating A twice with respect to x, we find that

$$d^2A/dx^2 = a/\{(a-1-x)(1+x)\},$$

while A and dA/dx both vanish when $x = 0$. Therefore, provided

$$x < 1,$$

i.e., provided $$f(\nu)/\nu < 1/a,$$

we have $$d^2A/dx^2 > \tfrac{1}{2},$$

and $$A = \tfrac{1}{2}a^2\zeta^2\,[d^2A/dx^2]_{x=\theta a\zeta} \quad (0 < \theta < 1),$$

so that $$A > \tfrac{1}{4}a^2\zeta^2.$$

Hence it is easily deduced that

$$a^{-\nu}p(\nu, m) < M\{a/\nu\}^{\frac{1}{2}}\exp\{-\tfrac{1}{4}a\,[f(\nu)]^2/\nu\},$$

where M is a constant independent of a, ν, and $f(\nu)$, if the necessary provisions are complied with. Similar arguments can be applied to the terms for which

$$\mu(\nu) \leqslant -f(\nu),$$

provided $$\nu \geqslant 1, \quad \nu/a - f(\nu) \geqslant 1, \quad \nu - \nu/a + f(\nu) \geqslant 1.$$

All these conditions are covered by the conditions of the Lemma. Since the total number of terms $\leqslant \nu$, the truth of the Lemma has been established.

7. Lemma II leads at once to the following theorem :*—

THEOREM I.—*Any member of the set K', which is such that*

$$|\mu(\nu)| \nless \{4\tau\nu\log(\nu)/a\}^{\frac{1}{2}},$$

for all values of $\nu \geqslant n$, where τ is any number $> \frac{3}{2}$, can be enclosed in a

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* *Loc. cit.*, Theorem 1·45.

set of intervals whose total length is less than

$$Ma^{\frac{1}{2}}n^{\frac{3}{2}-\tau},$$

where M is a constant independent of a and n.

We omit the proof of this theorem, for it is a mere repetition of the proof of the corresponding theorem in Messrs. Hardy and Littlewood's paper.

From this theorem, it follows at once that the numbers θ between 0 and Λ, *or between* 0 *and* 1, which are such that their expressions in the form (4) do not satisfy the inequality

$$|\mu(\nu)| < \{4\tau\nu \log(\nu)/a\}^{\frac{1}{2}},$$

for *all* values of $\nu \geqslant n$, form a set of measure less than

$$Ma^{\frac{1}{2}}n^{\frac{3}{2}-\tau} \quad (\tau > \tfrac{3}{2}).$$

8. We have now to transform this result into a theorem about the distribution of the points $(\lambda_i \theta)$ over the interval (0, 1).

We have $$(\lambda_\nu \theta) = \frac{\chi_\nu}{a} + \lambda_\nu \sum_{i=\nu+1}^{\infty} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\},$$

so that $$0 < (\lambda_\nu \theta) - \chi_\nu/a < \lambda_\nu I_\nu < \gamma/a,$$

where γ is a constant depending only on β.

Now let δ be the length of any interval included in the interval (0, 1), and let a_1, a_2 be the numbers corresponding to its end-points. For any given value of a, we can choose integers p_1, p_2, q_1, q_2, such that

$$p_1/a \geqslant a_1, \qquad \{q_1+\gamma\}/a < a_1,$$

$$\{p_2+\gamma\}/a < a_2, \qquad q_2/a \geqslant a_2,$$

where p_1, q_2 are the least possible integers, and p_2, q_1 are the greatest possible integers satisfying their respective inequalities. It follows that

$$a\delta = p_2 - p_1 + O(1) = q_2 - q_1 + O(1).$$

Now, if $$p_1 \leqslant \chi_i \leqslant p_2,$$

the point $(\lambda_i \theta)$ must fall inside the interval δ, while if it is not true that

$$q_1 < \chi_i < q_2,$$

the point $(\lambda_i\theta)$ cannot lie in the interval δ. But if θ does not belong to any one of a set of intervals whose total length is less than

$$M\{q_2-q_1-1\}a^{\frac{1}{2}}n^{\frac{3}{2}-\tau},$$

and if B_ν be the number of times than any integer b, $(q_2 > b > q_1)$, occurs among the first ν χ's $(\nu \geqslant n)$, then B_ν satisfies the relation

$$B_\nu = \nu/a+\epsilon\{4\tau\nu\log(\nu)/a\}^{\frac{1}{2}} \quad (|\epsilon| < 1). \tag{11}$$

Let Δ_ν denote the number of the first ν members $(\lambda_i\theta)$ that fall inside the interval δ. Then it follows from (11) that, *if* θ *does not belong to any one of a set of intervals whose total length is less than*

$$Ma^{\frac{3}{2}}n^{\frac{3}{2}-\tau} \quad (\tau > \tfrac{3}{2}),$$

then

$$\Delta_\nu = \{a\delta+O(1)\}\{\nu/a+\epsilon[4\tau\nu\log(\nu)/a]^{\frac{1}{2}}\} \quad (|\epsilon| < 1). \tag{12}$$

We have so far been considering the sequence obtained by omitting n_1 terms from the original sequence, where

$$n_1 = O\{\log(a)\}. \tag{13}$$

Taking these terms into account, we see that we have established the truth of the following statement:—

If $\{\lambda_n\}$ *be any sequence of positive numbers such that for all values of* n,

$$\lambda_n/\lambda_{n-1} > \beta^n \quad (\beta > 1),$$

then, if θ *be any number between* 0 *and* 1 *which does not belong to a set of intervals whose total length is less than*

$$Ma^{\frac{3}{2}}n^{\frac{3}{2}-\tau},$$

it follows that

$$\Delta_\nu = \{a\delta+O(1)\}\{\nu/a+\epsilon[4\tau\nu\log(\nu)/a]^{\frac{1}{2}}\}+O\{a^{\frac{1}{2}}\log(a)\} \quad (|\epsilon| < 1), \tag{14}$$

for any value of $\nu \geqslant n$, *and for any value of* $n > H\log(a)$. *M, H, and the constants implied by the O's, depend solely upon* β.

We must now choose the relative order of n and a in such a way that the error term in the relation (14) is made as small as possible. The two most important terms are of orders n/a and $[an\log(n)]^{\frac{1}{2}}$, when $\nu = n$; to make these of the same order (we shall thus achieve the best result), we must take

$$a = g\{n/\log(n)\}^{\frac{1}{3}},$$

where g is a constant. This being so, let N be any large integer, and let

$$n = \nu = N,$$

$$a = [\{N/\log(N)\}^{\frac{1}{3}}].$$

The necessary conditions will all be satisfied if N is greater than a certain number n_0 depending only on β. Thus it appears that if θ does not belong to a set of intervals whose total length is less than

$$MN^{2-\tau},$$

then

$$\Delta_N = N + \epsilon[\tau N^{\frac{2}{3}}\{\log(N)\}^{\frac{1}{3}}] \quad (|\epsilon| < \eta), \tag{15}$$

for all values of $N \geqslant n_0$, where M, η, n_0 depend only upon β.

We may now repeat the arguments used to establish Theorem I,* and deduce therefrom the following theorem :—

THEOREM II.—*The numbers* θ *between* 0 *and* 1 *for which it is not true that*

$$\Delta_n = \delta n + \epsilon\tau n^{\frac{2}{3}}[\log(n)]^{\frac{1}{3}} \quad (|\epsilon| < \eta), \tag{16}$$

for all values of $n \geqslant N \geqslant n_0$, *can be enclosed in a set of intervals whose total length is less than*

$$MN^{3-\tau} \quad (\tau > 3).$$

M, η, *and* n_0 *depend only upon* β.

Theorem III, that follows, is an obvious deduction from Theorem II.

THEOREM III.—*If* $\{\lambda_n\}$ *is any sequence of positive numbers satisfying*

$$\lambda_n/\lambda_{n-1} \geqslant \beta^n \quad (\beta > 1),$$

for all values of $n \geqslant n_0$, *if* δ *is the length of any interval included in the interval* (0, 1), *if* θ *is any number between* 0 *and* 1, *and if* Δ_n *denotes the number of the first* n *numbers* $(\lambda_\nu\theta)$ *that fall inside the interval* δ, *then for almost all values of* θ,

$$\Delta_n - \delta n = O\{n^{\frac{2}{3}}[\log(n)]^{\frac{1}{3}}\}. \tag{17}$$

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* *Loc. cit.*, p. 186.

9. We now proceed to extend this result to less heavily restricted sequences. Let $\{\lambda_n\}$ be any sequence of positive numbers satisfying

$$\lambda_n/\lambda_{n-1} \geqslant \beta > 1,$$

for all values of n. [If necessary, a finite number of terms may be omitted from the beginning.] We split this sequence up into an infinite number of subsequences by placing in the first subsequence all terms whose suffices can be expressed in the form

$$\tfrac{1}{2}s\,(s+1) \quad (s \textit{ integral, and } \geqslant 1)\,;$$

in the second, all those whose suffices can be expressed in the form

$$1+\tfrac{1}{2}s\,(s+1) \quad (s \textit{ integral, and } \geqslant 1)\,;$$

and in the r-th ($r \geqslant 2$), all those whose suffices can be expressed in the form

$$r-1+\tfrac{1}{2}(s+r-2)(s+r-1) \quad (s \textit{ integral, and } \geqslant 1).$$

Let n be any integer, and N any integer satisfying

$$\tfrac{1}{2}n\,(n+1) \leqslant N < \tfrac{1}{2}\,(n+1)(n+2),$$

and let us consider the first N terms of the given sequence. Of these N terms, n terms will be regarded as belonging to the first subsequence, and, in general, $n-r+1$ terms to the r-th subsequence, while a certain number will be left unattached. The system of division is perhaps made clearer by the following diagram showing the terms in the first few subsequences.

Number of Subsequence.	Number of Terms taken.	Suffix Formula.	Suffices.
1	n	$\frac{1}{2}s\,(s+1)$	1, 3, 6, 10, ...
2	$n-1$	$1+\frac{1}{2}s\,(s+1)$	2, 4, 7, 11, ...
3	$n-2$	$2+\frac{1}{2}\,(s+1)(s+2)$	5, 8, 12, ...
4	$n-3$	$3+\frac{1}{2}\,(s+2)(s+3)$	9, 13, ...
⋮	⋮	⋮	⋮ ⋮ ⋮

Suppose we make use of all the subsequences which contain $[\sqrt{n}]$, or more, terms in the above scheme. The total number of such subsequences will be

$$n-[\sqrt{n}]+1,$$

and the total number of terms accounted for by them will be

$$\tfrac{1}{2}\{n+[\sqrt{n}]\}\{n-[\sqrt{n}]+1\},$$

or

$$\tfrac{1}{2}n^2+O(n).$$

There will therefore be at most $O(n)$ terms left unattached among the total number N.

If $\{\lambda_s\}$ denote any one of these subsequences, the terms of $\{\lambda_s\}$ satisfy

$$\lambda_s/\lambda_{s-1} \geqslant \beta^s \quad (\beta > 1),$$

for all values of s, *the value of* β *being the same for all the subsequences*. It follows from this that, if n, and so N, be chosen sufficiently large (*i.e.*, if $n \geqslant n_0$, where n_0 depends only upon β), Theorem II can be applied to each subsequence. The longest set of intervals of which exception must be made is the set belonging to the last subsequence, and its total length is less than

$$Mn^{\frac{1}{2}(3-\tau)};$$

the total number of such excepted sets is less than n. Consequently if θ does not belong to any one of a set of intervals whose total length is less than

$$Mn^{\frac{1}{2}(5-\tau)},$$

i.e.,

$$MN^{\frac{1}{4}(5-\tau)},$$

$$(18)\quad \Delta_N = \delta\left\{\sum_{r=0}^{n-[\sqrt{n}]}(n-r)\right\}+O(n)+\epsilon\tau\left\{\sum_{r=0}^{n-[\sqrt{n}]}(n-r)^{\frac{2}{3}}[\log(n-r)]^{\frac{1}{3}}\right\} \quad (|\epsilon|<\eta),$$

$$= \delta N+O(n)+\epsilon\tau n^{\frac{5}{3}}[\log(n)]^{\frac{1}{3}} \quad (|\epsilon|<\eta),$$

$$= \delta N+\epsilon\tau N^{\frac{5}{6}}[\log(N)]^{\frac{1}{3}} \quad (|\epsilon|<\eta).$$

Without further comment, we may enunciate the following extensions of Theorems II and III:—

THEOREM IV.—*If* $\{\lambda_n\}$ *is any sequence of positive numbers satisfying*

$$\lambda_n/\lambda_{n-1} \geqslant \beta > 1,$$

for all values of $n \geqslant n_0$, *then the numbers* θ *for which it is not true that*

$$\Delta_n = \delta n+\epsilon\tau n^{\frac{5}{6}}[\log(n)]^{\frac{1}{3}} \quad (|\epsilon|<\eta),$$

for all values of $n \geqslant N \geqslant n_1$, *can be enclosed in a set of intervals whose* *total length is less than*

$$MN^{\frac{1}{4}(9-\tau)} \quad (\tau > 9).$$

where M, η, *and* n_1 *depend only upon* β.

THEOREM V.—*If* $\{\lambda_n\}$ *is any sequence of positive numbers satisfying*

$$\lambda_n/\lambda_{n-1} \geqslant \beta > 1,$$

and the other conditions of Theorem III are unaltered, then for almost all values of θ *between* 0 *and* 1,

$$\Delta_n - \delta n = O\{n^{\frac{5}{6}}[\log(n)]^{\frac{1}{3}}\}. \tag{19}$$

10. In the preceding section, a particular subdivision of the given sequence has been worked out in detail in order to show how Theorems II and III may be extended to the less restricted sequences contemplated in Theorems IV and V. The use of a more powerful subdivision and the application of identical arguments enables us to extend Theorems IV and V (or Theorems II and III) to sequences which satisfy the inequalities

$$\lambda_n/\lambda_{n-1} \geqslant \beta^{n^{-1+\xi}} \quad (n \geqslant n_0,\ \xi > 0,\ \beta > 1).$$

Choose an integer k, so that

$$k\xi > 1,$$

and let n, N be large integers such that

$$\frac{n(n+1)\dots(n+k-1)}{k!} \leqslant N < \frac{(n+1)(n+2)\dots(n+k)}{k!}.$$

Then we divide up the original subsequence so that the first subsequence contains terms whose suffices are of the form

$$s(s+1)\dots(s+k-1)/k! \quad (s \text{ integral, and} \geqslant 1),$$

and take the first n terms of this subsequence, and so on as in the last section. The r-th subsequence will have suffices of the form

$$a_r + (s+\beta_r)(s+\beta_r+1)\dots(s+\beta_r+k-1)/k!,$$

where a_r, β_r are integers depending on r. If we take all the subsequences which have more than $[\sqrt{n}]$ terms, it may be proved that the number of terms taken will be

$$N + O(N^{1-\mu}),$$

where μ is a positive number depending on k. If λ_s and λ_{s-1} are consecutive terms of the r-th subsequence

$$\lambda_s/\lambda_{s-1} \geqslant \beta^{f(s)},$$

where $f(s) = \dfrac{(s+\beta_r)\ldots(s+\beta_r+k-2)}{(k-1)!}\left[\dfrac{k!}{a_r k!+(s+\beta_r)\ldots(s+\beta_r+k-1)}\right]^{1-\zeta}.$

It may be verified that, whatever be the value of r,

$$f(s) \geqslant (s+\beta_r)^{\frac{1}{2}(k\zeta-1)} \geqslant 1,$$

provided that at most a finite number of terms be omitted from the subsequence. We can therefore apply Theorems IV and V to each subsequence in turn, obtaining finally the following theorem :*—

THEOREM VI.—*If* $\{\lambda_n\}$ *be any sequence of positive numbers satisfying*

$$\lambda_n/\lambda_{n-1} \geqslant \beta^{n^{-1+\zeta}} \quad (\zeta > 0,\ \beta > 1),$$

and the other conditions of Theorem III are unaltered, then for almost all values of θ *between* 0 *and* 1,

$$\Delta_n - \delta n = O\{n^{1-\mu}\},$$

where μ *is some positive number depending on* ζ.

11. We shall conclude the paper by considering the extension of the foregoing results to the m-dimensional set of points whose m coordinates are

$$(\lambda_n\theta_1),\ (\lambda_n\theta_2),\ \ldots,\ (\lambda_n\theta_m).$$

The method† by which this extension is to be made will be sufficiently explained by considering the extension of the foregoing theorems to the 2-dimensional set of points

$$(\lambda_n\theta_1), \quad (\lambda_n\theta_2).$$

Let θ_1, θ_2 be expressed in the forms (§ 3),

$$\theta_1 = \sum_{i=1}^{\infty} \frac{1}{\lambda_i}\left\{\phi_i^{(1)} + \frac{\chi_i^{(1)}}{a} + x_i^{(1)}\right\},$$

$$\theta_2 = \sum_{i=1}^{\infty} \frac{1}{\lambda_i}\left\{\phi_i^{(2)} + \frac{\chi_i^{(2)}}{a} + x_i^{(2)}\right\}.$$

---

* We could also have proved the same result by a continued application of the particular subdivision $k = 2$. This is clearly equivalent to confining ourselves to values of $k = 2^{\alpha}$, where α is an integer.

† This is the method used by Messrs. Hardy and Littlewood to establish Theorem 1·491, of which the following theorem is the analogue.

We suppose that $$\lambda_i/\lambda_{i-1} \geqslant G\beta^i \quad (i \geqslant 2),$$

and that $$a = [\sqrt{G}].$$

The point of coordinates (θ_1, θ_2) will be any point in a square of side Λ ($\geqslant 1$). We correlate this point (θ_1, θ_2) with the point

$$\psi_{1,2} = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^2} \left\{ \phi_i^{(1)} \left\{ \left[\frac{\lambda_i}{a\lambda_{i-1}}\right] + 1 \right\} + \phi_i^{(2)} + \frac{\chi_i^{(1)} a + \chi_i^{(2)}}{a^2} \right\}.^{*}$$

It is easily verified that $\psi_{1,2}$ may be any point along a line of length Λ' ($\geqslant 1$). Moreover, a set of intervals on the line of $\psi_{1,2}$, of total length L, may be shown to correspond to a set of areas in the (θ_1, θ_2) plane of total area ML, where M is independent of the particular intervals and areas concerned.

We now apply Theorem I to show that if $B_\nu^{1,2}$ be the number of times that the particular combination $b^{(1)}a + b^{(2)}$ occurs in first ν terms of the expression for $\psi_{1,2}$, then

$$B_\nu^{1,2} = \nu/a^2 + \epsilon \left\{ 4\tau\nu \log(\nu)/a^2 \right\}^{\frac{1}{2}} \quad (|\epsilon| < 1),$$

for all $(\psi_{1,2})$'s which do not belong to a set of intervals of total length $Ma\nu^{\frac{3}{2}-\tau}$.

From this it follows at once that, if $\Delta_\nu^{1,2}$ be the number of the first ν points whose coordinates are $(\lambda_n\theta_1)$, $(\lambda_n\theta_2)$ that are included in any rectangle (included in the unit square) whose sides are of lengths δ_1, δ_2, then

$$\Delta_\nu^{1,2} = \left\{ a\delta_1 + O(1) \right\} \left\{ a\delta_2 + O(1) \right\} \left\{ \nu/a^2 + \epsilon \left[4\tau\nu \log(\nu)/a^2 \right]^{\frac{1}{2}} \right\} \quad (|\epsilon| < 1),$$

unless $\{\theta_1, \theta_2\}$ belongs to a set of points of plane measure less than

$$Ma^3\nu^{\frac{3}{2}-\tau}.$$

From this point on, all the arguments can be repeated as before, and the restriction

$$\lambda_i/\lambda_{i-1} \geqslant G\beta^i$$

lightened in the manner of § 10. It is, moreover, clear that the arguments can be extended equally well to the case of the m-dimensional set of points

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[*] There is no need to insert the term x_i, as we are not considering $(\lambda_i^2\psi_{1,2})$.

$\{\theta_1, \theta_2, \ldots, \theta_m\}$. We may, therefore, without further preface, enunciate our final theorem.

THEOREM VII.—*If* $\{\lambda_n\}$ *be any sequence of positive numbers satisfying*

$$\lambda_n/\lambda_{n-1} \geqslant \beta^{n^{-1+\zeta}} \quad (\zeta > 0,\ \beta > 1),$$

and if Δ_n *be the number of the first n points*

$$(\lambda_\nu\theta_1),\ (\lambda_\nu\theta_2),\ \ldots,\ (\lambda_\nu\theta_m),$$

which lie inside an m-dimensional "rectangle" of "area" δ, *then for almost all the points* $\{\theta_1, \theta_2, \ldots, \theta_m\}$,

$$\Delta_n - \delta n = O\{n^{1-\mu}\},$$

where μ *is some positive number depending on* ζ.

THE DYNAMICAL THEORY OF THE TIDES IN A ZONAL OCEAN

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1. *Introduction.*

This paper is an extension of one already published,[*] and deals with the tides in an ocean bounded by two circles of latitude on a rotating globe.

The introduction of two boundaries into the problem causes very great complexity; and it has been found necessary to omit the discussion of many points of interest, in particular, the nature and magnitude of the free periods of oscillation.

I am again much indebted to Dr. T. H. Havelock for his advice and assistance.

Originally, in discussing the results, I had made numerical comparisons between the dynamical tides and the uncorrected "equilibrium" tides. But, at the suggestion of a referee, to whom the paper was submitted, I have substituted the "corrected equilibrium" values. A remarkable agreement, exhibited and explained on p. 223 results. At his suggestion I have also added a note making a comparison between the dynamical tide heights found in the previous paper for a polar basin, and the corresponding "corrected equilibrium" values.

2. *The Equations for the Tides and their Solutions.*

The equations which express the small oscillations of a liquid on a rotating sphere are, in the usual notation,

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[*] *Proc. London Math. Soc.*, p. 31 of the present volume.

$$\left.\begin{aligned} u &= \frac{\sigma}{4m(f^2-\mu^2)}\left\{\sqrt{(1-\mu^2)}\,\frac{d\zeta'}{d\mu}-\frac{s}{f}\,\frac{\mu\zeta'}{\sqrt{(1-\mu^2)}}\right\} \\ v &= \frac{\sigma}{4m(f^2-\mu^2)}\left\{\frac{\mu}{f}\sqrt{(1-\mu^2)}\,\frac{d\zeta'}{d\mu}-\frac{s\zeta'}{\sqrt{(1-\mu^2)}}\right\} \\ \frac{\sigma a\zeta}{h} &= -\frac{d}{d\mu}\left\{u\sqrt{(1-\mu^2)}-\frac{sv}{\sqrt{(1-\mu^2)}}\right\} \end{aligned}\right\}. \tag{1}$$

In these we have taken out the time-and-longitude factor, and put $\mu = \cos\theta =$ sine of latitude.

The part of the tide-producing potential which depends upon μ is, for the three species :

$(a)\quad H_0(\mu^2-\tfrac{1}{3})$,

$(b)\quad H_1\mu\sqrt{(1-\mu^2)}$,

and $(c)\quad H_2(1-\mu^2)$.

The form of these quantities suggests the substitution

$$\zeta = (1-\mu^2)^{s/2}\,z,$$

$$\bar{\zeta} = (1-\mu^2)^{s/2}\,\bar{z},$$

and
$$\zeta' = (1-\mu^2)^{s/2}\,z',$$

where $s = 0$, 1 or 2, according to the species of tide.

The first pair of equations (1) then becomes

$$\left.\begin{aligned} u\sqrt{(1-\mu^2)} &= \frac{\sigma(1-\mu^2)^{s/2}}{4m(f^2-\mu^2)}\left\{(1-\mu^2)\frac{dz'}{d\mu}-s\mu z'\left(1+\frac{1}{f}\right)\right\} \\ \frac{v}{\sqrt{(1-\mu^2)}} &= \frac{\sigma(1-\mu^2)^{s/2}}{4m(f^2-\mu^2)}\left\{\frac{\mu}{f}(1-\mu^2)\frac{dz'}{d\mu}-sz'\left(1+\frac{\mu^2}{f}\right)\right\} \end{aligned}\right\}. \tag{2}$$

On substituting these in the third equation (1), we have

$$\beta z = \frac{d}{d\mu}\left[\frac{s\mu z'\left(1+\dfrac{1}{f}\right)-(1-\mu^2)\dfrac{dz'}{d\mu}}{f^2-\mu^2}\right]+\frac{s^2z'+s\mu\left(1-\dfrac{1}{f}\right)\dfrac{dz'}{d\mu}}{f^2-\mu^2}; \tag{3}$$

where
$$\beta = \frac{4ma}{h} = \frac{4\omega^2a^2}{gh}.$$

The coefficients of equation (3) are regular in the vicinity of $\mu = 0$;

but they have singularities at the points

$$\mu = \pm f,$$

$$\mu = \pm 1,$$

and $$\mu = \infty .$$

The points $\mu = \pm f$ may be shown to be "apparent singularities."* At the points $\mu = \pm 1$, we have indices 0 and 1; while the point $\mu = \infty$ does not appear in the problem before us.

In choosing the form of the solution, the nature of the zone must be considered. If the pole $\mu = +1$ is included, a series of the type

$$z = (\mu - 1)\left[A_0 + A_1(\mu-1) + A_2(\mu-1)^2 + \ldots\right]$$

would be most useful. As such a series would usually be convergent up to the other singularity $\mu = -1$, this would be a comprehensive form including all polar basins, even those covering more than a hemisphere. The period equation would be given by the expression which states that the meridional velocity is zero along a certain circle of latitude. For a zone bounded by two circles of latitude, however, this series is not convenient; as, since the indices differ by an integer, the companion integral will contain a logarithm. In spite of the obvious advantages of such a series in the case of a polar basin, no use has been made of it; for the recurrence relation between the coefficients is very complex and difficult to handle: even in the simplest case (the tides of the first species) this relation contains four terms.

If series of powers of μ be used, we get two comparatively simple integrals, one an odd and the other an even function of μ. These are convergent up to the singularities $\mu = \pm 1$, and so can be applied to any zone bounded by two circles of latitude. But if we wish to include the poles, a second and rather troublesome condition must be added. In what follows, series of the latter class alone are used; and the basins are, consequently, such as do not include the poles.

By putting, in (3),

$$s\mu z'\left(1 + \frac{1}{f}\right) - (1-\mu^2)\frac{dz'}{d\mu} = X(f^2 - \mu^2), \tag{4}$$

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[*] *Loc. cit.*, p. 34.

we find
$$\beta z = \frac{sX + \frac{s}{f^2}\frac{dz'}{d\mu}}{\mu\left(1+\frac{1}{f}\right)} + \frac{dX}{d\mu}. \tag{5}$$

Now assume
$$z' = \sum_0^\infty A_n \mu^n,$$

$$\bar{z} = \sum_0^\infty B_n \mu^n,$$

and
$$X = \sum_0^\infty C_n \mu^n.$$

We then find

$$\left\{ s+n\left(1+\frac{1}{f}\right)\right\} C_n = \left(1+\frac{1}{f}\right)\beta\left(A_{n-1}+B_{n-1}\right) - \frac{s}{f^2}(n+1)A_{n+1};$$

and, on substituting in (4) and eliminating the C's,

$$\begin{aligned} &A_{n+2}\frac{(n+1)(n+2)}{s+(n+1)(1+1/f)} \\ &\quad + A_n\left\{ -s - \frac{ns(1-1/f)+n(n-1)}{s+(n-1)(1+1/f)} + \frac{\beta f^2}{s+(n+1)(1+1/f)}\right\} \\ &\quad - A_{n-2}\frac{\beta}{s+(n-1)(1+1/f)} \\ &= -B_n\frac{\beta f^2}{s+(n+1)(1+1/f)} + B_{n-2}\frac{\beta}{s+(n-1)(1+1/f)}. \end{aligned} \tag{6}$$

If the ocean is bounded by circles of latitude, we must have, at each, $u=0$; or

$$\sum_{n=0}^{\infty}\left[\frac{\beta(1+1/f)(A_{n-1}+B_{n-1}) - s/f^2 (n+1) A_{n+1}}{s+n(1+1/f)}\right]\mu^n = 0. \tag{7}$$

But if the ocean extends up to and includes the pole, this condition is to be replaced by the statement that the series $\Sigma A_n \mu^n$ is convergent with its first and second derivatives for $\mu = \pm 1$.

The values of B entering into the tide-raising potential are confined to B_0, B_1, and B_2; hence, in discussing the convergence of the series

$\Sigma A_n \mu^n$, we may omit the right-hand side of (6). We then find

$$A_{n+2}/A_n = \left\{ s + \frac{ns(1-1/f)+n(n-1)}{s+(n-1)(1+1/f)} - \frac{\beta f^2}{s+(n+1)(1+1/f)} \right.$$

$$\left. - \frac{\beta}{s+(n-1)(1+1/f)} \frac{1}{A_n/A_{n-2}} \right\} \frac{s+(n+1)(1+1/f)}{(n+1)(n+2)}.$$

Whatever finite value f may have, it is clear that A_{n+2}/A_n tends to the limit 0 or 1. In the first event, the series $\Sigma A_n \mu^n$ is valid up to and including the poles. In the second we have, more precisely,

$$\operatorname*{Limit}_{m \to \infty} A_{2m+2}/A_{2m} = 1 - \frac{1}{m} + \frac{\omega_m}{m^2},$$

where ω_m is less than a fixed finite quantity for all values of m greater than a given value. Hence, in this case, the series is only valid so long as $|\mu| < 1$; that is, up to, but not at, the pole. It is evident then, that if we wish to determine a series that will express the height of the tide at the poles we must determine the arbitrary quantities so that the condition

$$\operatorname*{Limit}_{n \to \infty} A_{n+2}/A_n = 0, \tag{8}$$

is fulfilled.

It is further clear from (6) that two independent series are represented, viz.,

(i) $A_0 + A_2\mu^2 + A_4\mu^4 + \ldots + A_{2r}\mu^{2r} + \ldots,$

and (ii) $A_1\mu + A_3\mu^3 + A_5\mu^5 + \ldots + A_{2r+1}\mu^{2r+1} + \ldots.$

The zonal oceans that can be discussed by means of the preceding analysis fall into two sets which are best examined separately as follow:

(i) *A Zonal Ocean bounded by Two Parallels of Latitude Symmetrically Placed with regard to the Equator.*

For convenience, write

$$L_n = \left\{ -s - \frac{ns(1-1/f)+n(n-1)}{s+(n-1)(1+1/f)} + \frac{\beta f^2}{s+(n+1)(1+1/f)} \right\} \frac{s+(n+1)(1+1/f)}{(n+1)(n+2)},$$

$$M_n = \frac{\beta}{s+(n-1)(1+1/f)} \frac{s+(n+3)(1+1/f)}{(n+3)(n+4)},$$

and

$$a'_n = \frac{-s/f^2 \,.\, n\mu_1^{n-1}}{s+(n-1)(1+1/f)} + \frac{\beta(1+1/f)\mu_1^{n+1}}{s+(n+1)(1+1/f)}. \tag{9}$$

Also let the boundaries be the circles of latitude arc sin μ_1 and arc sin $(-\mu_1)$.

In seeking for comprehensive expressions that will represent the facts of all three tide-species, the differences of the three forms of tide-producing potential must be noted. For the first species B_0 and B_2 both appear, for the second B_1, and for the third B_0, only. The introduction of the two coefficients for the long period tides complicates the general process, so that a slight variation of the method, given later, has been found advisable. But for the second and third species, where only one coefficient appears, we shall give but one expression, and we shall suppose that only B_0 is present. To cover species 2, this will simply necessitate placing the additional quantities in the right-hand side of the equations for the odd terms instead of the right-hand side of the equations for the even terms. With this understanding the following analysis may be regarded as comprehending the two cases $s = 1$ and $s = 2$.

Let
$$B_0 = H_s = \kappa A_0.$$

Then from (6) and (7), the equations to be satisfied are

$$\left.\begin{array}{l}\left(\dfrac{\kappa\beta f^2}{2}+L_0\right)A_0 \qquad\qquad +A_2 = 0 \\ \left\{\dfrac{-\beta(s+3+3/f)\kappa}{(s+1+1/f)\,3.4}-M_0\right\}A_0+L_2A_2+A_4 = 0 \\ \qquad\qquad -M_2A_2+L_4A_4+A_6 = 0 \\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\end{array}\right\}, \tag{10a}$$

$$\left.\begin{array}{r}L_1A_1+A_3 = 0 \\ -M_1A_1+L_3A_3+A_5 = 0 \\ -M_3A_3+L_5A_5+A_7 = 0 \\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\end{array}\right\}, \tag{10b}$$

$$\left.\begin{array}{l}\left\{\dfrac{(1+1/f)\beta\kappa\mu_1}{s+1+1/f}+a_0'\right\}A_0+a_1'A_1+a_2'A_2+\ldots = 0 \\ \left\{-\dfrac{(1+1/f)\beta\kappa\mu_1}{s+1+1/f}-a_0'\right\}A_0+a_1'A_1-a_2'A_2+\ldots = 0\end{array}\right\}. \tag{10c}$$

This group of equations represents both the free and the forced oscillations. For, in the case of the former, $\kappa = 0$ and f can then be determined; in the case of the latter, f is known and κ (or A_0) can be determined.

It is easily seen that, as in the case of an ocean wholly covering the globe, there are two possible systems of free oscillations: the first is characterized by symmetry with regard to the equator and has a non-zero height there; the second is asymmetric with regard to the equator and has a zero height there.

By a process of step-by-step elimination of the quantities A from (10a) and (10c), we have, in the limit, the following infinite determinantal equation

$$\begin{vmatrix} \dfrac{(1+1/f)\beta\kappa\mu_1}{s+1+1/f}+a'_0 & a'_2 & a'_4 & a'_6 & \cdots \\ \dfrac{\kappa\beta f^2}{2}+L_0 & 1 & 0 & 0 & \cdots \\ -\dfrac{(s+3+3/f)\beta\kappa}{(s+1+1/f)3.4}-M_0 & L_2 & 1 & 0 & \cdots \\ 0 & -M_2 & L_4 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \qquad (11)$$

Taking advantage of the notation

$$\Delta_n \equiv \begin{vmatrix} a'_n & a'_{n+2} & a'_{n+4} & \cdots \\ L_n & 1 & 0 & \cdots \\ -M_n & L_{n+2} & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

to which may be added the relation between successive Δ's,

$$\Delta'_n = a'_n - L_n\Delta'_{n+2} + M_n\Delta'_{n+4};$$

we find the following reduced form of equation (11)

$$\Delta'_0 = -\frac{\beta\kappa\mu_1(1+1/f)}{s+1+1/f} + \frac{\beta f^2\kappa}{2}\Delta'_2 - \frac{(s+3+3/f)\beta\kappa}{(s+1+1/f)\,3.4}\Delta'_4,$$

or $$A_0\Delta'_0 = \left[-\frac{\beta\mu_1(1+1/f)}{s+1+1/f} + \frac{\beta f^2}{2}\Delta'_2 - \frac{(s+3+3/f)\beta}{(s+1+1/f)3.4}\Delta'_4\right]H_s. \qquad (12)$$

Equation (12) shows that the free periods are given by the transcendental

$$\Delta'_0 = 0.$$

A corresponding equation can be written down from (10b) and (10c) for the other system of free oscillations.

(ii) *A Zonal Sea bounded by Two Circles of Latitude not Symmetrically Placed with regard to the Equator.*

Let the circles of latitude be arc sin (μ_1) and arc sin (μ_2). The equations to be satisfied are then (10a), (10b), and

$$\left.\begin{aligned}\left\{\frac{(1+1/f)\beta\kappa\mu_1}{s+1+1/f}+a_0'\right\}A_0+a_1'A_1+a_2'A_2+\ldots&=0\\ \left\{\frac{(1+1/f)\beta\kappa\mu_2}{s+1+1/f}+a_0''\right\}A_0+a_1''A_1+a_2''A_2+\ldots&=0\end{aligned}\right\}. \tag{13}$$

By the same process as before, we find that, in order that (10a), (10b), and (13) may be consistent, we must have

$$A_0\begin{vmatrix}\dfrac{(1+1/f)\beta\kappa\mu_1}{s+1+1/f}+a_0' & a_2' & a_4' & a_6' & \ldots\\ \dfrac{\beta\kappa f^2}{2}+L_0 & 1 & 0 & 0 & \ldots\\ -\dfrac{(s+3+3/f)\beta\kappa}{(s+1+1/f)\,3.4}-M_0 & L_2 & 1 & 0 & \ldots\\ 0 & -M_2 & L_4 & 1 & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots\end{vmatrix}$$

$$+A_1\begin{vmatrix}a_1' & a_3' & a_5' & a_7' & \ldots\\ L_1 & 1 & 0 & 0 & \ldots\\ -M_1 & L_3 & 1 & 0 & \ldots\\ 0 & -M_3 & L_5 & 1 & \ldots\\ \ldots & \ldots & \ldots & \ldots & \end{vmatrix}=0; \tag{14}$$

together with a similar equation, in which μ_1 is replaced by μ_2.

In the same manner as before these reduce to

$$\left.\begin{aligned}A_0\Delta_0'+A_1\Delta_1'&=\left[-\frac{(1+1/f)\beta\mu_1}{s+1+1/f}+\frac{\beta f^2}{2}\Delta_2'-\frac{(s+3+3/f)\beta.\Delta_4'}{(s+1+1/f)\,3.4}\right]H_s\\ A_0\Delta_0''+A_1\Delta_1''&=\left[-\frac{(1+1/f)\beta\mu_2}{s+1+1/f}+\frac{\beta f^2}{2}\Delta_2''-\frac{(s+3+3/f)\beta.\Delta_4''}{(s+1+1/f)\,3.4}\right]H_s\end{aligned}\right\}. \tag{15}$$

For the free oscillations, the right-hand members d[illegible] remains a pair of simultaneous transcendental equations which determine f and the ratio A_1/A_0. There will be a double infinity of free periods. Equations (10a) and (10b) complete the solution by determining all the coefficients A in terms of A_0, which is left arbitrary. In the case of the forced oscillations, (15) determine A_0 and A_1 uniquely, and the solution is completed again in the same manner.

If one boundary be the equator itself, say $\mu_2 = 0$, we see from (9) that all the quantities α''_n, excepting α''_1, are zero; and

$$\alpha''_1 = -\frac{s/f^2}{s} = -\frac{1}{f^2}.$$

For such a basin, the period equations will be (14), and

$$A_0 \begin{vmatrix} 0 & 0 & 0 & \dots \\ L_0 & 1 & 0 & \dots \\ -M_0 & L_2 & 1 & \dots \\ \dots & \dots & \dots & \end{vmatrix} + A_1 \begin{vmatrix} -1/f^2 & 0 & 0 & \dots \\ L_1 & 1 & 0 & \dots \\ -M_1 & L_3 & 1 & \dots \\ \dots & \dots & \dots & \end{vmatrix} = 0. \qquad (16)$$

From (14) and (16) it is at once clear that $A_1 = 0$, and

$$\Delta'_0 = 0. \qquad (17)$$

Hence there is only a *single* system of free periods, and the corresponding oscillations are symmetrical. Equation (17) is precisely that already given for the free periods in Case (i) above. The effect of the boundary at the equator is, then, to cut out the asymmetrical system of oscillations.

Numerical examples have been worked out for two cases: a narrow basin bounded by circles of latitude 30° and 14° 30′, wholly on one side of the equator; and one bounded by circles of latitude 30° and −14° 30′, including the equator. From these a fair idea may be obtained of the effect of such boundaries in modifying the tides; and a comparison with the equilibrium and complete ocean theories can be made without involving hopelessly complex arithmetical work.

3. *Tides of the First Species.*

In the general solution worked out in § 2, it was assumed that the tide-producing potential contained only one term, and we took it to be B_0. In the case of the tides of the first species, however, the potential is $H_0(\mu^2 - \frac{1}{3})$, and hence two terms, B_0 and B_2, appear. The general method, with the two quantities B, becomes very complicated; but a slight modification, applicable to the first species only, will enable us to simplify it.

In equations (2) and (3), put $s = 0$. We then find

$$\frac{d}{d\mu}\left[\frac{(1-\mu^2)\dfrac{d\zeta'}{d\mu}}{f^2-\mu^2}\right]+\beta(\zeta'+\bar{\zeta}) = 0, \tag{18}$$

and

$$u = \frac{\sigma\sqrt{(1-\mu^2)}}{4m(f^2-\mu^2)}\,\frac{d\zeta'}{d\mu}. \tag{19}$$

If we now assume, in the vicinity of $\mu = 0$,

$$\frac{1}{f^2-\mu^2}\,\frac{d\zeta'}{d\mu} = \sum_{n=0}^{\infty} A_n\mu^n, \tag{20}$$

and substitute in (18), we find

$$\zeta' = D+f^2A_0\mu+f^2A_1\frac{\mu^2}{2}+\frac{f^2A_2-A_0}{3}\mu^3+\ldots+(f^2A_{n-1}-A_{n-3})\frac{\mu^n}{n}+\ldots; \tag{21}$$

and the general relation between the coefficients is

$$A_{n+1}+A_{n-1}\left(\frac{\beta f^2}{n(n+1)}-1\right)-\frac{\beta}{n(n+1)}A_{n-3} = -\frac{\beta}{n+1}B_n. \tag{22}$$

This holds for all values of n except $n = 0$, when the relation is

$$A_1+\beta D = -\beta B_0. \tag{23}$$

Since we are excluding the possibilities $\mu = \pm 1$, the boundary conditions are, from (19),

$$\Sigma A_n\mu_1^n = 0 \quad\text{and}\quad \Sigma A_n\mu_2^n = 0.$$

If we use the notation of § 2, but note that now

$$L_n = \frac{\beta f^2}{(n+1)(n+2)}-1 \quad\text{and}\quad M_n = \frac{\beta}{(n+3)(n+4)},$$

and form the determinants as before, we find

$$\left.\begin{aligned} A_1\Delta_1'+A_0\Delta_0' &= \frac{\beta}{3}H_0\Delta_3' \\ \text{and}\qquad A_1\Delta_1''+A_0\Delta_0'' &= \frac{\beta}{3}H_0\Delta_3'' \end{aligned}\right\}. \tag{24}$$

For the free oscillations we reject the right-hand members of (24), and solve the pair of simultaneous transcendental equations for f and A_1/A_0. It hardly seems possible to do this with any satisfaction. When we deal with the narrow basin ($\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{4}$), the values of f are very large, and even a rough approximation is difficult to obtain. In the other example chosen, however, ($\mu_1 = \frac{1}{2}$, $\mu_2 = -\frac{1}{4}$), the values of f are smaller and some idea of their order of value may be obtained by rejecting the fourth and higher powers of μ_1, and the third and higher powers of μ_2.

For the forced oscillations, A_1 and A_0 are determined from (24). It is to be noted that the determinants required are readily found from the recurrence relation

$$\Delta'_n = \mu_1^n - L_n \Delta'_{n+2} + M_n \Delta'_{n+4}.$$

Afterwards (22) and (23) enable us to find D and the remaining coefficients A; and, finally, the value of $\zeta, = \zeta' + \bar{\zeta}$, is determined from (21).

(i) *Zonal Ocean bounded by Circles of Latitude* 14°30′ *and* 30°.

The following values of the first (and lowest) frequency were obtained by the method just indicated. They must be regarded as very rough approximations indeed.

It is to be noted that $\beta f^2 \left(= \frac{4\omega^2 a^2}{gh} \frac{\sigma^2}{4\omega^2}\right)$ is independent of ω, and as M is not involved in the approximation we are using, these values of the frequencies are obtained without reference to the rotation. This in itself, however, will not greatly vitiate the results, as the effect of the rotation is known to be small in low latitudes, reducing to zero at the equator.

The value we obtain is $\beta f^2 = 54$. Hence

$$\beta = \quad 5, \quad 10, \quad 20, \quad 40;$$

$$\text{Depth} = 58{,}080, \; 29{,}040, \; 14{,}520, \; 7{,}260 \text{ feet};$$

$$f\left(= \frac{\sigma}{2\omega}\right) = \quad 3{\cdot}3, \quad 2{\cdot}3, \quad 1{\cdot}6, \quad 1{\cdot}2.$$

As an example of a forced tide we take the case of the lunar fortnightly, for which $f^2 = {\cdot}00133$.

The expressions for the lunar fortnightly tides in the same ocean are the following; each expression is to be multiplied by the factor $\cos(\sigma t + \epsilon)$.

$\beta = 5$; *depth* $= 58{,}080$ *feet* :

$$\zeta/H_0 = -{\cdot}146 - {\cdot}0002\mu + 1{\cdot}000\mu^2 + {\cdot}0518\mu^3 - {\cdot}182\mu^4 + {\cdot}0310\mu^5 + {\cdot}157\mu^6$$
$$+ {\cdot}0317\mu^7 + {\cdot}0946\mu^8 + {\cdot}0273\mu^9 + {\cdot}0869\mu^{10} \ldots .$$

$\beta = 10$; *depth* = 29,040 *feet* :

$$\zeta/H_0 = -{\cdot}146-{\cdot}0005\mu+1{\cdot}001\mu^2+{\cdot}104\mu^3-{\cdot}365\mu^4+{\cdot}0621\mu^5+{\cdot}313\mu^6$$
$$+{\cdot}0799\mu^7+{\cdot}1437\mu^8+{\cdot}0737\mu^9+{\cdot}1598\mu^{10}\ldots.$$

$\beta = 20$; *depth* = 14,520 *feet* :

$$\zeta/H_0 = -{\cdot}146-{\cdot}0009\mu+1{\cdot}002\mu^2+{\cdot}207\mu^3-{\cdot}725\mu^4+{\cdot}1211\mu^5+{\cdot}629\mu^6$$
$$+{\cdot}236\mu^7+{\cdot}1045\mu^8+{\cdot}229\mu^9+{\cdot}263\mu^{10}\ldots.$$

$\beta = 40$; *depth* = 7,260 *feet* :

$$\zeta/H_0 = -{\cdot}147-{\cdot}0017\mu+1{\cdot}004\mu^2+{\cdot}417\mu^3-1{\cdot}456\mu^4+{\cdot}246\mu^5+1{\cdot}385\mu^6$$
$$+{\cdot}7702\mu^7-{\cdot}418\mu^8+{\cdot}782\mu^9+{\cdot}458\mu^{10}\ldots.$$

(ii) *Zonal Ocean bounded by Circles of Latitude* 30° *and* −14° 30′.

The approximate frequency equation is

$$\beta f^2 = 14{\cdot}3.$$

Hence we have the following rough values of the lowest frequencies for the different depths :

$\beta =$	5,	10,	20,	40;
Depth =	58,080,	29,040,	14,520,	7,260 feet;
$f\left(=\frac{\sigma}{2\omega}\right) =$	$1{\cdot}7$,	$1{\cdot}2$,	${\cdot}85$,	${\cdot}60$.

It is at once obvious that there is no possibility of synchronism of the lunar fortnightly tide with any of the free tides.

Expressions for the lunar fortnightly tides in the same zonal ocean the series must each be multiplied by the factor $\cos(\sigma t+\epsilon)$.

$\beta = 5$; *depth* = 58,080 *feet* :

$$\zeta/H_0 = -{\cdot}0619+{\cdot}0001\mu+1{\cdot}0002\mu^2-{\cdot}0171\mu^3-{\cdot}0779\mu^4-{\cdot}0102\mu^5$$
$$+{\cdot}226\mu^6-{\cdot}0103\mu^7+{\cdot}160\mu^8-{\cdot}009\mu^9+{\cdot}143\mu^{10}\ldots.$$

$\beta = 10$; *depth* = 29,040 *feet* :

$$\zeta/H_0 = -{\cdot}0613+{\cdot}0001\mu+1{\cdot}0004\mu^2-{\cdot}0336\mu^3-{\cdot}155\mu^4-{\cdot}0202\mu^5$$
$$+{\cdot}449\mu^6-{\cdot}0279\mu^7+{\cdot}302\mu^8-{\cdot}0257\mu^9+{\cdot}306\mu^{10}\ldots.$$

$\beta = 20$; *depth* = 14,520 *feet* :

$$\zeta/H_0 = -\cdot 0601 + \cdot 0002\mu + 1\cdot 0007\mu^2 - \cdot 0655\mu^3 - \cdot 302\mu^4 - \cdot 0389\mu^5$$
$$+ \cdot 911\mu^6 - \cdot 0747\mu^7 + \cdot 546\mu^8 - \cdot 0724\mu^9 + \cdot 688\mu^{10} \ldots .$$

$\beta = 40$; *depth* = 7,260 *feet* :

$$\zeta/H_0 = -\cdot 0578 + \cdot 0005\mu + 1\cdot 0014\mu^2 - \cdot 123\mu^3 - \cdot 581\mu^4 - \cdot 0718\mu^5$$
$$+ 1\cdot 84\mu^6 - \cdot 228\mu^7 + \cdot 796\mu^8 - \cdot 231\mu^9 + 1\cdot 72\mu^{10} \ldots .$$

4. *Tides of Second Species.*

In preparing the formulæ for these, we must bear in mind that the general formulæ previously given must be modified by the transference of the tide-producing terms into the odd, instead of the even coefficients.

If in the quantities of (9) we put $s = 1$, and take the tide-producing potential as containing one term only $B_1, = H_1$, we find, in place of (10a) and (10b), after putting $H_1 = \kappa A_1$,

$$\left.\begin{array}{l} A_0 L_0 + A_2 \qquad\qquad = 0 \\ -A_0 M_0 + A_2 L_2 + A_4 = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\}, \tag{25a}$$

$$\left.\begin{array}{l} A_1\left(L_1 + \dfrac{\beta\kappa}{24}\right) + A_3 \qquad\qquad\qquad = 0 \\ A_1\left(-M_1 - \dfrac{\beta\kappa . 13}{140}\right) + L_3 A_3 + A_5 = 0 \\ -A_3 M_3 \qquad\qquad\qquad + L_5 A_5 + A_7 = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\}. \tag{25b}$$

The boundary condition (7) also becomes

$$\Sigma a_n' A_n = -\tfrac{3}{7}\beta H_1 \mu_1^2 ; \tag{26}$$

where $$a_n' = \frac{-1/f^2 n}{1 + (n-1)(1+1/f)}\mu_1^{n-1} + \frac{\beta(1+1/f)}{1+(n+1)(1+1/f)}\mu_1^{n+1}.$$

In place of equations (15), we find

$$\left.\begin{array}{l} A_0 \Delta_0' + A_1 \Delta_1' = H_1\left(-\dfrac{3\beta}{7}\mu_1^2 + \dfrac{\beta}{24}\Delta_3' - \dfrac{13\beta}{140}\Delta_5'\right) \\ A_0 \Delta_0'' + A_1 \Delta_1'' = H_1\left(-\dfrac{3\beta}{7}\mu_2^2 + \dfrac{\beta}{24}\Delta_3'' - \dfrac{13\beta}{140}\Delta_5''\right) \end{array}\right\}. \tag{27}$$

These equations are treated in precisely the same manner as those of the first species.

For the forced tides, we have taken as typical the luni-solar diurnal for which f is rigorously $\frac{1}{2}$. The numerical work is also simplified thereby. It is to be noticed that equations (25a) and (25b) are then satisfied by putting

$$A_0 = A_2 = A_4 = \ldots = A_3 = A_5 = \ldots = 0, \quad \text{and} \quad \kappa = -1;$$

for then both $L_1 + \dfrac{\beta\kappa}{24}$ and $M_1 + \dfrac{\beta\kappa . 13}{140}$ equal zero. Hence there is a solution

$$z' = -A_1\mu = -H_1\mu,$$

or

$$\zeta' = z'\sqrt{(1-\mu^2)} = -H_1\mu\sqrt{(1-\mu^2)};$$

and

$$\zeta = \zeta' + \bar{\zeta} = 0.$$

This is Laplace's celebrated theorem that the luni-solar diurnal tide is evanescent when the ocean is of uniform depth and covers the whole globe. It is to be noted, however, that the condition (26) is not satisfied by this solution.

For the free periods we reject the right-hand side of (27) and eliminate A_0 and A_1. The result is

$$\Delta_0' \Delta_1'' - \Delta_1' \Delta_0'' = 0.$$

The complicated way in which f enters into each of these determinants makes it hopeless to attempt to find even rough values of the free periods. As shown later on, all that can be done is to ascertain approximately those critical depths at which a free mode coincides with a selected forced mode.

The forced tides are, however, not so difficult to obtain.

(i) *Zonal Ocean bounded by Circles of Latitude* 30° *and* 14°20′.

Luni-solar diurnal tides; each expression must be multiplied by the factor $\cos(\sigma t + \phi + \epsilon)$.

$\beta = 5$; *depth* = 58,080 *feet* :

$$\zeta/H_1 = [-7{\cdot}26 - 9{\cdot}99\mu^2 - 14{\cdot}8\mu^4 - 16{\cdot}3\mu^6 - 17{\cdot}5\mu^8 - 18{\cdot}2\mu^{10} + \ldots$$
$$+ {\cdot}756\mu - {\cdot}1575\mu^3 + {\cdot}202\mu^5 + {\cdot}170\mu^7 + {\cdot}185\mu^9 \ldots]\sqrt{(1-\mu^2)}.$$

$\beta = 10$; *depth* = 29,040 *feet* :

$$\zeta/H_1 = [-{\cdot}5845 - {\cdot}438\mu^2 - 1{\cdot}49\mu^4 - 1{\cdot}54\mu^6 - 1{\cdot}82\mu^8 - 1{\cdot}97\mu^{10} + \ldots$$
$$+ {\cdot}858\mu - {\cdot}358\mu^3 + {\cdot}510\mu^5 + {\cdot}339\mu^7 + {\cdot}417\mu^9 \ldots]\sqrt{(1-\mu^2)}.$$

$\beta = 20$; *depth* $= 14{,}520$ *feet* :

$$\begin{aligned}\zeta/H_1 = [&-{\cdot}411+{\cdot}206\mu^2-1{\cdot}63\mu^4-1{\cdot}04\mu^6-1{\cdot}75\mu^8-1{\cdot}93\mu^{10}+\ldots\\ &+{\cdot}661\mu-{\cdot}553\mu^3+{\cdot}848\mu^5+{\cdot}322\mu^7+{\cdot}608\mu^9\ldots]\sqrt{(1-\mu^2)}.\end{aligned}$$

$\beta = 40$; *depth* $= 7{,}260$ *feet* :

$$\begin{aligned}\zeta/H_1 = [&-{\cdot}264+{\cdot}793\mu^2-2{\cdot}20\mu^4+{\cdot}228\mu^6-1{\cdot}98\mu^8-1{\cdot}61\mu^{10}+\ldots\\ &+{\cdot}395\mu-{\cdot}639\mu^3+1{\cdot}19\mu^5-{\cdot}0415\mu^7+{\cdot}833\mu^9\ldots]\sqrt{(1-\mu^2)}.\end{aligned}$$

(ii) *Zonal Ocean bounded by Circles of Latitude* 30° *and* $-14° 30'$.

Luni-solar diurnal tides: each expression must be multiplied by the factor $\cos(\sigma t+\phi+\epsilon)$.

$\beta = 5$; *depth* $58{,}080$ *feet* :

$$\begin{aligned}\zeta/H_1 = [&-1{\cdot}33-1{\cdot}83\mu^2-2{\cdot}72\mu^4-3{\cdot}24\mu^6-3{\cdot}45\mu^8-3{\cdot}59\mu^{10}+\ldots\\ &+1{\cdot}094\mu-{\cdot}229\mu^3+{\cdot}310\mu^5+{\cdot}131\mu^7+{\cdot}155\mu^9\ldots]\sqrt{(1-\mu^2)}.\end{aligned}$$

$\beta = 10$; *depth* $29{,}040$ *feet* :

$$\begin{aligned}\zeta/H_1 = [&-{\cdot}241-{\cdot}181\mu^2-{\cdot}569\mu^4-{\cdot}595\mu^6-{\cdot}699\mu^8-{\cdot}759\mu^{10}+\ldots\\ &+1{\cdot}184\mu-{\cdot}493\mu^3+{\cdot}703\mu^5+{\cdot}470\mu^7+{\cdot}542\mu^9\ldots]\sqrt{(1-\mu^2)}.\end{aligned}$$

$\beta = 20$; *depth* $14{,}520$ *feet* :

$$\begin{aligned}\zeta/H_1 = [&-{\cdot}196+{\cdot}0980\mu^2-{\cdot}344\mu^4-{\cdot}169\mu^6-{\cdot}323\mu^8-{\cdot}350\mu^{10}+\ldots\\ &+1{\cdot}472\mu-1{\cdot}23\mu^3+1{\cdot}76\mu^5+{\cdot}628\mu^7+1{\cdot}204\mu^9\ldots]\sqrt{(1-\mu^2)}.\end{aligned}$$

$\beta = 40$; *depth* $7{,}260$ *feet* :

$$\begin{aligned}\zeta/H_1 = [&-{\cdot}258+{\cdot}774\mu^2-2{\cdot}15\mu^4+{\cdot}305\mu^6-1{\cdot}904\mu^8-1{\cdot}50\mu^{10}\ldots\\ &+2{\cdot}914\mu-4{\cdot}86\mu^3+8{\cdot}74\mu^5-{\cdot}502\mu^7+5{\cdot}96\mu^9\ldots]\sqrt{(1-\mu^2)}.\end{aligned}$$

5. *Tides of Third Species.*

The formulæ already found, viz., (10*a*), (10*b*), (13) and (15), are ready for use in the case of the third species, if we put $s = 2$. Only one term appears in the tide-producing potential, viz., $B_0 = H_2$. The procedure is in every case precisely the same as before.

As a typical example of a forced tide of this species I have worked out the luni-solar semi-diurnal tide for the two oceans selected. For this tide

f is rigorously unity. The other semi-diurnal tides have values of f not differing much from this.

As in the case of the tides of the second species, it was found impossible to obtain even a rough estimate of the free periods.

(i) *Zonal Ocean bounded by Circles of Latitude* 30° *and* 14° 30′.

Luni-solar semi-diurnal tide; each expression must be multiplied by the factor $\cos(\sigma t+2\phi+\epsilon)$.

$\beta = 5$; *depth* = 58,080 *feet*:

$$\begin{aligned}\zeta/H_2 = [&\cdot849-2\cdot58\mu^2-2\cdot53\mu^4-3\cdot76\mu^6-4\cdot80\mu^8-5\cdot86\mu^{10}+\ldots\\&+2\cdot53\mu+2\cdot95\mu^3+4\cdot74\mu^5+6\cdot10\mu^7+7\cdot46\mu^9\ldots](1-\mu^2).\end{aligned}$$

$\beta = 10$; *depth* = 29,040 *feet*:

$$\begin{aligned}\zeta/H_2 = [&-\cdot554-1\cdot89\mu^2-2\cdot50\mu^4-3\cdot61\mu^6-4\cdot59\mu^8-5\cdot61\mu^{10}+\ldots\\&-\cdot311\mu-\cdot104\mu^3-\cdot363\mu^5-\cdot433\mu^7-\cdot545\mu^9\ldots](1-\mu^2).\end{aligned}$$

$\beta = 20$; *depth* = 14,520 *feet*:

$$\begin{aligned}\zeta/H_2 = [&-0\cdot042-2\cdot71\mu^2-1\cdot40\mu^4-2\cdot81\mu^6-2\cdot67\mu^8-3\cdot45\mu^{10}+\ldots\\&+\cdot668\mu-\cdot890\mu^3+1\cdot56\mu^5+\cdot769\mu^7+1\cdot30\mu^9\ldots](1-\mu^2).\end{aligned}$$

$\beta = 40$; *depth* = 7,260 *feet*:

$$\begin{aligned}\zeta/H_2 = [&-\cdot030+2\cdot49\mu^2-4\cdot35\mu^4+2\cdot20\mu^6-2\cdot89\mu^8-1\cdot02\mu^{10}+\ldots\\&+1\cdot47\mu-6\cdot87\mu^3+8\cdot35\mu^5-5\cdot98\mu^7+1\cdot81\mu^9\ldots](1-\mu^2).\end{aligned}$$

(ii) *Zonal Ocean bounded by Circles of Latitude* 30° *and* −14° 30′.

Luni-solar semi-diurnal tides; each expression must be multiplied by the factor $\cos(\sigma t+2\phi+\epsilon)$.

$\beta = 5$; *depth* = 58,080 *feet*:

$$\begin{aligned}\zeta/H_2 = [&-3\cdot52-4\cdot76\mu^2-8\cdot89\mu^4-12\cdot15\mu^6-15\cdot59\mu^8-19\cdot03\mu^{10}-\ldots\\&-1\cdot51\mu-1\cdot76\mu^3-2\cdot83\mu^5-3\cdot74\mu^7-4\cdot67\mu^9\ldots](1-\mu^2).\end{aligned}$$

$\beta = 10$; *depth* = 29,040 *feet*:

$$\begin{aligned}\zeta/H_2 = [&-\cdot509-1\cdot98\mu^2-2\cdot50\mu^4-3\cdot66\mu^6-4\cdot64\mu^8-5\cdot67\mu^{10}+\ldots\\&+\cdot215\mu+\cdot0718\mu^3+\cdot251\mu^5+\cdot299\mu^7+\cdot377\mu^9\ldots](1-\mu^2).\end{aligned}$$

$\beta = 20$; *depth* = 14,520 *feet* :

$$\zeta/H_2 = [-1{\cdot}818+9{\cdot}73\mu^2-6{\cdot}06\mu^4+5{\cdot}28\mu^6-2{\cdot}019\mu^8+3{\cdot}49\mu^{10}+\ldots$$
$$+11{\cdot}4\mu-15{\cdot}2\mu^3+11{\cdot}4\mu^5+{\cdot}500\mu^7+4{\cdot}56\mu^9\ldots](1-\mu^2).$$

$\beta = 40$; *depth* = 7,260 *feet* :

$$\zeta/H_2 = [-{\cdot}048-2{\cdot}23\mu^2+{\cdot}083\mu^4-4{\cdot}45\mu^6-2{\cdot}47\mu^8-4{\cdot}39\mu^{10}+\ldots$$
$$+1{\cdot}48\mu-6{\cdot}91\mu^3+8{\cdot}39\mu^5-6{\cdot}02\mu^7+1{\cdot}82\mu^9\ldots](1-\mu^2).$$

6. *Discussion of Results.*

(i) The fortnightly tide.

We shall compare the numerical values of the tide-heights found in the preceding sections with the theory of an ocean wholly covering the globe,* and with the "corrected equilibrium" theory. For the latter we proceed to find the correction to be applied for such a zonal ocean as we are discussing, of which the boundaries are μ_1 and μ_2. The complete expression for the "equilibrium" tide is

$$\bar{\zeta} = H_0(\mu^2-\tfrac{1}{3})\cos(\sigma_0 t+\epsilon_0)+C.$$

But over the whole zone we must have

$$\iint \bar{\zeta}\, dS = 0,$$

where dS is an element of surface.

On performing the integration we find

$$C = -\frac{H_0}{3}(\mu_2^2+\mu_2\mu_1+\mu_1^2-1)\cos(\sigma_0 t+\epsilon_0).$$

The entries in the table are calculated from these expressions.

TABLE I. *Ocean bounded by Circles of Latitude* 30° *and* 14° 30′.

	$\beta = 5$		$\beta = 10$		$\beta = 20$		$\beta = 40$	
	Latitudes.							
	30°	14° 13′	30°	14° 30′	30°	14 30′	30°	14° 30′
Present Theory	·103	−·084	·102	−·083	·100	−·083	·092	−·083
Complete Ocean Theory	—	—	−·020	−·176	—	—	·021	−·095
Corrected Equilibrium Theory ..	·104	−·083	·104	−·083	·104	−·083	·104	−·083

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* Darwin, *Proc. Roy. Soc.*, Vol. XLI (1886), p. 337. (Calculated for $\beta = 10$ and 40 only.)

TABLE II. *Ocean bounded by Circles of Latitude* 30° *and* −14° 30′.

	$\beta = 5$		$\beta = 10$		$\beta = 20$		$\beta = 40$	
	Latitudes.							
	30°	14° 30′	30°	14° 30′	30°	14° 30′	30°	14° 30′
Present Theory	·184	·0006	·183	·0012	·178	·0026	·170	·0048
Complete Ocean Theory	—	—	−·020	−·176	—	—	·021	−·095
Corrected Equilibrium Theory ...	·1875	zero	·1875	zero	·1875	zero	·1875	zero

(Each figure in the entries to be multiplied by H_0.)

The important fact that stands out from these tables is the close approximation of the "corrected equilibrium" tide to the true dynamical tide. This may also be exhibited analytically as follows.

From equations (1) we find for the tide of first species

$$\left.\begin{aligned} u &= \frac{\sigma}{4m(f^2-\mu^2)}\sqrt{(1-\mu^2)}\left\{\frac{d\zeta}{d\mu}-\frac{d\bar{\zeta}}{d\mu}\right\} \\ \frac{\sigma a\zeta}{h} &= -\frac{d}{d\mu}\left\{u\sqrt{(1-\mu^2)}\right\} \end{aligned}\right\}. \tag{28}$$

From these we find

$$\frac{h}{4ma}\frac{d^2}{d\mu^2}u\sqrt{(1-\mu^2)} = \frac{-u(f^2-\mu^2)}{\sqrt{(1-\mu^2)}}-\frac{\sigma}{4m}\frac{d\bar{\zeta}}{d\mu}.$$

On integrating,

$$\frac{h}{4ma}u\sqrt{(1-\mu^2)}+\int d\mu\int d\mu\left\{\frac{u(f^2-\mu^2)}{\sqrt{(1-\mu^2)}}\right\} = -\frac{\sigma}{4m}\int\bar{\zeta}\,d\mu+E_1\mu+E_2.$$

If U be the maximum value of $|u|$ in the zone under consideration, then

$$\int d\mu\int d\mu\left\{\frac{u(f^2-\mu^2)}{\sqrt{(1-\mu^2)}}\right\} \leqslant U\int d\mu\int d\mu\left\{\frac{(f^2-\mu^2)}{\sqrt{(1-\mu^2)}}\right\}$$

$$\leqslant U\left\{\frac{f^2\mu^2}{2}+\left(\frac{f^2}{2}-1\right)\frac{\mu^4}{12}+\dots\right\}.$$

Now by the original supposition of infinitely small motions upon which equations (1) were based, U is a quantity whose square may be neglected.

If then f^4 and μ^4 be at least of the same order as U, the term we are discussing may be neglected in comparison with that preceding it. Hence

$$\frac{h}{4ma} u\sqrt{(1-\mu^2)} = -\frac{\sigma}{4m}\int \bar{\zeta}\,d\mu + E_1\mu + E_2.$$

On applying the conditions $u = 0$ at $\mu = \mu_1$ and $\mu = \mu_2$, we have

$$E_2 = 0,$$

and

$$E_1 = \frac{\sigma}{4m(\mu_2-\mu_1)}\int_{\mu_1}^{\mu_2} \bar{\zeta}\,du.$$

Finally, by the second of equations (28),

$$\zeta = \bar{\zeta} - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \bar{\zeta}\,d\mu.$$

This is rigorously the expression for the "corrected equilibrium" tide height.

The term we have neglected was compared with the term

$$\frac{h}{4ma} u\sqrt{(1-\mu^2)},$$

and we tacitly assumed that the fraction $\dfrac{h}{4ma}$ was not small enough to reduce the order of the term. If, however, h be reduced greatly in value, the approximation we have found will be less satisfactory. This is the explanation of the fact that in the table the agreement of the theories is less close as β increases, that is, as h decreases.

(ii) Diurnal tide.

The tables give the tide heights at the corresponding latitudes.

TABLE III. *Ocean bounded by Circles of Latitude* 30° *and* 14° 30′.

	= 5		$\beta = 10$		$\beta = 20$		$\beta = 40$	
	Latitudes.							
	30°	14° 30′	30°	14° 30′	30°	14° 30′	30°	14° 30′
Present Theory...	$-9{\cdot}23H_1$	$-7{\cdot}51H_1$	$-{\cdot}364H_1$	$-{\cdot}396H_1$	$-{\cdot}168H_1$	$-{\cdot}240H_1$	$-{\cdot}045H_1$	$-{\cdot}129H_1$
Equilibrium Theory* ...	$+{\cdot}433H_1$	$+{\cdot}242H_1$	$+{\cdot}433H_1$	$+{\cdot}242H_1$	$+{\cdot}433H_1$	$+{\cdot}242H_1$	$+{\cdot}433H_1$	$+{\cdot}242H_1$

* The correction to the "equilibrium" theory in this case is zero.

TABLE IV. *Ocean bounded by Circles of Latitude* 30° *and* −14° 30′.

	$\beta = 5$		$\beta = 10$		$\beta = 20$		$\beta = 40$	
	Latitudes.							
	30°	−14° 30′	30°	−14° 30′	30°	−14° 30′	30°	−14° 30′
Present Theory...	$-1{\cdot}30 H_1$	$-1{\cdot}67 H_1$	$+{\cdot}193 H_1$	$-{\cdot}532 H_1$	$+{\cdot}475 H_1$	$-{\cdot}524 H_1$	$+{\cdot}843 H_1$	$-{\cdot}843 H_1$
Equilibrium Theory* ...	$+{\cdot}433 H_1$	$-{\cdot}242 H_1$	$+{\cdot}433 H_1$	$-{\cdot}242 H_1$	$+{\cdot}433 H_1$	$-{\cdot}242 H_1$	$+{\cdot}433 H_1$	$-{\cdot}242 H_1$

The luni-solar diurnal tide is, as already mentioned, evanescent in the complete ocean theory.

In Table III, the high magnitudes of the tide heights when $\beta = 5$ are due to approximate synchronism. While the complexity of the period equation makes it almost impossible to get even an approximate idea of the values of the free periods, yet with a little trouble we may find rough values of β (or h) which would cause synchronism between the free oscillations and the forced oscillations of a given character. So that, if in the equations [derived from (27)]

$$\left.\begin{aligned} A_0\Delta_0'+A_1\Delta_1' &= 0 \\ A_0\Delta_0''+A_1\Delta_0'' &= 0 \end{aligned}\right\}, \tag{29}$$

we substitute the suitable values for μ_1 and μ_2, put $f = \frac{1}{2}$, and take only the first terms in each determinant, we get an algebraic equation for β. By this method, great exactness is not attainable, but a sufficient indication of the position of the critical depths can be found.

Corresponding to Table III, we find $\beta = 3{\cdot}1$, 338.

Corresponding to Table IV, we find $\beta = 3{\cdot}8$, 46.

As will be shown presently, the first root is usually too small; the second is probably considerably wrong, and is only useful for showing that the higher roots indicate depths quite out of the range here considered. To give greater confidence to the results, I have worked out the values of $\Delta_0'\Delta_1''-\Delta_1'\Delta_0''$ for $\beta = 4$ and $\beta = 5$. The former gives $+{\cdot}4515$, and the latter $-{\cdot}2242$; showing that the critical depth lies between those given by $\beta = 4$ and 5. For the larger basin, the values of the same expression for $\beta = 4$ and $\beta = 5$ are, respectively, $+{\cdot}9169$ and $-1{\cdot}2239$. The critical depth is again between those given by $\beta = 4$ and $\beta = 5$. But as the variation of the function between values given by $\beta = 4$ and 5 is much more rapid in the latter than in the former case, the effect of a small departure from the critical depth in the latter case is to reduce much more rapidly the height of the tide; hence in Table IV, for $\beta = 5$, the heights are not far from normal.

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* See note on previous page.

Neither the theory of the ocean completely covering the globe nor the equilibrium theory offers any useful indication of the correct tide heights.

(iii) Semi-diurnal tide.

TABLE IV. *Ocean bounded by Circles of Latitude* 30° *and* 14° 30′.

	$\beta = 5$		$\beta = 10$		$\beta = 20$		$\beta = 40$	
	Latitudes.							
	30°	14° 30′	30°	14° 30′	30°	14° 30′	30°	14° 30′
Present Theory...	$1{\cdot}36H_2$	${\cdot}348H_2$	$-1{\cdot}09H_2$	$-{\cdot}734H_2$	$-{\cdot}436H_2$	$-{\cdot}063H_2$	$+{\cdot}363H_2$	$+{\cdot}361H_2$
Complete Ocean Theory	$1{\cdot}25H_2$	$1{\cdot}75H_2$	$5{\cdot}86H_2$	$8{\cdot}56H_2$	$-{\cdot}445H_2$	$-1{\cdot}56H_2$	$+6{\cdot}90H_2$	$-1{\cdot}02H_2$
Equilibrium Theory* ...	$+{\cdot}75H_2$	$+{\cdot}94H_2$	$+{\cdot}75H_2$	$+{\cdot}94H_2$	$+{\cdot}75H_2$	$+{\cdot}94H_2$	$+{\cdot}75H_2$	$+{\cdot}94H_2$

TABLE V. *Ocean bounded by Circles of Latitude* 30° *and* −14° 30′.

	$\beta = 5$		$\beta = 10$		$\beta = 20$		$\beta = 40$	
	Latitudes.							
	30°	−14° 30′	30°	−14° 30′	30°	−14° 30′	30°	−14° 30′
Present Theory...	$-4{\cdot}98H_2$	$-3{\cdot}23H_2$	$-{\cdot}837H_2$	$-{\cdot}655H_2$	$+3{\cdot}36H_2$	$-3{\cdot}61H_2$	$-{\cdot}443H_2$	$-{\cdot}429H_2$
Complete Ocean Theory	$1{\cdot}25H_2$	$1{\cdot}75H_2$	$5{\cdot}86H_2$	$8{\cdot}56H_2$	$-{\cdot}445H_2$	$-1{\cdot}56H_2$	$6{\cdot}90H_2$	$-1{\cdot}02H_2$
Equilibrium Theory* ...	${\cdot}75H_2$	${\cdot}94H_2$	${\cdot}75H_2$	${\cdot}94H_2$	${\cdot}75H_2$	${\cdot}94H_2$	${\cdot}75H_2$	${\cdot}94H_2$

The values of β roughly indicating critical depths for these tides are, for the first ocean, 2·2 and 84; for the second, 2·8 and 18. These numbers account for the higher magnitude of the tides, in Table V, where $\beta = 5$ and $\beta = 20$, than in the other cases.

Again, there seems no relationship between the figures given by the three theories.

---

* The correction to the ordinary "equilibrium" theory is zero in this case.

It is important to notice that the effect of the boundaries on the diurnal tide (which is zero in the complete ocean theory) is to make it to correspond in magnitude with the semi-diurnal and in many cases to predominate. This is fully exhibited in the following table, where we have taken the ratio $H_1/H_2 = 4{\cdot}5$.* The cases of exaggerated tides due to approximate synchronism are omitted.

		Lat. 30°.		Lat. 14° 30′.	
		Diurnal.	Semi-diurnal.	Diurnal.	Semi-diurnal.
First example.	$\beta = 10$	$1{\cdot}6H_2$	$1{\cdot}09H_2$	$1{\cdot}7H_2$	$\cdot73H_2$
	$\beta = 20$	$\cdot76H_2$	$\cdot44H_2$	$1{\cdot}02H_2$	$\cdot063H_2$
	$\beta = 40$	$\cdot20H_2$	$\cdot36H_2$	$\cdot58H_2$	$\cdot36H_2$
		Lat. 30°.		Lat. −14° 30′.	
		Diurnal.	Semi-diurnal.	Diurnal.	Semi-diurnal.
Second example.	$\beta = 10$	$\cdot87H_2$	$\cdot84H_2$	$2{\cdot}4H_2$	$\cdot66H_2$
	$\beta = 40$	$3{\cdot}8H_2$	$\cdot44H_2$	$3{\cdot}8H_2$	$\cdot43H_2$

Note on the "Corrected Equilibrium" Theory of the Tides in a Polar Basin.

In a previous paper (*Proceedings*, p. 31 of the present volume) I have worked out the dynamical theory of the tides in a polar basin. In comparing the results obtained with the "equilibrium" results, the uncorrected forms of the latter were taken. For the tides of the second and third species the correction is zero, but in the case of those of the first species the correction makes an important difference.

From p. 223 of the present paper, the corrected form of the long period "equilibrium" tide is

$$\bar{\zeta} = H_0 \left\{ (\mu^2 - \tfrac{1}{3}) - \tfrac{1}{3}(\mu_1^2 + \mu_1 \mu_2 + \mu_2^2 - 1) \right\},$$

where μ_1 and μ_2 are the boundaries of the zone. If this be changed into a form suitable for use in a polar basin by putting $\nu^2 = 1 - \mu^2$, and by taking $\mu_2 = 1$, $\mu_1 = \sqrt{(1-\nu_1^2)}$, the result is

$$\bar{\zeta} = H_0 \left\{ (\tfrac{2}{3} - \nu^2) - \tfrac{1}{3}(1 - \nu_1^2) - \tfrac{1}{3}\sqrt{(1 - \nu_1^2)} \right\}.$$

---

* Darwin, *Scientific Papers*, Vol. I, pp. 20 and 21 (tides K_1 and K_2).

We now repeat the tables on p. 62 of the former paper, substituting the corrected form of the equilibrium tide.

Boundary $\nu_1 = \frac{1}{4}$, *Lat.* 75° 30′.

	Tide Height at Pole.		Tide Height at Boundary.	
	Dynamical Theory.	Corrected Equilibrium Theory.	Dynamical Theory.	Corrected Equilibrium Theory.
$\beta = 5$	$\cdot 0306 H_0$	$\cdot 0315 H_0$	$-\cdot 0307 H_0$	$-\cdot 0310 H_0$
$\beta = 10$	$\cdot 0298 H_0$	$\cdot 0315 H_0$	$-\cdot 0305 H_0$	$-\cdot 0310 H_0$
$\beta = 20$	$\cdot 0284 H_0$	$\cdot 0315 H_0$	$-\cdot 0293 H_0$	$-\cdot 0310 H_0$
$\beta = 40$	$\cdot 0260 H_0$	$\cdot 0315 H_0$	$-\cdot 0270 H_0$	$-\cdot 0310 H_0$

Boundary $\nu_1 = \frac{1}{2}$, *Lat.* 60°.

	Tide Height at Pole.		Tide Height at Boundary.	
	Dynamical Theory.	Corrected Equilibrium Theory.	Dynamical Theory.	Corrected Equilibrium Theory.
$\beta = 5$	$\cdot 117 H_0$	$\cdot 128 H_0$	$-\cdot 115 H_0$	$-\cdot 122 H_0$
$\beta = 10$	$\cdot 105 H_0$	$\cdot 128 H_0$	$-\cdot 112 H_0$	$-\cdot 122 H_0$
$\beta = 20$	$\cdot 0887 H_0$	$\cdot 128 H_0$	$-\cdot 103 H_0$	$-\cdot 122 H_0$
$\beta = 40$	$\cdot 0668 H_0$	$\cdot 128 H_0$	$-\cdot 0899 H_0$	$-\cdot 122 H_0$

It is noticeable that the "corrected equilibrium" theory gives a fair approximation to the dynamical theory when the basin is small and the depth great. The approximation is not so good (for values of $\nu = \mu$) as in the case of a zonal ocean near the equator. If the matter be examined analytically in the same manner as is done on p. 224 of this paper, the reasons for the less satisfactory approximation become obvious.

ON THE THEORY OF CLOSED CONVEX SURFACE

By TADAHIKO KUBOTA.

[Received November 4th, 1914.—Read November 12th, 1914.]

IT is well known that any surface of the second order is cut by a series of parallel planes in similar and similarly situated curves. The question, whether the surfaces of the second order form the only class of surfaces having this property, has induced me to make the following investigation, which may be looked upon as a continuation of my previous paper on Closed Convex Surface, published in the *Science Reports* of the Tohoku Imperial University, 1914. The result at which I arrive can be expressed as follows:—

If a closed convex surface be cut by every series of parallel planes in similar and similarly situated curves, then the surface must necessarily be a surface of the second order, that is, an ellipsoid.

I.

From the assumption that the closed convex surface is cut by every series of parallel planes in similar and similarly situated curves, I will deduce the following series of properties of the surface, so that I shall finally arrive at the proposed theorem.

All tangent planes which touch the surface at the points on a plane section of the surface, pass through one and the same point.*

In order to prove this property we take a plane section of the surface as xy plane of coordinates of rectangular axes, and take the origin of coordinates inside the surface. We further assume that the curve of the section is expressed by the following equations

---

* Let us assume that the surface has tangent planes everywhere.

$$x = f(\theta),$$

$$y = g(\theta),$$

where θ denotes the angle formed by the radius vector from the origin O to the point on the curve and x axis of coordinates. Then we have a parametric representation of the surface with respect to the rectangular axes in the following form :—

$$\begin{aligned} x &= [\rho(u)-\rho(0)+1]\, f(\theta)+a(u)-a(0),\\ y &= [\rho(u)-\rho(0)+1]\, g(\theta)+b(u)-b(0),\\ z &= u, \end{aligned}$$

where the function $\rho(u)-\rho(0)+1$ denotes the ratio of corresponding lengths in similar curves of the section by the planes $z = u$ and $z = 0$, and $a(u)-a(0)$, $b(u)-b(0)$ are the coordinates of the point corresponding to the origin by the transformation of similarity. We can now express the equation of the tangent plane of the surface at a point of the plane section by $z = 0$, as follows :—

$$\begin{vmatrix} \xi-f(\theta) & \left(\frac{\partial x}{\partial \theta}\right)_{u=0} & \left(\frac{\partial x}{\partial u}\right)_{u=0} \\ \eta-g(\theta) & \left(\frac{\partial y}{\partial \theta}\right)_{u=0} & \left(\frac{\partial y}{\partial u}\right)_{u=0} \\ \zeta & \left(\frac{\partial z}{\partial \theta}\right)_{u=0} & \left(\frac{\partial z}{\partial u}\right)_{u=0} \end{vmatrix} = 0,$$

i.e.,

$$\begin{vmatrix} \xi-f(\theta) & f'(\theta) & \rho'(0)\, f(\theta)+a'(0) \\ \eta-g(\theta) & g'(\theta) & \rho'(0)\, g(\theta)+b'(0) \\ \zeta & 0 & 1 \end{vmatrix} = 0,$$

where ξ, η, ζ denote current coordinates.

This equation shows us that all tangent planes which touch the surface at the points on the curve of the section pass through one and the same point

$$\left(-\frac{a'(0)}{\rho'(0)},\quad -\frac{b'(0)}{\rho'(0)},\quad -\frac{1}{\rho'(0)}\right).$$

It immediately follows that all points of contact of the tangent planes passing through an external point lie on a determinate plane.

This theorem, which is evidently the converse of the preceding one, can be proved in the following manner.

Let us assume that C is the locus of points of contact of tangent planes passing through the external point P. We have to prove that this curve must necessarily be a plane curve. On the curve C we take any three points A_1, A_2, A_3, and draw the plane passing through these three points, then the plane will cut the surface in a plane curve C_1. Since, by the preceding theorem, every tangent plane at a point on the curve C_1 passes through one and the same point P, it follows that the curve C_1 must be a portion of the curve C, that is, either the curve C is itself a plane curve or it must break up into a certain number of plane curves. If the second case occur, we take two of these plane curves, say $C_1 C_2$, into which the curve C is broken up.

Applying the preceding theorem again to the plane section intersecting both of the curves $C_1 C_2$, we can infer that all tangent planes of the surface will pass through a point, *i.e.* P. This is evidently contradictory to the fact that we can easily construct a tangent plane which does not pass through the point by drawing the plane perpendicular to the normal of the surface from the point P at the foot of this normal. Therefore C must necessarily be a plane curve. (Q.E.D.)

Establishing the correspondence between an external point P and the plane passing through the curve of points of contact, one and only one plane cutting the surface corresponds to every external point of the surface. I will call this plane "the polar plane" of the point P with respect to the surface for the sake of convenience. It is especially important to remark that polar planes are only defined for external points in this case.

Now we take a straight line g, which does not meet the surface and construct the polar plane π of the point P which lies on the straight line g. Since the surface is convex and closed, we can draw two and only two tangent planes passing through the straight line g; let us denote the points of contact by H, K.

The plane π passes always through the two points H, K, and we finally have the following result:—

If the point P move on a straight line g, then the polar plane π will turn round the straight line joining two points of contact of two tangent planes through the straight line g.

Let a be a plane cutting the surface in a closed convex curve $\mathfrak{A}$, and P be any external point on the plane a, then the polar plane π of the point P with respect to the surface will cut the plane a in a straight line p which is the joining line of the two points of contact of the two tangents drawn from the point P to the curve $\mathfrak{A}$. Call this straight line "the polar line" of the point P with respect to the curve $\mathfrak{A}$. The remark that polar lines are only defined for external points is also necessary.

If the point P move on a straight line lying on the plane α, then the polar plane will turn round a straight line, and therefore the polar line p will pass through one and the same point on the plane α. Finally, we get the following result:—

If the point P move on a straight line which lies on the plane α *and does not cut the curve* $\mathfrak{A}$, *then its polar line will pass through one and the same point lying inside the curve* $\mathfrak{A}$.

Let g be a straight line, and G be the point of concurrence of polar lines of points on the straight line g. We take this point G as the origin of coordinates, and take the perpendicular from the point G to the straight line g as the x-axis of coordinates.

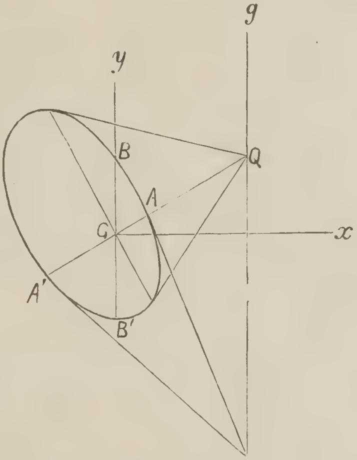

FIG. 1.

Any straight line passing through G will cut the curve at the two points A, A', and two tangents at these points will intersect on the straight line g.

We will denote the point of intersection of the straight line AA' and the straight line g by Q. Since the x axis is perpendicular to the straight line g, the equation of the straight line g must be of the form

$$x = \gamma,$$

where γ denotes a constant. If we denote the absolute value of the ratio

$$GA : AQ$$

by r, then r is a function of the angle

$$\sphericalangle\, xGQ = \vartheta,$$

and we will denote the function by f. That is,

$$\text{absolute value of } \frac{GA}{AQ} = r = f(\vartheta),$$

$$\text{absolute value of } \frac{GA'}{A'Q} = r = f(\vartheta + \pi),$$

where $$-\frac{\pi}{2} \leqslant \vartheta \leqslant \frac{\pi}{2},$$

and $f(\vartheta)$ is not necessarily analytic.

Thus we get the following expressions for the coordinates x, y of the point A :—

$$x = \gamma \frac{f(\vartheta)}{1+f(\vartheta)},$$

$$y = \gamma \frac{f(\vartheta)}{1+f(\vartheta)} \operatorname{tg} \vartheta,$$

and for the coordinates x', y' of the point A'

$$x' = -\gamma \frac{f(\vartheta+\pi)}{1-f(\vartheta+\pi)},$$

$$y' = -\gamma \frac{f(\vartheta+\pi)}{1-f(\vartheta+\pi)} \operatorname{tg} \vartheta.$$

Now we can determine the equation of the tangent at the point P and get the following equation :—

$$\begin{vmatrix} \xi - x & \dfrac{f'(\vartheta)}{[1+f(\vartheta)]^2} \\ \eta - y & \dfrac{f'(\vartheta)}{[1+f(\vartheta)]^2} \operatorname{tg} \vartheta - \dfrac{f(\vartheta)}{1+f(\vartheta)} \sec^2 \vartheta \end{vmatrix} = 0,$$

where ξ, η denote the current coordinates.

From this equation it follows that the ordinate of the point of intersection of the tangent with the straight line g is equal to

$$\gamma \left[\operatorname{tg} \vartheta - \frac{f(\vartheta)}{f'(\vartheta)} \sec^2 \vartheta \right].$$

In the same way we get the following expression as the point of intersection of the tangent at the point A' with the straight line g,

$$\gamma \left[\operatorname{tg} \vartheta - \frac{f(\vartheta+\pi)}{f'(\vartheta+\pi)} \sec^2 \vartheta \right].$$

Since these two tangents intersect on the straight line g, we have the following relation

$$\frac{f'(\vartheta+\pi)}{f(\vartheta+\pi)} = \frac{f'(\vartheta)}{f(\vartheta)} \quad \left(-\frac{\pi}{2} \leqslant \vartheta \leqslant \frac{\pi}{2} \right);$$

integrating this equation we get

$$f(\vartheta+\pi)=Cf(\vartheta)\quad\left(-\frac{\pi}{2}\leqq\vartheta\leqq\frac{\pi}{2}\right),$$

where C denotes a constant.

Let us now consider the ratio

$$\frac{f(\vartheta+\pi)}{f(\vartheta)}=\frac{GA'}{GA}:\frac{QA'}{QA}=\text{absolute value of }(A'A,\,GQ)$$

where $(A'A,\,GQ)$ denotes the anharmonic ratio of the four points A', A, G, Q, and then consider the limiting cases where

$$\operatorname{Lim}\vartheta=-\frac{\pi}{2},\qquad\operatorname{Lim}\vartheta=\frac{\pi}{2},$$

respectively, and we get

$$\operatorname*{Lim}_{\vartheta=-\frac{1}{2}\pi}\frac{f(\vartheta+\pi)}{f(\vartheta)}=\frac{GB}{GB'},$$

where B, B' denote the two points of intersection of the curve with the positive and negative parts of the y axis respectively. Similarly we get the following relation

$$\operatorname*{Lim}_{\vartheta=\frac{1}{2}\pi}\frac{f(\vartheta+\pi)}{f(\vartheta)}=\frac{GB'}{GB}.$$

Since these two values must be equal to C, we finally get

$$C^2=1,$$

from which it follows that

$$C=1,$$

as C is always positive. We finally have this

$$f(\vartheta+\pi)=f(\vartheta).$$

Thus the chord AA' is divided harmonically by the points G and Q.

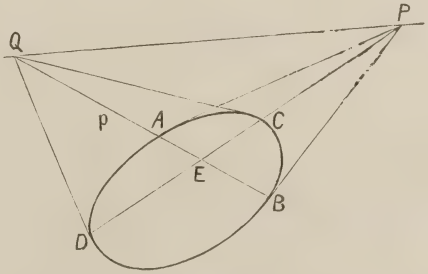

FIG. 2.

Now let P be any external point in the plane α, and p be its polar line, which will cut the curve $\mathfrak{A}$ in the points A, B. Passing through the point P we draw a straight line cutting the curve at the points C, D, and cutting the straight line at the point E. Now we draw two tangents at the points C and D, and let their points of intersection be denoted by Q. If a point move on the straight line PQ, then its polar line will always pass through the point E. Therefore the four points D, E, C, P must form a harmonic range. If we draw the straight line $PC'D'$ near to the straight line PCD in such a manner that it will cut the curve at the points C', D' and the straight line AB at the point E', then the four points P, C', E', D' will also form a harmonic range, and therefore the four straight lines CC', AB, DD' will intersect at one point. In the limiting case where the straight line $PC'D'$ moves indefinitely near to PCD, so that C', D' subsequently tend to C, D respectively, the straight lines CC' and DD' will be tangents at the points C, D respectively. Therefore the straight line AB must pass through the point Q.

Finally we have the following property :—

If we draw a straight line PCD through an external point P on the plane α in such a manner that the straight line will cut the curve at the points C, D, and the polar line AB of the point P at the point E, then the four points D, E, C, P will form a harmonic range, and two tangents at the points C, D will intersect in a point Q lying on the straight line AB. Also the four points B, E, A, Q will form a harmonic range.

Let us now take the triangle PAB as the triangle of reference, and let x_1, x_2, x_3 be the homogeneous coordinates of the point C, and, finally, let us denote the coordinates of the point Q by $(0, \xi_1, \xi_2)$, then we have by the harmonic property

$$\xi_3 x_2 + \xi_2 x_3 = 0. \tag{1}$$

Now let the homogeneous equation of the curve $\mathfrak{A}$, expressed in homogeneous coordinates, be written in the form

$$f(x_1, x_2, x_3) = 0, \tag{2}$$

where $f(x_1, x_2, x_3)$ is not necessarily analytic, then the equation of the tangent at the point C, *i.e.*, of the straight line CQ, will be

$$X_1 \frac{\partial f}{\partial x_1} + X_2 \frac{\partial f}{\partial x_2} + X_3 \frac{\partial f}{\partial x_3} = 0,$$

where X_1, X_2, X_3 denote the current coordinates.

This equation must then be satisfied by the coordinates $(0, \xi_1, \xi_2)$, *i.e.*,

$$\xi_2 \frac{\partial f}{\partial x_2} + \xi_3 \frac{\partial f}{\partial x_3} = 0.$$

If we compare this equation with (1), we get the following equation

$$x_2 \frac{\partial f}{\partial x_2} - x_3 \frac{\partial f}{\partial x_3} = 0. \tag{3}$$

Now let us assume, without any loss of generality, that the equation (2) is put into the following form

$$f(x_1, x_2, x_3) = \frac{x_3}{x_1} - \phi\left(\frac{x_2}{x_1}\right).$$

It therefore follows, by (3),

$$\phi'\left(\frac{x_2}{x_1}\right)\frac{x_2}{x_1} + \phi\left(\frac{x_2}{x_1}\right) = 0,$$

and we finally get $\phi\left(\frac{x_2}{x_1}\right) = C\frac{x_1}{x_2}.$

Thus the equation of the curve must be of the form

$$x_3 x_2 = Cx_1^2.$$

Therefore the curve must be of the second order, and, in fact, an ellipse, since it is a closed curve.

Since the surface is thus cut by every plane in an ellipse, it follows by applying the theorem proved by me in my last-named paper, that the surface must necessarily be an ellipsoid. Thus the proposed theorem is completely established.

II.

Although it seems to be somewhat superfluous, I should like to treat the following theorem as it is closely connected with the preceding.

If a surface of revolution be cut by a series of parallel planes, which are parallel to the axis of rotation in similar curves, then the surface is necessarily a surface of the second degree.

We take the axis of rotation as the axis of z, and denote the perpendicular distance of a point on the surface from the z axis by r, and denote the angle formed by this perpendicular and the axis of x by ϑ as usual, then we have a parametric representation of the surface as follows:—

$$x = r\cos\vartheta,$$

$$y = r\sin\vartheta,$$

$$r^2 = \phi(z).$$

Now we shall assume that the surface is cut by a series of planes perpendicular to x axis in similar curves, that is, by the planes

$$x = \text{constant} = C,$$

then we have the following equation as the equation of the curve of the section

$$\phi(z) - y^2 = C^2. \tag{1}$$

On the other hand, the most general infinitesimal Lie's transformation of similarity is of the form

$$\delta z = (\kappa y + lz + m)\,\delta t,$$

$$\delta y = (-\kappa z + ly + n)\,\delta t,$$

where κ, l, m, n denote certain constants ; this transformation is written in the form

$$Uf = (\kappa y + lz + m)\frac{\partial f}{\partial z} + (-\kappa z + ly + n)\frac{\partial f}{\partial y},$$

$$U[\phi(z) - y^2] = \phi'(z)(\kappa y + lz + m) - 2y(-\kappa z + ly + m) = F(y, z).$$

This function must be equal to a constant for all values of y, z satisfying (1), and therefore $F(y, z)$ must be of the form

$$\Phi[\phi(z) - y^2],$$

and therefore F must satisfy the following partial differential equation

$$2y\frac{\partial F}{\partial z} + \phi'(z)\frac{\partial F}{\partial y} = 0,$$

i.e., $2y[\phi''(z)(\kappa y + lz + m) + \phi'(z)\,l + 2\kappa z]$

$$+\phi'(z)[\phi'(z)\,\kappa - 2(-\kappa z + ly + n) - 2yl] = 0.$$

If we put the coefficients of y^2, y, 1 separately equal to zero, then we get the following series of equations respectively

$$k\phi''(z) = 0,$$

$$2\phi''(z)(lz + m) + 2\phi'(z)\,l + 4\kappa z - 2\phi'(z)\,l - 2l\phi'(z) = 0,$$

$$\phi'(z)\,\kappa + 2\kappa z - 2n = 0.$$

From these equations we get

$$\kappa = 0 \quad \text{and} \quad n = 0,$$

and, if $l \neq 0$, then $$\phi''(z)(lz+m)-\phi'(z)\,l = 0,$$

$$\frac{\phi''(z)}{\phi'(z)} = \frac{l}{lz+m}.$$

Integrating this differential equation, we get

$$\phi(z) = p(lz+m)^2+q,$$

where p and q denote constants.

If $l = 0$, then we get $$\phi''(z)m = 0,$$

$$\phi(z) = pz+q.$$

In both cases the surface is a surface of the second degree, which was to be proved.

FREE AND FORCED LONGITUDINAL TIDAL MOTION IN A LAKE

By J. PROUDMAN.

[Read June 11th, 1914.—Received October 24th, 1914.]

1. The lake considered in this paper is such that the size and shape of its transverse section vary only slowly. The variations may be in any manner, subject only to certain very general conditions which are satisfied in many actual lakes.

The determination of the ordinary longitudinal seiches in such a lake was reduced by Prof. Chrystal[*] to the solution of a linear differential equation of the second order in the normal form, with certain boundary conditions.

The determination of the "temperature seiche," discovered by E. R. Watson and E. M. Wedderburn, has been shown by the latter of these authors[†] to be similarly reducible.

Again, the discussion of the vibrations of a string of any law of density forms a problem which is reducible to the same equations and conditions. For the special case in which the string is of uniform density the original method of John and Daniel Bernoulli was to replace it by one in which the mass was concentrated at a finite number of equidistant points.[‡] This, when the masses at the points are no longer equal, is the method used in the present paper to suggest the form of the general solution. It involves the regarding of the differential equation as the limiting form of a difference equation, a method which has been very much used.

In the first place the determination of the forced motion in the lake due to a periodic change in atmospheric pressure[§] which varies along the

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[*] "Hydrodynamical Theory of Seiches," *Trans. Roy. Soc. Edin.*, Vol. XLI, p. 599 (1905).

[†] "The Temperature Seiche," *ibid.*, Vol. XLVII, p. 619 (1910); "Temperature Observations . . .," *ibid.*, Vol. XLVIII, p. 629 (1912).

[‡] The analysis is given by Lord Rayleigh, *Theory of Sound*, Vol. I, p. 172.

[§] The effect of a number of types of pressure disturbances on a special lake has been considered by Prof. Chrystal, *Trans. Roy. Soc. Edin.*, Vol. XLVI, p. 499 (1908).

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lake, as well as to a periodic bodily disturbing force which is directed everywhere along the length of the lake, is reduced to the solution, with the same boundary conditions, of a differential equation which is an extension of that of Prof. Chrystal. Of course we are merely concerned with a very simple linear boundary problem, of which Prof. Chrystal's is the "homogeneous" case.

A general equation for the free periods, and a general expression, in terms of the period, for the motion in any free mode, are obtained, as well as a complete expression for the periodic forced motion.

General Equations.

2. Following Prof. Chrystal, let x denote the area of the surface of the lake from one end up to a transverse section which is at a distance s, measured along the length of the lake, from some fixed transverse section. Then, if a denotes the total area of the surface of the lake, x will range from 0 to a. Let the transverse section corresponding to x have an area $A(x)$ and be of breadth $b(x)$ at the free surface, so that

$$\frac{dx}{ds} = b(x).$$

Now let V denote the total volume of water which has passed the section at x up to time t; then V will vanish at $x = 0$ and $x = a$. Let ξ denote the forward displacement of a particle in the section at x, and ζ the elevation of the free surface at this section. It is supposed that ξ and ζ are functions only of x and t. We shall then have

$$\xi = \frac{V}{A(x)}, \qquad \zeta = -\frac{\partial V}{\partial x},$$

the latter being the equation of continuity.

Let Π denote the atmospheric pressure, and S the bodily disturbing force per unit mass, supposed to act everywhere along the length of the lake. Π and S are supposed to be functions only of x and t.

When the motion is "tidal" in character, the pressure intensity in the water will be

$$\Pi + g\rho\,(\zeta + \text{depth below mean surface}),$$

where ρ is the density, supposed uniform, of the water, and g is the acceleration due to gravity. The dynamical equation then gives, on neglecting squares and products of displacements,

$$\frac{\partial^2 \xi}{\partial t^2} = -\frac{1}{\rho}\frac{\partial \Pi}{\partial s} - g\frac{\partial \zeta}{\partial s} + S, \tag{1}$$

which, on substituting from the above, gives

$$\frac{\partial^2 V}{\partial t^2} = -\frac{b(x)\,A(x)}{\rho}\frac{\partial \Pi}{\partial x} + gb(x)\,A(x)\frac{\partial^2 V}{\partial x^2} + A(x)\,S. \tag{2}$$

If we write

$$p(x) = b(x)\,A(x), \qquad F = \frac{1}{g\rho}\frac{\partial \Pi}{\partial x} - \frac{1}{gb(x)}S, \tag{3}$$

(2) may be written
$$\frac{\partial^2 V}{\partial x^2} - \frac{1}{gp(x)}\frac{\partial^2 V}{\partial t^2} = F. \tag{4}$$

The function $p(x)$ can never be negative, and can only vanish at the ends of the lake. For the possibility of the type of motion we are considering $p(x)$ must be a continuous function of x, and we shall further require that dp/dx shall not vanish at an end point for which p vanishes.*

Suppose now the motion to be periodic with speed σ, and take

$$\lambda = \frac{\sigma^2}{g}.$$

Then V is determined by the equation

$$\frac{d^2 V}{dx^2} + \frac{\lambda}{p(x)}V = F, \tag{5}$$

with the condition that V shall vanish at $x = 0$ and $x = a$.

For a free mode we have $F = 0$; let V, λ be then denoted by V_n, λ_n respectively, so that we have

$$\frac{d^2 V_n}{dx^2} + \frac{\lambda_n}{p(x)}V_n = 0. \tag{6}$$

It is to be remarked that, owing to the approximations made in deriving the equations, only the smaller values of λ will admit of interpretation.

When we speak of a solution of (5) or (6), we shall always suppose it understood that the solution vanishes at $x = 0$ and $x = a$.

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* Chrystal and Wedderburn have calculated the function $p(x)$ for Lochs Earn and Treig, and in each case this condition is satisfied: *Trans. Roy. Soc. Edin.*, Vol. XLI, p. 823 (1905).

The determination of the possible values of λ_n for which (6) possesses a solution, together with that of the corresponding solution, has been considered by Picard,* and, in fact, he has given direct processes by which they may be calculated. The processes of the present paper are different.

Solution of a Difference Equation.

3. In accordance with the method indicated in § 1, let us consider the determination of the $m-1$ constants

$$v(1),\ v(2),\ \ldots,\ v(m-1),$$

which are subject to the relations

$$v(r-1)+\left\{\frac{a^2}{m^2}\frac{\lambda}{p(r)}-2\right\}v(r)+v(r+1)=\frac{a^2}{m^2}f(r), \tag{7}$$

for $$r=1,\ 2,\ \ldots,\ m-1,$$

with $$v(0)=v(m)=0.$$

The difference equation (7) is suggested by the differential equation (5) which may be regarded as a limiting form of it as $m\to\infty$.

For our purpose we shall require the determinant

$$\begin{vmatrix} 2-\dfrac{a^2}{m^2}\dfrac{\lambda}{p(\mu+1)}, & -1, & 0, & \ldots, & 0 \\ -1, & 2-\dfrac{a^2}{m^2}\dfrac{\lambda}{p(\mu+2)}, & -1, & \ldots, & 0 \\ 0, & -1, & 2-\dfrac{a}{m^2}\dfrac{\lambda}{p(\mu+3)}, & \ldots, & 0 \\ \ldots\ \ldots & \ldots\ \ldots & \ldots\ \ldots & \ldots\ \ldots & \ldots\ \ldots \\ 0, & 0, & 0, & \ldots, & 0 \\ 0, & 0, & 0, & \ldots, & -1 \\ 0, & 0, & 0, & \ldots, & 2-\dfrac{a^2}{m^2}\dfrac{\lambda}{p(\nu-1)} \end{vmatrix},$$

which we shall call $\Delta(\mu,\ \nu,\ \lambda)$.

$\Delta(\mu,\ \nu,\ 0)$ is easily evaluated and found to be $\nu-\mu$, and then, on expanding $\Delta(\mu,\ \nu,\ \lambda)$ in powers of λ, it is easily seen to be given by

$$\Delta(\mu,\ \nu,\ \lambda)=(\nu-\mu)+\sum_{n=1}^{\nu-\mu-1}(-\lambda)^n S_n(\mu,\ \nu), \tag{8}$$

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* *Traité d'Analyse*, t. III, ch. VI.

where $S_n(\mu, \nu)$ is given by either of the two equal multiple sums

$$\frac{a^{2n}}{m^{2n}} \sum_{s_n=\mu+n}^{\nu-1} \sum_{s_{n-1}=\mu+n-1}^{s_n-1} \dots \sum_{s_2=\mu+2}^{s_3-1} \sum_{s_1=\mu+1}^{s_2-1} \frac{(\nu-s_n)(s_n-s_{n-1})\dots(s_1-\mu)}{p(s_1)\,p(s_2)\dots p(s_n)}, \tag{9}$$

$$\frac{a^{2n}}{m^{2n}} \sum_{s_n=\mu+1}^{\nu-n} \sum_{s_{n-1}=s_n+1}^{\nu-n+1} \dots \sum_{s_2=s_3+1}^{\nu-2} \sum_{s_1=s_2+1}^{\nu-1} \frac{(\nu-s_1)(s_1-s_2)\dots(s_n-\mu)}{p(s_1)\,p(s_2)\dots p(s_n)}. \tag{10}$$

The cofactor of the element in the r-th row and s-th column of the determinant $\Delta(0, m, \lambda)$ is, for the respective cases in which $r <, =, > s$,

$$\Delta(0, r, \lambda)\,\Delta(s, m, \lambda), \quad \Delta(0, r, \lambda)\,\Delta(r, m, \lambda), \quad \Delta(0, s, \lambda)\,\Delta(r, m, \lambda), \tag{11}$$

where $\Delta(0, 1, \lambda) = \Delta(m-1, m, \lambda) = 1$.

Now, when $f(r) = 0$ for all values of r concerned, the condition for the existence of a solution of the problem in question is

$$\Delta(0, m, \lambda) = 0. \tag{12}$$

This is an algebraic equation of the $(m-1)$-th degree in λ; let its roots be denoted by

$$\lambda_1, \lambda_2, \dots, \lambda_{m-1}.$$

Corresponding to λ_n we have, by the ordinary theory,

$$\frac{v(1)}{\Delta(0, 1, \lambda_n)} = \frac{v(2)}{\Delta(0, 2, \lambda_n)} = \dots = \frac{v(r)}{\Delta(0, r, \lambda_n)} = \dots = \frac{v(m-1)}{\Delta(0, m-1, \lambda_n)}, \tag{13}$$

and

$$\frac{v(1)}{\Delta(1, m, \lambda_n)} = \frac{v(2)}{\Delta(2, m, \lambda_n)} = \dots = \frac{v(r)}{\Delta(r, m, \lambda_n)} = \dots = \frac{v(m-1)}{\Delta(m-1, m, \lambda_n)}, \tag{14}$$

and the determination is unique. We notice that when $\Delta(0, m, \lambda) = 0$, $\Delta(0, r, \lambda)/\Delta(r, m, \lambda)$ must be independent of r.

When $f(r)$ is not zero for all values of r concerned, and λ is prescribed, we shall have, provided

$$\Delta(0, m, \lambda) \neq 0,$$

$$v(r) = -\frac{1}{\Delta(0, m, \lambda)} \left\{ \Delta(r, m, \lambda) \sum_{s=1}^{r-1} \Delta(0, s, \lambda) f(s) + \Delta(0, r, \lambda)\,\Delta(r, m, \lambda)\, f(r) \right.$$
$$\left. + \Delta(0, r, \lambda) \sum_{s=r+1}^{m-1} \Delta(s, m, \lambda) f(s) \right\}, \tag{15}$$

for $r = 1, 2, \dots, m-1$.

Solution of the Differential Equation.

4. We now proceed to solve the original problem on the lines suggested by the preceding section, giving, however, an independent justification.

From the expressions (9) and (10) for $S_n(\mu, \nu)$ we are led to consider the two multiple integrals

$$\int_{s_n=\xi}^{\eta}\int_{s_{n-1}=\xi}^{s_n}\cdots\int_{s_2=\xi}^{s_3}\int_{s_1=\xi}^{s_2}\frac{(\eta-s_n)(s_n-s_{n-1})\dots(s_2-s_1)(s_1-\xi)}{p(s_1)\,p(s_2)\,\dots\,p(s_n)}\,ds_1\,ds_2\,\dots\,ds_n, \tag{16}$$

$$\int_{s_n=\xi}^{\eta}\int_{s_{n-1}=s_n}^{\eta}\cdots\int_{s_2=s_3}^{\eta}\int_{s_1=s_2}^{\eta}\frac{(\eta-s_1)(s_1-s_2)\dots(s_{n-1}-s_n)(s_n-\xi)}{p(s_1)\,p(s_2)\,\dots\,p(s_n)}\,ds_1\,ds_2\,\dots\,ds_n, \tag{17}$$

for $n>0$, where $0\leqslant\xi\leqslant\eta\leqslant a$. The integrands of these are always positive, so that if the integrals exist we may change the order of integration. Doing this for (16), we obtain

$$\int_{s_1=\xi}^{\eta}\int_{s_2=s_1}^{\eta}\cdots\int_{s_{n-1}=s_{n-2}}^{\eta}\int_{s_n=s_{n-1}}^{\eta}\frac{(\eta-s_n)(s_n-s_{n-1})\dots(s_2-s_1)(s_1-\xi)}{p(s_1)\,p(s_2)\dots p(s_n)}\,ds_1\,ds_2\dots ds_n\;,$$

which only differs by a change in notation from (17). The integrals obviously exist for $0<\xi\leqslant\eta<a$, since the integrands are then continuous; let their common value be denoted by

$$I_n(\xi, \eta),$$

and take

$$I_0(\xi, \eta)=\eta-\xi. \tag{18}$$

The equality of (16) and (17) was to be expected from that of (9) and (10).

From (16) we have

$$I_n(\xi, \eta)=\int_{\xi}^{\eta}\frac{\eta-s}{p(s)}\,I_{n-1}(\xi, s)\,ds, \tag{19}$$

and this may be written as the double integral

$$\int_{s=\xi}^{\eta}\int_{s'=s}^{\eta}\frac{I_{n-1}(\xi, s)}{p(s)}\,ds\,ds', \tag{20}$$

which, by interchanging the order of integration, may be written

$$\int_{s'=\xi}^{\eta}\int_{s=\xi}^{s'}\frac{I_{n-1}(\xi, s)}{p(s)}\,ds\,ds'. \tag{21}$$

Similarly, from (17), we obtain

$$I_n(\xi, \eta) = \int_\xi^\eta \frac{s-\xi}{p(s)} I_{n-1}(s, \eta)\, ds \tag{22}$$

$$= \int_{s=\xi}^{\eta} \int_{s'=\xi}^{s} \frac{I_{n-1}(s, \eta)}{p(s)}\, ds\, ds' \tag{23}$$

$$= \int_{s'=\xi}^{\eta} \int_{s=s'}^{\eta} \frac{I_{n-1}(s, \eta)}{p(s)}\, ds\, ds'. \tag{24}$$

These relations are true for $n > 0$.

For a particular value of ξ, $I_n(\xi, \eta)$ increases as η increases, while for a particular value of η, $I_n(\xi, \eta)$ decreases as ξ increases.

We now proceed to show that $I_n(0, a)$ will exist, provided that

$$p(x) > kx(a-x), \tag{25}$$

where k is any positive constant. It will obviously be sufficient to show the existence for the case in which $p(x) = x(a-x)$. Now

$$\int_0^x \frac{x-s}{a-s} \left(\frac{s}{a}\right)^n ds = \sum_{r=2}^{\infty} \frac{1}{(n+r-1)(n+r)} \frac{x^{n+r}}{a^{n+r-1}} \leqslant \frac{1}{n+1} \frac{x^{n+2}}{a^{n+1}},$$

for $0 \leqslant x \leqslant a$, so that we have in succession, on using (19),

$$I_0(0, x) = x,$$

$$I_1(0, x) = \int_0^x \frac{x-s}{a-s}\, ds \leqslant \frac{x^2}{a},$$

$$I_2(0, x) \leqslant \int_0^x \frac{x-s}{a-s} \frac{s}{a}\, ds \leqslant \frac{1}{2} \frac{x^3}{a^2},$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$I_n(0, x) \leqslant \frac{1}{(n-1)!} \int_0^x \frac{x-s}{a-s} \left(\frac{s}{a}\right)^{n-1} ds \leqslant \frac{1}{n!} \frac{x^{n+1}}{a^n}.$$

We thus see that when (25) is satisfied, $I_n(0, a)$ will exist and have a value which is $< a/n!\, k^n$. The only restriction imposed by (25), further than those already required, is that dp/dx shall not vanish at an end for which p vanishes.

From (24), (21) we obtain respectively

$$\frac{\partial}{\partial \xi} I_n(\xi, \eta) = -\int_\xi^\eta \frac{I_{n-1}(s, \eta)}{p(s)}\, ds, \qquad \frac{\partial}{\partial \eta} I_n(\xi, \eta) = \int_\xi^\eta \frac{I_{n-1}(\xi, s)}{p(s)}\, ds, \tag{26}$$

for $0 < \xi \leqslant \eta \leqslant a$ and $0 \leqslant \xi \leqslant \eta < a$ respectively, while from (18) we obtain

$$\frac{\partial}{\partial \xi} I_0(\xi, \eta) = -1, \quad \frac{\partial}{\partial \eta} I_0(\xi, \eta) = 1. \tag{27}$$

We next obtain

$$\frac{\partial^2}{\partial \xi^2} I_n(\xi, \eta) = \frac{1}{p(\xi)} I_{n-1}(\xi, \eta), \quad \frac{\partial^2}{\partial \eta^2} I_n(\xi, \eta) = \frac{1}{p(\eta)} I_{n-1}(\xi, \eta), \tag{28}$$

for $n > 0$ and the same ranges, while

$$\frac{\partial^2}{\partial \xi^2} I_0(\xi, \eta) = 0, \quad \frac{\partial^2}{\partial \eta^2} I_0(\xi, \eta) = 0. \tag{29}$$

The conditions necessary for the validity of the operations involved in the derivation of (26) and (28) are satisfied.

The above relations may be used for the determination of $I_n(\xi, \eta)$ instead of (16) and (17). For instance, $I_n(\xi, \eta)$, regarded as a function of the parameter ξ and the variable η, may be determined step by step from the differential equations in (28) and (29), together with the conditions at the initial point $\eta = \xi$,

$$I_0(\xi, \eta) = 0, \quad \frac{\partial}{\partial \eta} I_0(\xi, \eta) = 1,$$

$$I_n(\xi, \eta) = 0, \quad \frac{\partial}{\partial \eta} I_n(\xi, \eta) = 0 \quad (n > 0),$$

which are derived from (26) and (27).

5. Now take

$$R(\xi, \eta, \lambda) = \sum_{n=0}^{\infty} (-\lambda)^n I_n(\xi, \eta). \tag{30}$$

This is suggested by (8); we shall expect the condition for a solution of (6) to be

$$R(0, a, \lambda_n) = 0,$$

and the solution itself to be given either by

$$V_n = R(0, x, \lambda_n) \quad \text{or} \quad V_n = R(x, a, \lambda_n).$$

When (25) holds, we see from the upper limit obtained for $I_n(0, a)$, that (30) will be absolutely convergent, as well as uniformly convergent with respect to ξ, η, for all values of λ.

We shall then have

$$\begin{aligned}\frac{\partial}{\partial\xi}R(\xi,\eta,\lambda) &= \sum_{n=0}^{\infty}(-\lambda)^n\frac{\partial}{\partial\xi}I_n(\xi,\eta)\\ &= -1+\lambda\sum_{n=0}^{\infty}(-\lambda)^n\int_{\xi}^{\eta}\frac{I_n(s,\eta)}{p(s)}\,ds\\ &= -1+\lambda\int_{\xi}^{\eta}\frac{R(s,\eta,\lambda)}{p(s)}\,ds, \qquad (31)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial\eta}R(\xi,\eta,\lambda) &= \sum_{n=0}^{\infty}(-\lambda)^n\frac{\partial}{\partial\eta}I_n(\xi,\eta)\\ &= 1-\lambda\sum_{n=0}^{\infty}(-\lambda)^n\int_{\xi}^{\eta}\frac{I_n(\xi,s)}{p(s)}\,ds\\ &= 1-\lambda\int_{\xi}^{\eta}\frac{R(\xi,s,\lambda)}{p(s)}\,ds. \qquad (32)\end{aligned}$$

We have here used the relations (26) and (27).

We next obtain

$$\frac{\partial^2}{\partial\xi^2}R(\xi,\eta,\lambda) = -\frac{\lambda}{p(\xi)}R(\xi,\eta,\lambda), \qquad \frac{\partial^2}{\partial\eta^2}R(\xi,\eta,\lambda) = -\frac{\lambda}{p(\eta)}R(\xi,\eta,\lambda). \qquad (33)$$

We now see that $R(\xi, x, \lambda)$ *and* $R(x, \eta, \lambda)$ *are each solutions of the differential equation*

$$\frac{d^2V}{dx^2}+\frac{\lambda}{p(x)}V = 0, \qquad (34)$$

for $0 \leqslant \xi \leqslant x < a$, *and* $0 < x \leqslant \eta \leqslant a$ *respectively, and that*

$$R(\xi, x, \lambda) = 0, \qquad \frac{\partial}{\partial x}R(\xi, x, \lambda) = 1,$$

for $x = \xi$, *while* $R(x, \eta, \lambda) = 0, \quad \frac{\partial}{\partial x}R(x,\eta,\lambda) = -1,$

for $x = \eta$.

Again, these conditions, which are "conditions of Cauchy," may be used to determine the functions $R(\xi, x, \lambda)$, $R(x, \eta, \lambda)$, and this determination may be carried out in a manner similar to that of Picard mentioned in § 2, which is based on successive approximations. If we assume, as a solution of (34), for example,

$$V = \sum_{n=0}^{\infty}\lambda^n v_n,$$

with the conditions $V = 0$, $dV/dx = 1$, at the initial point $x = \xi$, we obtain by Picard's method a differential equation and initial conditions for

v_n which, on comparison with those of § 4, show that

$$v_n = (-1)^n I_n(\xi, x).$$

Picard himself does not use these "conditions of Cauchy," but takes his primary solution to have a prescribed value at each of the end points. The functions $R(\xi, x, \lambda)$, $R(x, \eta, \lambda)$ are included in a solution given by Forsyth,[*] while the function $R(\xi, x, \lambda)$ has been obtained by Liapounoff[†] for the case in which the function $p(x)$ is periodic.

Since $R(\xi, x, \lambda)$ and $R(x, \eta, \lambda)$ are two solutions of (34), we know that

$$\frac{\partial}{\partial x} R(\xi, x, \lambda)\, R(x, \eta, \lambda) - \frac{\partial}{\partial x} R(x, \eta, \lambda)\, R(\xi, x, \lambda),$$

will be independent of x; we have, in fact,

$$\frac{\partial}{\partial x} R(\xi, x, \lambda)\, R(x, \eta, \lambda) - \frac{\partial}{\partial x} R(x, \eta, \lambda)\, R(\xi, x, \lambda) = R(\xi, \eta, \lambda), \tag{35}$$

on putting either $x = \xi$ or $x = \eta$. When $R(\xi, \eta, \lambda) = 0$, we see further from (35) that $R(\xi, x, \lambda)/R(x, \eta, \lambda)$ will be independent of x. This was to be expected from § 3. We may put $\xi = 0$, $\eta = a$, if $0 < x < a$.

6. With regard to the tidal problem, the solution for the *free modes* is now seen, as was expected, to be given by

$$V_n = R(0, x, \lambda_n) \quad \text{or} \quad V_n = R(x, a, \lambda_n), \tag{36}$$

where λ_n is a root of the equation

$$R(0, a, \lambda) = 0. \tag{37}$$

This is the *period equation;* the left-hand side is an integral function in λ, and we know that the roots are real, positive, infinite in number, and isolated.[‡]

The ratio of the two solutions in (36) is seen from the end of the preceding section to be independent of x.

For the *forced motion,* we assume, on the suggestion of (15), when

$$R(0, a, \lambda) \neq 0,$$

$$V(x) = -\frac{1}{R(0, a, \lambda)} \left\{ R(x, a, \lambda) \int_0^x R(0, s, \lambda)\, F(s)\, ds \right.$$
$$\left. + R(0, x, \lambda) \int_x^a R(s, a, \lambda)\, F(s)\, ds \right\}. \tag{38}$$

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[*] *A Treatise on Differential Equations*, 3rd ed., p. 120, Ex.

[†] *Ann. Fac. Sci. Toulouse*, (2), t. IX, p. 403.

[‡] See, for instance, Picard, *l.c.*

This vanishes at $x = 0$ and $x = a$. We have further

$$\frac{d}{dx} V(x) = -\frac{1}{R(0, a, \lambda)} \left\{ \frac{\partial}{\partial x} R(x, a, \lambda) \int_0^x R(0, s, \lambda) F(s) ds \right.$$
$$\left. + \frac{\partial}{\partial x} R(0, x, \lambda) \int_x^a R(s, a, \lambda) F(s) ds \right\} . \quad (39)$$

and, consequently,

$$\frac{d^2}{dx^2} V(x) = \frac{\lambda}{p(x)} \frac{1}{R(0, a, \lambda)} \left\{ R(x, a, \lambda) \int_0^x R(0, s, \lambda) F(s) ds \right.$$
$$\left. + R(0, x, \lambda) \int_x^a R(s, a, \lambda) F(s) ds \right\}$$
$$- \frac{1}{R(0, a, \lambda)} \left\{ \frac{\partial}{\partial x} R(x, a, \lambda) R(0, x, \lambda) \right.$$
$$\left. - \frac{\partial}{\partial x} R(0, x, \lambda) R(x, a, \lambda) \right\} F(x). \quad (40)$$

Equations (39) and (40) are valid for $0 \leqslant x \leqslant a$. Therefore

$$\frac{d^2}{dx^2} V(x) + \frac{\lambda}{p(x)} V(x) = F(x),$$

on using (35), so that (38) provides the solution of (5) required.

ON CYCLOTOMIC QUINQUISECTION

By W. BURNSIDE.

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IF p is a prime and q a factor of $p-1$, there is an equation of degree q with rational coefficients, each of whose roots is the sum of $(p-1)/q$ of the primitive p-th roots of unity, no such p-th root occurring in more than one of the sums. The equation is an Abelian equation with a cyclical group. The determination of the system of relations expressing any rational function of the roots of this equation as a linear function of the roots, has been called by Cayley the problem of cyclotomic q-section. He worked out the theory completely in Vol. XI (Old Series) of our *Proceedings* for the cases $q = 3$ and $q = 4$. In a short note in Vol. XII, he refers to the case of quinquisection, and gives in tabular form the solution for primes under 100. It is not indicated how this table was constructed, nor is the problem of quinquisection in its general form really attacked at all. So far as I am aware the problem has not been directly dealt with since. The nature of the algebraic "field," determined by the equation of the q-th degree above referred to, has formed the subject of various investigations during the last twenty years; but the determination of the system of integers on which all rational relations between the roots depend, has not, I believe, been considered.

This problem is completely solved here for the case $q = 5$. It is shewn to depend on the two diophantine equations

$$12^2p = [4p-16-25(A+B)]^2+5.15^2(A-B)^2+2.15^2C^2+2.15^2D^2,$$

$$0 = [4p-16-25(A+B)](A-B)+3(C^2+4CD-D^2).$$

In the first paragraph of the paper certain general formula are proved which hold when q is any odd prime. These could easily be extended to the case in which q is any number. In the succeeding paragraphs the problem of quinquisection is dealt with.

1. *Notation.*—p is an odd prime,

q is an odd prime factor of $p-1$, and $p-1 = qt$,

w is an assigned primitive p-th root of unity,

a is an assigned primitive root of the congruence

$$a^{p-1} \equiv 1 \pmod{p},$$

β is an assigned primitive root of the congruence

$$\beta^{q-1} \equiv 1 \pmod{q}.$$

Each of the $p-1$ primitive p-th roots of unity is included just once in the form

$$w^{a^{i+xq}} \quad (i = 0, 1, \ldots, q-1\,;\; x = 0, 1, \ldots, t-1).$$

Put

$$A_i = \sum_{x=0}^{x=t-1} w^{a^{i+xq}} \quad (i = 0, 1, \ldots, q-1).$$

Each A_i consists of the sum of t distinct primitive p-th roots of unity, and each primitive p-th root occurs just once in one of the A_i's. When w is replaced by w^{a^q}, each A_i remains unaltered. When w is replaced by w^a the A_i's undergo the cyclical permutation

$$(A_0 A_1 \ldots A_{q-1}).$$

If w' is any root occurring in A_i, then

$$A_i = \sum_{x=0}^{x=t-1} w'^{a^{xq}}.$$

In particular, since t is even, if A_i contains w' it will also contain w'^{-1}.

When i is replaced by βi, the A_i's undergo the permutation

$$\begin{pmatrix} A_0 & A_1 & A_2 & \ldots & A_{q-1} \\ A_0 & A_\beta & A_{2\beta} & \ldots & A_{(q-1)\beta} \end{pmatrix},$$

where the suffixes are reduced (mod q). This leaves A_0 unchanged and gives a regular circular permutation of the other A_i's.

If A_i and A_j are two distinct A's, and if the product $A_i A_j$ is formed without reduction, *i.e.*, without taking account of the relation

$$1 + w + w^2 + \ldots + w^{p-1} = 0,$$

it will consist of the product of t^2 primitive p-th roots, because if w' occurs in A_i, then w'^{-1} does *not* occur in A_j. Moreover, since $A_i A_j$ is unaltered when w is replaced by w^{a^q}, the product can be arranged as the sum of a number of A's. Hence

$$A_i A_j = \sum_{k=1}^{k=t} c_{ijk} A_k,$$

where the c's are zeroes or positive integers, such that

$$\sum_{k=1}^{k=t} c_{ijk} = t.$$

The product A_i^2 will contain t units and t^2-t primitive roots of unity, so that

$$A_i^2 = t + \sum_{k=1}^{k=t} c_{iik} A_k,$$

where again the c's are zeroes or positive integers, and

$$\sum_{k=1}^{k=t} c_{iik} = t-1.$$

Let w' be a root occurring in A_i, and suppose that of the t distinct roots in $w'A_j$ just x belong to A_k. Since both A_j and A_k are unaltered, when w^{α^q} is written for w^{α} it follows that of the t distinct roots in $w'^{\alpha^q} A_j$ just x belong to A_k. Hence

$$x = c_{ijk}.$$

Let $$w^{\alpha^{j+a_1 q}}, \quad w^{\alpha^{j+a_2 q}}, \quad \ldots, \quad w^{\alpha^{j+a_x q}}$$

be the roots occurring in A_j which, when multiplied by w^{α^i}, give roots occurring in A_k, so that

$$w^{\alpha^i+\alpha^{j+a_1 q}} = w^{\alpha^{k+b_1 q}}, \quad \ldots, \quad w^{\alpha^i+\alpha^{j+a_x q}} = w^{\alpha^{k+b_x q}},$$

while no further relation such as

$$w^{\alpha^i+\alpha^{j+a_{x+1} q}} = w^{\alpha^{k+b_{x+1} q}}$$

holds. Then

$$w^{-\alpha^i+\alpha^{k+b_1 q}} = w^{\alpha^{j+a_1 q}}, \quad \ldots, \quad w^{-\alpha^i+\alpha^{k+b_x q}} = w^{\alpha^{j+a_x q}},$$

and no further similar relation holds. Since $w^{-\alpha^i}$ is a root occurring in A_i, this is equivalent to the equation

$$c_{ijk} = c_{ikj},$$

i.e., the numerical value of the symbol c_{ijk} is independent of the sequence of the letters in the suffix. Moreover the same reasoning clearly holds when two of the letters in the suffix are the same, *i.e.*,

$$c_{iik} = c_{iki}.$$

Further, since writing w^a for w changes A_i into A_{i+1}, the relation

$$A_{i+1}A_{j+1} \sum_{k=0}^{k=t} c_{ijk}A_{k+1}$$

follows from
$$A_iA_j = \sum_{k=0}^{k=t} c_{ijk}A_k.$$

Hence, since the equation for the A's is irreducible,

$$c_{i+1,\,j+1,\,k+1} = c_{ijk}.$$

The complete multiplication table of the A's, viz., the set of relations

$$A_i^2 = t + \sum_1^t c_{iik}A_k,$$

$$A_iA_j = \sum_1^t c_{ijk}A_k$$

$$(i, j = 0, 1, 2, \ldots, q-1)$$

is materially simplified by these relations among the c's. Moreover, the fact that this system of relations is invariant when the A's and the c's simultaneously undergo the permutations

$$\begin{pmatrix} A_i \\ A_{i\beta} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_{ijk} \\ c_{i\beta,\,j\beta,\,k\beta} \end{pmatrix}$$

indicates that any system of equations connecting the reduced c's must be invariant for a certain cyclical permutation of the reduced c's.

2. *Quinquisection.*—I consider now in particular the case $q = 5$. The complete multiplication table of the A's arises in this case from the three relations

$$\begin{aligned} A_0^2 &= \lambda + c_{000}A_0 + c_{001}A_1 + c_{002}A_2 + c_{003}A_3 + c_{004}A_4, \\ A_0A_1 &= \phantom{\lambda +{}} c_{010}A_0 + c_{011}A_1 + c_{012}A_2 + c_{013}A_3 + c_{014}A_4, \\ A_0A_2 &= \phantom{\lambda +{}} c_{020}A_0 + c_{021}A_1 + c_{022}A_2 + c_{023}A_3 + c_{024}A_4, \end{aligned}$$

by the cyclical permutation $(A_0A_1A_2A_3A_4)$, where λ stands for $\frac{1}{5}(p-1)$. Taking account of the previously obtained relations between the c's, we may write

$$c_{000} = a, \qquad c_{001} = c_{010} = a, \qquad c_{002} = c_{020} = b,$$

$$c_{003} = c_{022} = c, \qquad c_{004} = c_{001} = d, \qquad c_{012} = c_{014} = c_{021} = e,$$

$$c_{013} = c_{023} = c_{024} = f.$$

The relations are then

$$\left.\begin{aligned} A_0^2 &= \lambda+\alpha A_0+aA_1+bA_2+cA_3+dA_4 \\ A_0A_1 &= \quad aA_0+dA_1+eA_2+fA_3+eA_4 \\ A_0A_2 &= \quad bA_0+eA_1+cA_2+fA_3+fA_4 \end{aligned}\right\}, \qquad \text{(i)}$$

where

$$\left.\begin{aligned} \alpha+a+b+c+d &= \lambda-1 \\ a+d+2e+f &= \lambda \\ b+c+e+2f &= \lambda \end{aligned}\right\}. \qquad \text{(ii)}$$

These give

$$\left.\begin{aligned} \alpha &= \lambda-1-a-b-c-d \\ e &= \tfrac{1}{3}(\lambda-2a-2d+b+c) \\ f &= \tfrac{1}{3}(\lambda+a+d-2b-2c) \end{aligned}\right\}. \qquad \text{(iii)}$$

When $q = 5$, we may take the β of § 1 to be 2. If i is replaced by βi, the A's undergo the permutation $(A_1A_2A_4A_3)$, and it is found that equations (i) remain unaltered, if the coefficients undergo the permutation $(abdc)(ef)$. It follows that whatever set of relations are found between the coefficients, they must be changed into themselves by the permutation $(abdc)(ef)$. This is clearly true of equations (ii).

The equations (i) and those derived from them by the permutation $(A_0A_1A_2A_3A_4)$ may be used to express any rational integral function of the A's as a linear function. In particular, if ϵ is a primitive fifth root of unity, they give

$$(A_0+\epsilon A_1+\epsilon^2A_2+\epsilon^3A_3+\epsilon^4A_4)^2 = f(\epsilon)(A_0+\epsilon^2A_1+\epsilon^4A_2+\epsilon A_3+\epsilon^3A_4),$$

where

$$f(\epsilon) = \alpha+2e+2f+(b+2a+2f)\epsilon+(d+2b+2e)\epsilon^2+(a+2c+2e)\epsilon^3 \\ +(c+2d+2f)\epsilon^4.$$

Hence

$$(A_0+\epsilon A_1+\epsilon^2A_2+\epsilon^3A_3+\epsilon^4A_4)^4 = [f(\epsilon)]^2 f(\epsilon^2)(A_0+\epsilon^4A_1+\epsilon^3A_2+\epsilon^2A_3+\epsilon A_4).$$

But equations (i) also give

$$(A_0+\epsilon A_1+\epsilon^2A_2+\epsilon^3A_3+\epsilon^4A_4)(A_0+\epsilon^4A_1+\epsilon^3A_2+\epsilon^2A_3+\epsilon A_4) = p,$$

so that

$$(A_0+\epsilon A_1+\epsilon^2A_2+\epsilon^3A_3+\epsilon^4A_4)^5 = p\,[f(\epsilon)]^2 f(\epsilon^2).$$

Writing ϵ^{-1} for ϵ, and multiplying the two equations together,

$$p^3 = [f(\epsilon)]^2 f(\epsilon^2) f(\epsilon^3)\,[f(\epsilon^4)]^2,$$

whence

$$p = f(\epsilon)\,f(\epsilon^4).$$

This equation can be written in the form

$$p = (a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3 + a_4\epsilon^4)(a_1\epsilon^4 + a_2\epsilon^3 + a_3\epsilon^2 + a_4\epsilon).$$

When ϵ is replaced by ϵ^2, the coefficients undergo the permutation $(a_1 a_2 a_4 a_3)$. Apart from this the integers a_1, a_2, a_3, a_4 are unique. For the only modification possible in the two factors, if the coefficients are to be integers, is to multiply one factor by an arbitrary unit (of the field of the fifth roots of unity) and the other by the reciprocal unit. When this is done it is found that the coefficients entering in the two factors are no longer the same, unless the unit is a power of ϵ.

Now

$$f(\epsilon) = (2a+b-\alpha-2e)\,\epsilon + (2b+d-\alpha-2f)\,\epsilon^2 + (2c+a-\alpha-2f)\,\epsilon^3 + (2d+c-\alpha-2e)\,\epsilon^4$$

$$= [a+d+\tfrac{1}{2}(b+c)-\alpha-2e](\epsilon+\epsilon^4) + [\tfrac{1}{2}(a+d)+b+c-\alpha-2f](\epsilon^2+\epsilon^3)$$
$$+[a-d+\tfrac{1}{2}(b-c)](\epsilon-\epsilon^4) + [b-c-\tfrac{1}{2}(a-d)](\epsilon^2-\epsilon^3).$$

Put $$A+D = 2a, \qquad B+C = 2b,$$
$$A-D = 2d, \qquad B-C = 2c,$$

so that A, B, C, D are integers, and equations (iii) become

$$\alpha = \lambda-1-A-B, \quad e = \tfrac{1}{3}(\lambda-2A+B), \quad f = \tfrac{1}{3}(\lambda+A-2B).$$

Then $$f(\epsilon) = \left(\frac{4-p}{3} + \frac{10A}{3} + \frac{5B}{6}\right)(\epsilon+\epsilon^4) + \left(\frac{4-p}{3} + \frac{5A}{6} + \frac{10B}{3}\right)(\epsilon^2+\epsilon^3)$$
$$+\left(D+\frac{C}{2}\right)(\epsilon-\epsilon^4) + \left(C-\frac{D}{2}\right)(\epsilon^2-\epsilon^3)$$

$$= P(\epsilon+\epsilon^4) + Q(\epsilon^2+\epsilon^3) + R(\epsilon-\epsilon^4) + S(\epsilon^2-\epsilon^3), \text{ say},$$

$$f(\epsilon^4) = P(\epsilon+\epsilon^4) + Q(\epsilon^2+\epsilon^3) - R(\epsilon-\epsilon^4) - S(\epsilon^2-\epsilon^3),$$

and $$p = P^2(\epsilon^2+\epsilon^3+2) + Q^2(\epsilon+\epsilon^4+2) - 2PQ$$
$$-R^2(\epsilon^2+\epsilon^3-2) - S^2(\epsilon+\epsilon^4-2) - 2RS(\epsilon^2+\epsilon^3-\epsilon-\epsilon^4).$$

Since P, Q, R, S are rational, this involves

$$p = \tfrac{3}{2}P^2 + \tfrac{3}{2}Q^2 - 2PQ + \tfrac{5}{2}R^2 + \tfrac{5}{2}S^2,$$
$$0 = P^2 - Q^2 - R^2 + S^2 - 4RS.$$

When for P, Q, R, S their values in terms of A, B, C, D are entered, these equations become

$$12^2 p = [4p-16-25(A+B)]^2+5.15^2[A-B]^2+2.15^2C^2+2.15^2D^2, \quad \text{(iv)}$$

$$0 = [4p-16-25(A+B)][A-B]+3[C^2+4CD-D^2]. \quad \text{(v)}$$

It follows from the preceding remarks concerning the factors of p, that an integral solution of (iv) and (v) exists, and that apart from the obvious substitution

$$A' = B, \quad B' = A, \quad C' = -D, \quad D' = C,$$

it is unique.

From the multiplication table of the A's, and the relation

$$A_0+A_1+A_2+A_3+A_4 = -1,$$

the equation of the fifth degree which they satisfy can be calculated, the only symmetric function which involves rather lengthy arithmetic being the product. I find the equation to be

$$x^5+x^4-\tfrac{2}{5}(p-1)x^3+\left[\tfrac{1}{3}p(A+B)-\frac{2(p-1)(2p+3)}{3.5^2}\right]x^2$$

$$+\left[\tfrac{1}{9}p\left(\frac{p-1}{5}+A+B\right)^2-pAB-\frac{(p-1)^2}{5^3}\right]x$$

$$+\tfrac{1}{5}p\left[\frac{1}{5.6^3}\left(5(A+B)-\frac{4p-4}{5}\right)^3+\frac{1}{6^2}\left(\frac{2p-2}{5}-A-B\right)^2\right.$$

$$\left.+\tfrac{1}{4}(A-B)^2+\tfrac{1}{8}(A-B)(D^2-C^2)\right]-\frac{(p-1)^3}{5^5} = 0.$$

The coefficients are, as they should be, invariant for the substitution

$$A' = B, \quad B' = A, \quad C' = -D, \quad D' = C.$$

I have verified the numerical values of the above coefficients in the two cases $p = 11$ and $p = 41$, which make it probable that they are correct.

3. If the prime p is not too great, the determination of the integral solutions of (iv) and (v) does not involve lengthy calculations. When $C^2+4CD-D^2$ has a given value m, while C and D are integers, C^2+D^2 is less than $m/\sqrt{5}$. Hence combining the two equations

$$|14p-16-25(A+B)|+15\sqrt{5}\,|A-B| < 12\sqrt{p}.$$

If $p \equiv 1 \pmod 3$, $A+B \equiv 0 \pmod 3$, while if $p \equiv -1 \pmod 3$, $A+B \equiv 1 \pmod 3$. Hence the successive values of first term on the right in this inequality differ by 75, and the number of possibilities for $(A+B)$ and $|A-B|$ is not very great. It is to be noticed that $|A-B|$ cannot be zero, and that $|A-B|$ and $A+B$ are of the same parity.

Finally, the four solutions of the two equations given by applying the substitution

$$A' = B, \quad B' = A, \quad C' = -D, \quad D' = C,$$

to any one of them correspond to the permutation $(abdc)(ef)$ of the original coefficients.

As an example I take the case $p = 271$. Here $A+B = 3E$, and the inequality is

$$|356-25E| + 5\sqrt{5}\,|A-B| < 4\sqrt{(271)}.$$

This gives

$$E = 13, \quad |A-B| = 1 \text{ or } 3,$$

$$E = 14, \quad |A-B| = 2 \text{ or } 4,$$

$$E = 15, \quad |A-B| = 1 \text{ or } 3.$$

The values of C^2+D^2 in the six cases are 65, 45, 75, 55, 76 and 46 respectively. The only possible one is the first, and this gives

$$C^2+D^2 = 65,$$

$$C^2+4CD-D^2 = \pm 31,$$

according as $A-B$ is $\mp$. The $+$ sign must be taken for $A-B$, and then $C = 1$, $D = 8$. Hence

$$A+B = 39, \quad A-B = 1, \quad C = 1, \quad D = 8,$$

giving

$$a = 14, \quad b = 10, \quad c = 9, \quad d = 6,$$

$$a = 14, \quad e = 11, \quad f = 12.$$

For primes less than 100, the table is

p	A	B	C	D
11	0	1	1	0
31	1	2	4	−1
41	4	3	3	0
61	5	4	4	−1
71	7	6	2	3

4. It is found on trial that the equations (iv) and (v) are the conditions that equations (i) and those derived from them by the permutation $(A_0 A_1 A_2 A_3 A_4)$ should, with the relations (ii), form a consistent multiplication table for a set of symbols A_i $(i = 0, 1, 2, 3, 4)$, such that $A_i A_j = A_j A_i$. It was, in fact, in this way that I was originally led to them. This result is still true, if in (iv) and (v), p is replaced by $5\lambda + 1$, without making any assumption at all with regard to λ, a, a, b, c, d, e, f, except that they obey the ordinary laws of arithmetic. If, however, (iv) and (v) are arrived at in this way, there is clearly no direct way of shewing that, when $\lambda = (p-1)/5$, and the other letters are positive integers, (iv) and (v) have only one system of solutions.

I have carried the case $q = 7$ so far as to assure myself that it is not quite parallel with that of $q = 5$. A set of three simultaneous Diophantine relations occur, but they are *not* sufficient to ensure that the equations expressing the products of the A's form a consistent multiplication table.

ON LAMÉ'S DIFFERENTIAL EQUATION AND ELLIPSOIDAL HARMONICS

By E. T. WHITTAKER.

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1. *Introductory.*

The object of the present paper is in the first place to establish the following theorem :—

The solutions of the homogeneous integral equation

$$y(x) = \lambda \int_0^{4K} P_n(k \operatorname{sn} x \operatorname{sn} s)\, y(s)\, ds \tag{1}$$

(where n is a positive integer, P_n is Legendre's function, and k is the modulus, and $4K$ the period of the elliptic functions sn) *are solutions of Lamé's differential equation*

$$\frac{d^2y}{dx^2} = \{n(n+1)\, k^2 \operatorname{sn}^2 x + A\}\, y, \tag{2}$$

and, in fact, are precisely those solutions of Lamé's differential equation which are rational in $\operatorname{sn} x$.

It is well known that the differential equation (2) has *no* solutions which are rational in $\operatorname{sn} x$, except when the constant A has one of a certain sequence of values—the "Eigenwerte" or "characteristic values" or "auto-values" of A. The integral equation (1) does not involve A at all, but it involves the constant λ, which does not occur in the differential equation, and the integral equation (1) has no solution (other than zero) except when λ has one of a certain sequence of Eigenwerte. The theorem asserts that the solutions of (1) corresponding to the Eigenwerte of λ are the same as the solutions of (2) corresponding to the Eigenwerte of A.

This theorem is applied in the later part of the paper in order to solve a problem which has been present to my mind for some years past—namely, that of expressing the "ellipsoidal harmonics" (that is to say, the solution of Laplace's equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

which are appropriate to the ellipsoid), in the form which, as I showed in 1902,* represents the general solution of Laplace's equation, namely,

$$V=\int_0^{2\pi} f(x\cos\theta+y\sin\theta+iz,\ \theta)\,d\theta.$$

It may be recalled that most of the well-known particular solutions of Laplace's equation can be readily expressed in this form; for instance, the fundamental spherical harmonic, cylindrical harmonic, and elliptic-cylindrical harmonic solutions can be expressed respectively by the equations

$$r^n P_n^m(\cos\theta)\cos m\phi=\text{constant}\times\int_0^{2\pi}(x\cos\theta+y\sin\theta+iz)^n\cos m\theta\,d\theta,$$

$$e^{-kz}J_m(k\rho)\cos m\phi=\text{constant}\times\int_0^{2\pi}e^{ik(x\cos\theta+y\sin\theta+iz)}\cos m\theta\,d\theta,$$

$$e^{-kz}\,ce_n(\xi)\,ce_n(i\eta)=\text{constant}\times\int_0^{2\pi}e^{ik(x\cos\theta+y\sin\theta+iz)}\,ce_n(\theta)\,d\theta.$$

But the ellipsoidal harmonics, which are products of Lamé's functions, resisted all attempts to express them in this form; and, indeed, I satisfied myself that it was impossible to obtain such an expression of them by use of any of the properties of Lamé's functions then known. The discovery of the integral equation (1), however, has now enabled me to show (in § 4) that *any ellipsoidal harmonic in the rectangular coordinates* (x, y, z) *can be expressed in the form*

$$\int_0^{4K} P_n\left\{\frac{1}{k'c}(k'x\,\mathrm{sn}\,s+y\,\mathrm{cn}\,s+iz\,\mathrm{dn}\,s)\right\}E(s)\,ds, \tag{3}$$

where P_n denotes Legendre's function, and $E(s)$ denotes a Lamé's function. This expression leads (in § 5) to a very *general property possessed*

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* *Mathematische Annalen*, Vol. 57 (1903), pp. 333–355.

by all Lamé's functions, which can be expressed by the equation

$$E(\alpha)\,E(\beta)\,E(\gamma)$$

$$= \lambda \int_0^{4K} P_n\left(k^2 \operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}s - \frac{k^2}{k'^2}\operatorname{cn}\alpha \operatorname{cn}\beta \operatorname{cn}\gamma \operatorname{cn}s \right.$$

$$\left. - \frac{1}{k'^2}\operatorname{dn}\alpha \operatorname{dn}\beta \operatorname{dn}\gamma \operatorname{dn}s\right) E(s)\,ds, \qquad (4)$$

where α, β, and γ are any three quantities. By assigning special constant values to β and γ, an infinite number of integral equations for Lamé's functions can be obtained, of which that obtained in the beginning of the present paper is one.

2. *Derivation of the Fundamental Integral Equation.*

We shall now proceed to establish the result numbered (1) above.

For this purpose we shall require a property possessed by the Legendrian function $P_n(k \operatorname{sn} x \operatorname{sn} s)$, namely, that *it is a solution of the partial differential equation*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial s^2} = n(n+1)\,k^2(\operatorname{sn}^2 x - \operatorname{sn}^2 s)\,u. \qquad (5)$$

To prove this, we have by direct differentiation,

$$\frac{\partial^2}{\partial x^2} P_n(k \operatorname{sn} x \operatorname{sn} s) - \frac{\partial^2}{\partial s^2} P_n(k \operatorname{sn} x \operatorname{sn} s)$$

$$= P_n''(k \operatorname{sn} x \operatorname{sn} s)\, k^2(\operatorname{cn}^2 x \operatorname{dn}^2 x \operatorname{sn}^2 s - \operatorname{cn}^2 s \operatorname{dn}^2 s \operatorname{sn}^2 x)$$

$$+ 2k^3 P_n'(k \operatorname{sn} x \operatorname{sn} s) \operatorname{sn} x \operatorname{sn} s\,(\operatorname{sn}^2 x - \operatorname{sn}^2 s)$$

$$= k^2(\operatorname{sn}^2 x - \operatorname{sn}^2 s)\left[(k^2 \operatorname{sn}^2 x \operatorname{sn}^2 s - 1)\, P_n'(k \operatorname{sn} x \operatorname{sn} s)\right.$$

$$\left. + 2k \operatorname{sn} x \operatorname{sn} s\, P_n'(k \operatorname{sn} x \operatorname{sn} s)\right]$$

$$= n(n+1)\,k^2(\operatorname{sn}^2 x - \operatorname{sn}^2 s)\,P_n(k \operatorname{sn} x \operatorname{sn} s),$$

which establishes the result.

Now, let $I(x)$ denote the integral

$$\int_0^{4K} P_n(k \operatorname{sn} x \operatorname{sn} s)\, y(s)\,ds,$$

where $y(s)$ denotes a solution of Lamé's equation

$$\frac{d^2y}{ds^2} = \{n(n+1)\,k^2\,\mathrm{sn}^2\,s + A\}\,y, \tag{6}$$

the solution in question being rational in sn s.

Then we have

$$\frac{d^2I}{dx^2} - \{n(n+1)\,k^2\,\mathrm{sn}^2\,x + A\}\,I$$

$$= \int_0^{4K}\left[\frac{\partial^2}{\partial x^2}P_n(k\,\mathrm{sn}\,x\,\mathrm{sn}\,s) - n(n+1)k^2\,\mathrm{sn}^2\,x\,P_n(k\,\mathrm{sn}\,x\,\mathrm{sn}\,s) - AP_n(k\,\mathrm{sn}\,x\,\mathrm{sn}\,s)\right] \times y(s)\,ds.$$

Using the above property (5) of the Legendrian function, the right-hand side of this equation becomes

$$\int_0^{4K}\left[\frac{\partial^2}{\partial s^2}P_n(k\,\mathrm{sn}\,x\,\mathrm{sn}\,s) - n(n+1)k^2\,\mathrm{sn}^2\,s\,P_n(k\,\mathrm{sn}\,x\,\mathrm{sn}\,s) - AP_n(k\,\mathrm{sn}\,x\,\mathrm{sn}\,s)\right] \times y(s)\,ds.$$

Substituting from equation (6), this becomes

$$\int_0^{4K}\left[y(s)\,\frac{\partial^2}{\partial s^2}P_n(k\,\mathrm{sn}\,x\,\mathrm{sn}\,s) - P_n(k\,\mathrm{sn}\,x\,\mathrm{sn}\,s)\,\frac{d^2y}{ds^2}\right]ds,$$

or

$$\left[y(s)\,\frac{\partial}{\partial s}\,P_n(k\,\mathrm{sn}\,x\,\mathrm{sn}\,s) - P_n(k\,\mathrm{sn}\,x\,\mathrm{sn}\,s)\,\frac{dy}{ds}\right]_0^{4K},$$

and this is zero, since $P_n(k\,\mathrm{sn}\,x\,\mathrm{sn}\,s)$ and $y(s)$ have $4K$ as a period.

The integral I is therefore a solution of the equation

$$\frac{d^2y}{dx^2} - \{n(n+1)\,k^2\,\mathrm{sn}^2\,x + A\}\,y = 0\,; \tag{7}$$

and from its form it is evidently a rational function of sn x. But there cannot be more than one independent solution of the equation (7) which is rational in sn x: and this is the solution which, when s was taken as variable, was denoted by $y(s)$. Therefore I must be a constant multiple of this solution $y(x)$; that is to say, *the solutions of Lamé's equation* (7) *which are rational in* sn x *satisfy the homogeneous integral equation*

$$y(x) = \lambda\int_0^{4K} P_n(k\,\mathrm{sn}\,x\,\mathrm{sn}\,s)\,y(s)\,ds.$$

3. *A Particular Case.*

As an illustration we shall verify this result for one of the simplest of Lamé's functions, namely,

$$E(x) = k^2 \operatorname{sn}^2 x - t,$$

where $$t = \tfrac{1}{3}\{1+k^2+\sqrt{(1-k^2+k^4)}\}.$$

As we shall find, the verification by direct integration is by no means obvious, but requires the recurrence formulæ between the integrals of elliptic functions, as well as the algebraic properties of t.

For this particular function we have $n=2$, so the theorem to be verified is

$$E(x) = \lambda \int_0^{4K} (3k^2 \operatorname{sn}^2 x \operatorname{sn}^2 s - 1)(k^2 \operatorname{sn}^2 s - t)\, ds. \tag{8}$$

Now we have $\dfrac{d}{ds}(\operatorname{sn} s \operatorname{cn} s \operatorname{dn} s) = 1-2(1+k^2)\operatorname{sn}^2 s + 3k^2 \operatorname{sn}^4 s.$

Integrating with respect to s between the limits 0 and $4K$, we see that if the integrals $\int_0^{4K} ds$ and $\int_0^{4K} \operatorname{sn}^2 s\, ds$ are denoted by I and J respectively, we have

$$\int_0^{4K} \operatorname{sn}^4 s\, ds = \frac{1}{3k^2}\{2(1+k^2)J - I\}.$$

Substituting in equation (8), the integral on the right hand becomes

$$tI - k^2(1+3t \operatorname{sn}^2 x) J + k^2 \operatorname{sn}^2 x \{2(1+k^2)J - I\},$$

or $$-I(k^2 \operatorname{sn}^2 x - t) - k^2 J\{1+3t \operatorname{sn}^2 x - 2(1+k^2)\operatorname{sn}^2 x\}.$$

But since t satisfies the equation

$$3t^2 - 2t(1+k^2) + k^2 = 0,$$

we have $$3t - 2(1+k^2) = -\frac{k^2}{t};$$

and therefore the expression may be written

$$-I(k^2 \operatorname{sn}^2 x - t) - \frac{k^2 J}{t}(k^2 \operatorname{sn}^2 x - t),$$

or $$-\left(I + \frac{k^2 J}{t}\right) E(x),$$

which establishes the result.

4. *Expression of the General Ellipsoidal Harmonic.*

We shall now obtain an expression for the general ellipsoidal harmonic in the form

$$\int_0^{2\pi} f(x\cos\theta+y\sin\theta+iz,\ \theta)\,d\theta.$$

Let

$$\frac{x^2}{\lambda^2}+\frac{y^2}{\lambda^2-b^2}+\frac{z^2}{\lambda^2-c^2}=1$$

be a family of confocal quadrics. The *ellipsoidal harmonics* associated with this family are defined to be the product of one of the quantities

$$1,\quad x,\quad y,\quad z,\quad yz,\quad zx,\quad xy,\quad xyz,$$

by an expression of the type

$$\left(\frac{x^2}{t_1}+\frac{y^2}{t_1-b^2}+\frac{z^2}{t_1-c^2}-1\right)\left(\frac{x^2}{t_2}+\frac{y^2}{t_2-b^2}+\frac{z^2}{t_2-c^2}-1\right)\\ \dots\left(\frac{x^2}{t_k}+\frac{y^2}{t_k-b^2}+\frac{z^2}{t_k-c^2}-1\right)$$

(where t_1, t_2, ..., t_k are constants), such that the whole product satisfies Laplace's equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0.$$

It is a well known theorem that there are always $(2n+1)$ of these ellipsoidal harmonics of any degree n in (x, y, z). We shall denote any one of them by $G_n(x, y, z)$.

The terms of highest degree in $G_n(x, y, z)$ will evidently be

$$\left\{\begin{matrix} & x & yz & \\ 1 & y & zx & xyz \\ & z & xy & \end{matrix}\right\}\left(\frac{x^2}{t_1}+\frac{y^2}{t_1-b^2}+\frac{z^2}{t_1-c^2}\right)\left(\frac{x^2}{t_2}+\frac{y^2}{t_2-b^2}+\frac{z^2}{t_2-c^2}\right)\\ \dots\left(\frac{x^2}{t_k}+\frac{y^2}{t_k-b^2}+\frac{z^2}{t_k-c^2}\right),$$

the large bracket indicating that some one of the quantities inside it is to be taken as a multiplier of the product outside. These terms of highest degree we shall denote by $H_n(x, y, z)$. It is obvious that $H_n(x, y, z)$ satisfies Laplace's equation on its own account; and being homogeneous of degree n in (x, y, z), it will be expressible in the form

$$H_n(x, y, z)=\int_0^{2\pi}(x\cos\theta+y\sin\theta+iz)^n f(\theta)\,d\theta,$$

where $f(\theta)$ is not yet known.

Now it is known* that

$$G_n(x, y, z) = \left\{ 1 - \frac{D^2}{2(2n-1)} + \frac{D^4}{2.4(2n-1)(2n-3)} - \dots \right\} H_n(x, y, z),$$

where D^2 stands for $\alpha^2 \frac{\partial^2}{\partial x^2} + \beta^2 \frac{\partial^2}{\partial y^2} + \gamma^2 \frac{\partial^2}{\partial z^2}$,

and (α, β, γ) are the semi-axes of any one of the quadrics of the family.

Substituting for $H_n(x, y, z)$ the above value, we have

$$G_n(x, y, z)$$
$$= \int_0^{2\pi} \left\{ A^n - \frac{n(n-1)}{2(2n-1)} A^{n-2} B^2 + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} A^{n-4} B^4 - \dots \right\} f(\theta)\, d\theta,$$

where $A = x \cos\theta + y \sin\theta + iz$,

and $B = \sqrt{(\alpha^2 \cos^2\theta + \beta^2 \sin^2\theta - \gamma^2)} = \sqrt{(c^2 - b^2 \sin^2\theta)}$,

so $$G_n(x, y, z) = \int_0^{2\pi} P_n\left(\frac{A}{B}\right) f(\theta)\, d\theta,$$

where P_n denotes Legendre's function; a constant factor and a B^n having been absorbed into the $f(\theta)$.

Thus *we have the ellipsoidal harmonic expressed in the form*

$$G_n(x, y, z) = \int_0^{2\pi} P_n\left(\frac{x\cos\theta + y\sin\theta + iz}{\sqrt{(c^2 - b^2\sin^2\theta)}}\right) f(\theta)\, d\theta,$$

where $f(\theta)$ is a function as yet undetermined.

The form of this result suggests a change of variable. Let s be a new variable introduced in place of θ, and defined by the equation

$$\sin\theta = \frac{\operatorname{cn} s}{\operatorname{dn} s},$$

where the elliptic functions are formed with the modulus $k = b/c$. Then we have

$$\frac{\cos\theta}{\sqrt{(c^2 - b^2\sin^2\theta)}} = \frac{\operatorname{sn} s}{c}, \quad \frac{\sin\theta}{\sqrt{(c^2 - b^2\sin^2\theta)}} = \frac{\operatorname{cn} s}{k'c}, \quad d\theta = -\frac{k'ds}{\operatorname{dn} s},$$

and the formula for $G^n(x, y, z)$ becomes

$$G_n(x, y, z) = \int_0^{4K} P_n\left\{\frac{1}{k'c}(k'x \operatorname{sn} s + y \operatorname{cn} s + iz \operatorname{dn} s)\right\} \phi(s)\, ds, \qquad (9)$$

where $\phi(s)$ is a function which we shall now proceed to determine.

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* W. D. Niven, *Phil. Trans.*, Vol. 182 (1891), p. 245.

In place of (x, y, z) introduce coordinates (α, β, γ) defined by the equations

$$x = ck^2 \operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma,$$

$$y = -c\frac{k^2}{k'} \operatorname{cn}\alpha \operatorname{cn}\beta \operatorname{cn}\gamma,$$

$$z = \frac{ic}{k'} \operatorname{dn}\alpha \operatorname{dn}\beta \operatorname{dn}\gamma,$$

the elliptic functions being formed with the same modulus as before.

Then the equations

$$\alpha = \text{constant}, \quad \beta = \text{constant}, \quad \gamma = \text{constant},$$

represent respectively the three families of surfaces of the confocal family: and it is known from the way in which ellipsoidal harmonics are built up of Lamé's functions that the ellipsoidal harmonic $G_n(x, y, z)$ considered either as a function of α alone, or as a function of β alone, or as a function of γ alone, satisfies Lamé's differential equation.

Now G in its new form is

$$G_n(x, y, z) = \int_0^{4K} P_n\left(k^2 \operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}s - \frac{k^2}{k'^2} \operatorname{cn}\alpha \operatorname{cn}\beta \operatorname{cn}\gamma \operatorname{cn}s \right.$$
$$\left. - \frac{1}{k'^2} \operatorname{dn}\alpha \operatorname{dn}\beta \operatorname{dn}\gamma \operatorname{dn}s\right) \phi(s)ds,$$

and this satisfies Lamé's equation when regarded as a function of α, whatever constant values may be assigned to β and γ. Suppose then that we assign the value K to β, and the value $K+iK'$ to γ, so that

$$\operatorname{cn}\beta = 0, \quad \operatorname{sn}\beta = 1, \quad \operatorname{dn}\gamma = 0, \quad \operatorname{sn}\gamma = \frac{1}{k}.$$

Thus

$$\int_0^{4K} P_n(k \operatorname{sn}\alpha \operatorname{sn}s)\, \phi(s)\, ds$$

is a solution of Lamé's equation in α. Comparing this with the result of § 2, we see that

$$\phi(s) = E(s),$$

where $E(s)$ is a Lamé's function, and indeed is the particular Lamé's function associated with the ellipsoidal harmonic considered. Thus substituting this value of $\phi(s)$ in equation (9), *we have the required expression for a general ellipsoidal harmonic, namely,*

$$G_n(x, y, z) = \text{constant} \times \int_0^{2\pi} P_n\left\{\frac{1}{k'c}(k'x \operatorname{sn}s + y \operatorname{cn}s + iz \operatorname{dn}s)\right\} E(s)\, ds. \quad (10)$$

5. *A General Property of Lamé's Functions.*

If in equation (10) we express the ellipsoidal harmonic as a product of three Lamé's functions, the equation becomes

$$E(\alpha)\, E(\beta)\, E(\gamma)$$
$$= \text{constant} \times \int_0^{4K} P_n \left(k^2 \operatorname{sn} \alpha \operatorname{sn} \beta \operatorname{sn} \gamma \operatorname{sn} s - \frac{k^2}{k'^2} \operatorname{cn} \alpha \operatorname{cn} \beta \operatorname{cn} \gamma \operatorname{cn} s \right.$$
$$\left. - \frac{1}{k'^2} \operatorname{dn} \alpha \operatorname{dn} \beta \operatorname{dn} \gamma \operatorname{dn} s \right) E(s)\, ds.$$

If in this equation (which includes as a particular case the integral equation obtained in § 2) we regard any two of the quantities α, β, γ—say β and γ—as constants, we obtain a homogeneous integral equation for the Lamé's function $E(\alpha)$, with a symmetrical nucleus. It is obvious that an infinite number of homogeneous integral equations satisfied by the Lamé's function $E(\alpha)$ can be obtained by assigning special values to β and γ, or differentiating, &c., with respect to one of them.

THE MEAN VALUE OF THE MODULUS OF AN ANALYTIC FUNCTION

By G. H. HARDY.

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1. Suppose that $f(x)$ is an analytic function of the complex variable x, regular for $|x| < \rho$, and that $M(r)$ denotes, as usual, the maximum of $|f(x)|$ on the circle $|x| = r < \rho$. Then it is known that $M(r)$ possesses the following properties:—

(i) $M(r)$ is a steadily increasing function of r;

(ii) $\log M(r)$ is a convex function of $\log r$, so that

$$\log M(r) \leqslant \frac{\log (r_2/r)}{\log (r_2/r_1)} \log M(r_1) + \frac{\log (r/r_1)}{\log (r_2/r_1)} \log M(r_2),$$

if
$$0 < r_1 \leqslant r \leqslant r_2 < \rho.$$

Further, when $f(x)$ is an integral function, so that $\rho = \infty$, it is known that

(iii) $M(r)$ tends to infinity with (r), and, unless $f(x)$ is a polynomial, more rapidly than any power of r.*

It was suggested to me by Dr. H. Bohr and Prof. E. Landau, rather more than a year ago, that the property (i) is possessed also by the *mean* value of $|f(x)|$ on the circle $|x| = r$, *i.e.*, by the function

$$\mu(r) = \frac{1}{2\pi} \int_0^{\pi} |f(re^{i\theta})| \, d\theta.$$

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* The theorems (i) and (iii) are classical. Theorem (ii) was discovered independently by Blumenthal (*Jahresbericht der Deutschen Math.-Vereinigung*, Vol. 16, p. 97), Faber (*Math. Annalen*, Vol. 63, p. 549), and Hadamard (*Bulletin de la Soc. Math. de France*, Vol. 24, p. 186). The first statement of the theorem was due to Hadamard and the first proof to Blumenthal. The theorem is a corollary of one concerning the associated radii of convergence of a power series in two variables, due to Fabry (*Comptes Rendus*, Vol. 134, p. 1190), and Hartogs (*Math. Annalen*, Vol. 62, p. 1).

In the attempt to prove this I have been led to prove a good deal more, in particular that the function $\mu(r)$, and the more general function

$$\mu_\delta(r) = \frac{1}{2\pi}\int_0^\pi |f(re^{i\theta})|^\delta \, d\theta,$$

where δ is any positive number, possesses *all* the properties (i)–(iii) characteristic of $M(r)$. It should be observed that this is obvious when $\delta = 1$ and $\sqrt{\{f(x)\}}$ is one-valued for $r < \rho$; for then we have

$$\sqrt{\{f(x)\}} = b_0 + b_1 x + b_2 x^2 + \ldots,$$

say, and $$\mu(r) = |b_0|^2 + |b_1|^2 r^2 + |b_2|^2 r^4 + \ldots .$$

2. The argument of the following paragraphs depends on two lemmas concerning conjugate functions*.

Suppose that $$x = \xi + i\eta,$$

and that $$X = \Xi + i\mathrm{H}$$

is a function of x regular for all values of x under consideration. Then Ξ and H are real conjugate functions of ξ and η.

Let ψ be a real function of Ξ and H, and so of ξ and η, with continuous second derivatives. Then the lemmas in question are expressed by the formulæ

$$\text{(A)} \qquad \frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = \left(\frac{\partial^2\psi}{\partial\Xi^2} + \frac{\partial^2\psi}{\partial\mathrm{H}^2}\right) M^2,$$

$$\text{(B)} \qquad \left(\frac{\partial\psi}{\partial\xi}\right)^2 + \left(\frac{\partial\psi}{\partial\eta}\right)^2 = \left\{\left(\frac{\partial\psi}{\partial\Xi}\right)^2 + \left(\frac{\partial\psi}{\partial\mathrm{H}}\right)^2\right\} M^2,$$

where $$M = \left|\frac{dX}{dx}\right| = \sqrt{\left\{\left(\frac{\partial\Xi}{\partial\xi}\right)^2 + \left(\frac{\partial\mathrm{H}}{\partial\xi}\right)^2\right\}} = \sqrt{\left\{\left(\frac{\partial\Xi}{\partial\eta}\right)^2 + \left(\frac{\partial\mathrm{H}}{\partial\eta}\right)^2\right\}}.$$

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* The use of these lemmas was suggested to me by Dr. Bromwich, at a time when the paper contained only a part of its present contents. The whole argument has been reconstructed in consequence of this suggestion, and is much more concise and elegant than it was before. I am also indebted to Dr. Bromwich and to a referee for a number of minor suggestions. The lemmas themselves are given in Clerk-Maxwell's *Electricity and Magnetism*, Vol. 1, p. 289, and Dr. Bromwich informs me that they are due to Lamé ("Mémoire sur les Lois de l'Équilibre du Fluide Éthéré", *Journal de l'École Polytechnique*, Vol. 3, cahier 23).

The formula (A) and (B) may be proved as follows. From the equations

$$\frac{\partial\psi}{\partial\xi} = \frac{\partial\psi}{\partial\Xi}\frac{\partial\Xi}{\partial\xi} + \frac{\partial\psi}{\partial\mathrm{H}}\frac{\partial\mathrm{H}}{\partial\xi}, \quad \dots, \quad \dots, \quad \dots,$$

$$\frac{dX}{dx} = \frac{\partial X}{\partial\xi} = -i\frac{\partial X}{\partial\eta} = \frac{\partial\Xi}{\partial\xi} + i\frac{\partial\mathrm{H}}{\partial\xi} = -i\frac{\partial\Xi}{\partial\eta} + \frac{\partial\mathrm{H}}{\partial\eta},$$

it is easy to deduce that

(1) $$\frac{\partial\psi}{\partial\xi} - i\frac{\partial\psi}{\partial\eta} = \left(\frac{\partial\psi}{\partial\Xi} - i\frac{\partial\psi}{\partial\mathrm{H}}\right)\mu,$$

(2) $$\frac{\partial\psi}{\partial\xi} + i\frac{\partial\psi}{\partial\eta} = \left(\frac{\partial\psi}{\partial\Xi} + i\frac{\partial\psi}{\partial\mathrm{H}}\right)\bar{\mu},$$

where $\mu = \frac{dX}{dx}$ and $\bar{\mu}$ is the conjugate of μ. The formula (B) follows at once by multiplication. To prove (A) we operate on (1) with the operator

$$\frac{\partial}{\partial\xi} + i\frac{\partial}{\partial\eta},$$

and apply (2), observing that $$\left(\frac{\partial}{\partial\xi} + i\frac{\partial}{\partial\eta}\right)\mu = 0.$$

3. Suppose now that $X = f(x)$ is regular for $|x| < \rho$, and that D is an annular region, defined by inequalities of the form

$$0 < r_1 \leqslant r = |x| \leqslant r_2 < \rho,$$

and including no zeros of $f(x)$.

Let $$\log x = \log r + i\theta = \zeta = \rho + i\theta,$$

$$\log X = \log R + i\Theta = \mathrm{Z} = \mathrm{P} + i\Theta,$$

where $$r > 0, \quad R > 0, \quad -\pi < \theta \leqslant \pi, \quad -\pi < \Theta \leqslant \pi.$$

Then P and Θ are conjugate functions of ρ and θ, with second derivatives continuous for all values of ρ and θ which correspond to values of x in D.

Let us take $$\psi = F(R) = \phi(\mathrm{P}),$$

where $F(R)$ is a function with a continuous second differential coefficient. Applying Lemma A, we obtain

(1) $$\frac{\partial^2\phi}{\partial\rho^2} + \frac{\partial^2\phi}{\partial\theta^2} = \frac{\partial^2\phi}{\partial\mathrm{P}^2}M^2,$$

where

(2) $$M^2 = \left|\frac{d\mathrm{Z}}{d\zeta}\right|^2 = \left(\frac{\partial\mathrm{P}}{\partial\rho}\right)^2 + \left(\frac{\partial\Theta}{\partial\rho}\right)^2.$$

Let us now suppose that $\log\phi(\mathrm{P})$ is a positive and convex function of P, so that

$$\frac{\partial^2}{\partial \mathrm{P}^2} \log \phi(\mathrm{P}) \geqslant 0,$$

or

$$\phi \frac{\partial^2 \phi}{\partial \mathrm{P}^2} \geqslant \left(\frac{\partial \phi}{\partial \mathrm{P}}\right)^2;$$

and let

(3)
$$\nu(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\mathrm{P})\, d\theta.$$

Then

$$\nu'(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \phi}{\partial \mathrm{P}} \frac{\partial \mathrm{P}}{\partial \rho}\, d\theta,$$

$$|\nu'(\rho)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \left|\frac{\partial \phi}{\partial \mathrm{P}}\right| \left|\frac{\partial \mathrm{P}}{\partial \rho}\right| d\theta \leqslant \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\left\{\phi \frac{\partial^2 \phi}{\partial \mathrm{P}^2}\right\}} \left|\frac{\partial \mathrm{P}}{\partial \rho}\right| d\theta,$$

and so, by Schwarz's inequality,

(4)
$$\{\nu'(\rho)\}^2 \leqslant \frac{1}{4\pi^2} \int_0^{2\pi} \phi\, d\theta \int_0^{2\pi} \frac{\partial^2 \phi}{\partial \mathrm{P}^2} \left|\frac{\partial \mathrm{P}}{\partial \rho}\right|^2 d\theta \leqslant \frac{\nu(\rho)}{2\pi} \int_0^{2\pi} \frac{\partial^2 \phi}{\partial \mathrm{P}^2} M^2 d\theta.$$

But

$$\nu''(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 \phi}{\partial \rho^2}\, d\theta$$

and

$$\frac{\partial^2 \phi}{\partial \rho^2} + \frac{\partial^2 \phi}{\partial \theta^2} = \frac{\partial^2 \phi}{\partial \mathrm{P}^2} M^2,$$

by (1). Hence

(5)
$$\nu''(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 \phi}{\partial \mathrm{P}^2} M^2 d\theta - \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 \phi}{\partial \theta^2}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 \phi}{\partial \mathrm{P}^2} M^2 d\theta,$$

since ϕ is a function of P or of R only, and R is periodic in θ. From (4) and (5) it follows that

(6)
$$\nu(\rho)\, \nu''(\rho) \geqslant \{\nu'(\rho)\}^2,$$

or that $\log \nu$ is a convex function of ρ.

We have thus proved

THEOREM I.—*If* $\log \{\phi(\log R)\}$

is a convex function of $\log R$, *then*

$$\log \nu(\log r) = \log \left\{\frac{1}{2\pi} \int_0^{2\pi} \phi(\log R)\, d\theta\right\}$$

is, throughout any interval of values of r *which includes no zeros of* $f(x)$, *a convex function of* $\log r$.

In particular, we may take

$$F(R) = \phi(\mathrm{P}) = e^{\delta\mathrm{P}} = R^{\delta},$$

in which case $\phi\phi'' = \phi'^2$. It follows that $\log \mu_\delta(r)$, and in particular $\log \mu(r)$, is a convex function of $\log r$, throughout any interval of values of r which includes no zeros of $f(x)$. This case is indeed the critical case of Theorem I, the condition that $\phi(\mathrm{P})$ should be a convex function of P being only just satisfied.

4. With Theorem I we may associate another theorem, in which less is postulated and less proved.

THEOREM II.—*If* $\phi(\log R)$ *is a convex function of* $\log R$, *then* $\nu(\log r)$ *is a convex function of* $\log r$.

For $\nu''(\rho)$ is positive, by (5) of § 3. The critical case of Theorem II is that in which $\phi(\log R) = \log R$. In this case we have, by a well known theorem of Jensen*,

$$\nu(\log r) = \frac{1}{2\pi}\int_0^{2\pi} \log R\, d\theta = \log \left| \frac{cr^n}{a_{m+1}a_{m+2}\ldots a_n} \right|,$$

where
$$f(x) = cx^m + \ldots,$$

and $a_{m+1}, a_{m+2}, \ldots, a_n$ are the zeros of $f(x)$, other than the origin, whose moduli are not greater than r. In this case $\nu(\log r)$ is a linear function of $\log r$ throughout any interval of values of r which includes no zeros of $f(x)$.

5. In order to proceed further with our investigations concerning $\mu_\delta(r)$, we must examine the behaviour of $\mu_\delta(r)$ for the exceptional values of r which correspond to zeros of $f(x)$, and for $r = 0$. I shall prove that

$$r\,\frac{d\mu_\delta(r)}{dr}$$

is continuous without exception.

Let
$$x_0 = \rho e^{i\phi} \quad (\rho > 0)$$

be a zero of $f(x)$. We have to prove that

$$\frac{d\mu_\delta(r)}{dr}$$

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* *Acta Mathematica*, Vol. 22, p. 359.

is continuous throughout an interval of values of r of the type

$$\rho-\eta \leqslant r \leqslant \rho+\eta.$$

I shall suppose, for simplicity, that x_0 is the only zero of modulus ρ. The proof is substantially the same when there are several such zeros. I shall prove that the integral

$$\frac{d\mu_\delta(r)}{dr} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial R^\delta}{\partial r}\, d\theta$$

is uniformly convergent throughout the interval $\rho-\eta \leqslant r \leqslant \rho+\eta$, if η is small enough.

We have $$f(x) = (x-x_0)^m f_1(x),$$

where m is a positive integer, and $f_1(x)$ has no zeros whose modulus lies between $\rho-\eta$ and $\rho+\eta$, so that $|f_1(x)|$ lies between positive constants H_1 and H_2.

Now, taking $\psi = F(R)$ in Lemma B, we have

$$\left(\frac{\partial F}{\partial \xi}\right)^2 + \left(\frac{\partial F}{\partial \eta}\right)^2 = \left\{\left(\frac{\partial F}{\partial \Xi}\right)^2 + \left(\frac{\partial F}{\partial \mathrm{H}}\right)^2\right\} \left|\frac{df}{dx}\right|^2.$$

In particular, if $$F(R) = R = \sqrt{(\Xi^2+\mathrm{H}^2)},$$

we have $$\left(\frac{\partial R}{\partial \xi}\right)^2 + \left(\frac{\partial R}{\partial \eta}\right)^2 = \left|\frac{df}{dx}\right|^2,$$

and $$\left|\frac{\partial R}{\partial r}\right| = \left|\cos\theta\frac{\partial R}{\partial \xi} + \sin\theta\frac{\partial R}{\partial \eta}\right| \leqslant \sqrt{\left\{\left(\frac{\partial R}{\partial \xi}\right)^2 + \left(\frac{\partial R}{\partial \eta}\right)^2\right\}} = \left|\frac{df}{dr}\right|.$$

But $$\frac{df}{dx} = m(x-x_0)^{m-1}f_1(x) + (x-x_0)^m\frac{df_1}{dx},$$

and so $$\left|\frac{df}{dx}\right| < K|x-x_0|^{m-1},$$

where K is a constant. Hence

(5) $$\left|\frac{\partial R}{\partial r}\right| < K|x-x_0|^{m-1}.$$

Also

(6) $$R^{\delta-1} < H_2^{\delta-1}|x-x_0|^{m(\delta-1)}$$

if $\delta > 1$, and

(6′) $$R^{\delta-1} < H_1^{\delta-1}|x-x_0|^{m(\delta-1)},$$

if $\delta < 1$. From (5) and (6) or (6′) it follows that

$$\left| R^{\delta-1} \frac{\partial R}{\partial r} \right| < K_1 \, |x-x_0|^{m\delta-1}, \tag{7}$$

where K_1 is a constant. If $m\delta-1 \geqslant 0$, we have

$$\left| R^{\delta-1} \frac{\partial R}{\partial r} \right| < K_2,$$

where K_2 is a constant, and then the integral

$$\int_0^{2\pi} R^{\delta-1} \frac{\partial R}{\partial r} \, d\theta$$

is obviously uniformly convergent. If, on the other hand, $m\delta-1 < 0$, we have

$$|x-x_0| = \sqrt{(r^2+\rho^2-2r\rho\cos\omega)},$$

where $\omega = \theta-\phi$, and so

$$|x-x_0| > K_3 \, |\sin \tfrac{1}{2}\omega|,$$

where K_3 is a constant. The uniform convergence of the integral then follows at once when we compare it with

$$\int_0^{2\pi} |\sin \tfrac{1}{2}\omega|^{m\delta-1} \, d\omega.$$

6. We have thus proved that $\log \mu_\delta(r)$ is a convex function of $\log r$ for all positive values of r save certain exceptional values, and that

$$\frac{d \log \mu_\delta(r)}{d \log r}$$

is continuous even for these values of r. It follows that $\log \mu_\delta(r)$ is a convex function of $\log r$ for all positive values of r without exception*. *A fortiori* is $\mu_\delta(r)$ a convex function of $\log r$, and

$$r \frac{d\mu_\delta(r)}{dr},$$

an increasing function of r.

It remains to consider the behaviour of $r \dfrac{d\mu_\delta(r)}{dr}$ as $r \to 0$. Suppose that the origin is a zero of $f(x)$ of order m. Then

$$R^\delta = r^{m\delta} R_1^\delta,$$

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* A series of continuous convex arcs, fitted together so as to have the same tangents at the points of junction, forms a single convex curve.

where R_1 is positive and has continuous derivatives. Hence

$$r\frac{d\mu_\delta(r)}{dr} = m\delta\, r^{m\delta}\int_0^{2\pi} R_1^\delta\, d\theta + r^{m\delta+1}\int_0^{2\pi}\frac{\partial R_1^\delta}{\partial r}\, d\theta,$$

which plainly tends to zero as $r \to 0$.

Thus $r\dfrac{d\mu_\delta(r)}{dr}$ is continuous and steadily increasing for all positive values of r, and tends to zero as $r \to 0$. It follows that

$$r\frac{d\mu_\delta(r)}{dr} \geqslant 0$$

for all positive values of r.

We have thus proved

THEOREM III.—*The integral*

$$\mu_\delta(r) = \frac{1}{2\pi}\int_0^{\pi} R^\delta\, d\theta \quad (\delta > 0)$$

is a positive, continuous, and steadily increasing function of r. *The same is true of*

$$r\frac{d\mu_\delta(r)}{dr}.$$

And $\log \mu_\delta(r)$, *and a fortiori* $\mu_\delta(r)$ *itself, is a convex function of* $\log r$.

7. The last theorem contains *inter alia* the answer to the question raised by Bohr and Landau. It should, however, be observed that the most appropriate measure of the "average increase" of $f(x)$ is not the mean value of R, or of any power of R, but of $\log R$; for the former means are not adequately affected by the occurrence of zeros of $f(x)$, or of arcs on which R is small.

8. It remains to discuss the analogues for $\mu_\delta(r)$ of the property (iii) of § 1.

We may suppose without loss of generality that $f(x)$ has infinitely many zeros. If it has not, it is of the form

$$P(x)\, e^{g(x)},$$

where $P(x)$ is a polynomial and $g(x)$ an integral function. Now

$$e^{\frac{1}{2}\delta g(x)} = b_0 + b_1 x + b_2 x^2 + \ldots,$$

say; and

$$\mu_\delta^{(1)}(r) = \frac{1}{2\pi}\int_0^{2\pi} |e^{g(x)}|^\delta \, d\theta = |b_0|^2 + |b_1|^2 r^2 + |b_2|^2 r^4 + \dots,$$

certainly tends to infinity more rapidly than any power of r. It follows immediately that the same is true of $\mu_\delta(r)$.

Suppose, then, that $f(x)$ has an infinity of zeros, and that $r_{m+1}, r_{m+2}, \dots, r_n$ are the moduli of those, other than the origin, whose moduli do not exceed r. Then, if $g(\theta)$ is any continuous function of θ, we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{g(\theta)} \, d\theta \geqslant e^{\frac{1}{2\pi}\int_0^{2\pi} g(\theta)\, d\theta};$$

and so $$\mu_\delta(r) = \frac{1}{2\pi}\int_0^{2\pi} R^\delta \, d\theta \geqslant e^{\frac{\delta}{2\pi}\int_0^{2\pi} \log R \, d\theta} = \left| \frac{cr^n}{r_{m+1} r_{m+2} \dots r_n} \right|^\delta,$$

by Jensen's theorem. It follows at once that $\mu_\delta(r)$ tends to infinity with r more rapidly than any power of r. We can indeed go further, and establish relations between the rate of increase of r_n, considered as a function of n, and $\mu_\delta(r)$, considered as a function of r, in every way analogous to those given by Jensen's theorem for $M(r)$.* For example, if the "real order" of $f(x)$ is ρ, we have

$$\mu_\delta(r) > e^{r^{\rho-\epsilon}}$$

for every positive ϵ and values of r surpassing all limit.

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* Lindelöf, *Acta Societatis Fennicae*, Vol. 31, No. 1; see also Borel, *Leçons sur les fonctions méromorphes*, p. 105.

OSCILLATIONS NEAR ONE OF THE ISOSCELES-TRIANGLE SOLUTIONS OF THE THREE BODY PROBLEM

By Daniel Buchanan.

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1. *Introduction.*

The isosceles-triangle solutions of the three body problem are the periodic solutions in which two of the masses are finite and equal, while the third body moves in a straight line and remains equidistant from the equal bodies. This straight line passes through the centre of gravity of the system, and is perpendicular to the plane of the initial motion of the equal bodies. Pavanini obtained the first of these solutions in 1907 by means of an elliptic integral.* In this solution the finite bodies move in a circle, and the third body is infinitesimal. Macmillan also obtained this solution independently in 1910, and further developed the solution as a power series in a parameter having coefficients which are periodic functions of the time.† Solutions for two additional cases, not treated by Pavanini and Macmillan, were obtained by the author of this paper in 1911.‡ In one solution the finite bodies move in ellipses, and in the other all three bodies are finite. When the third body becomes infinitesimal, this second solution reduces to one closely related to that obtained by Macmillan. In 1914, the author of this paper obtained another solution in which all three bodies are finite.§ When the third body becomes infinitesimal this solution reduces to the one in which the finite bodies move in ellipses.

In the present paper, periodic solutions are shown to exist when the finite bodies move in a circle, and the third body, an infinitesimal, oscillates about the line of the isosceles-triangle solutions. The solutions are constructed as power series in two parameters. One parameter is a function of the initial projection of the infinitesimal body from the plane of motion of the finite bodies. There is no direct physical interpretation of the other parameter.

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* *Annali di Matematica*, Ser. III, Vol. XIII (1907), pp. 179–202.

† *Astronomical Journal*, Nos. 625, 626 (1911).

‡ Moulton, *Periodic Orbits*, Chap. X.

§ *Transactions of the American Mathematical Society*, Vol. XVI, No. 2, April, 1915.

2. *The Differential Equations.*

Let m_1 and m_2 represent the two finite bodies of equal mass, and μ the infinitesimal body. Let the unit of mass be chosen so that

$$m_1 = m_2 = m = \tfrac{1}{2};$$

the linear unit so that the distance between m_1 and m_2 shall be unity, and the unit of time so that the Gaussian constant shall also be unity. Let the origin of coordinates be taken at the centre of mass, and the plane of motion of the finite bodies as the $\xi\eta$-plane. Let the coordinates of m_1, m_2 and μ be $\xi_1, \eta_1, 0$; $\xi_2, \eta_2, 0$; and ξ, η, ζ respectively. If m_1 and m_2 are started from the points $\frac{1}{2}, 0, 0$, and $-\frac{1}{2}, 0, 0$ respectively, so that they move in a circle, then

$$\xi_1 = -\xi_2 = \tfrac{1}{2}\cos(t-t_0), \qquad \eta_1 = -\eta_2 = \tfrac{1}{2}\sin(t-t_0),$$

and the differential equations of motion of the infinitesimal body are

$$\left.\begin{aligned} \xi'' &= -m\xi\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right]+m\left[\frac{\xi_1}{\rho_1^3}+\frac{\xi_2}{\rho_2^3}\right] \\ \eta'' &= -m\eta\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right]+m\left[\frac{\eta_1}{\rho_1^3}+\frac{\eta_2}{\rho_2^3}\right] \\ \zeta'' &= -m\zeta\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right] \\ \rho_1 &= \sqrt{\{(\xi-\xi_1)^2+(\eta-\eta_1)^2+\zeta^2\}} \\ \rho_2 &= \sqrt{\{(\xi-\xi_2)^2+(\eta-\eta_2)^2+\zeta^2\}} \end{aligned}\right\}. \tag{1}$$

If we refer the motion of the system to a set of rectangular coordinates rotating about the ζ-axis with the uniform velocity unity, that is, if we make in (1) the substitutions

$$\begin{aligned} \xi &= x\cos(t-t_0)-y\sin(t-t_0), \\ \eta &= x\sin(t-t_0)+y\cos(t-t_0), \\ \zeta &= z, \end{aligned}$$

then the differential equations (1) become

$$\left.\begin{aligned}
x''-2y'-x &= -mx\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]+\tfrac{1}{2}m\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right]\\
y''+2x'-y &= -my\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]\\
z'' &= -mz\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]\\
r_1 &= \sqrt{\{(\tfrac{1}{2}-x)^2+y^2+z^2\}}\\
r_2 &= \sqrt{\{(\tfrac{1}{2}+x)^2+y^2+z^2\}}
\end{aligned}\right\}. \qquad (2)$$

These differential equations have the Jacobian integral

$$(x')^2+(y')^2+(z')^2 = x^2+y^2+2m\left(\frac{1}{r_1}+\frac{1}{r_2}\right)+\text{constant}. \qquad (3)$$

When the infinitesimal body is given an initial projection along the z-axis, it will remain on the z-axis and therefore $x \equiv y \equiv 0$. The differential equation defining its motion then becomes

$$z'' = -\frac{8z}{(1+4z^2)^{\frac{3}{2}}}.$$

The periodic solution of this equation was found to be*

$$\left.\begin{aligned}
z = \psi = {} & a\sin(\tau-\tau_0)+\tfrac{3}{16}a^3[\sin 3(\tau-\tau_0)-\sin(\tau-\tau_0)]+\dots\\
& +a^{2j+1}[A_1^{(2j+1)}\sin(\tau-\tau_0)+\dots+A_{2j+1}^{(2j+1)}\sin(2j+1)(\tau-\tau_0)]+\dots\\
\tau-\tau_0 = {} & \frac{t-t_0}{\sqrt{\{\frac{1}{8}(1+\delta)\}}}, \qquad \delta = \tfrac{9}{2}a^2-\tfrac{141}{32}a^4+\tfrac{35}{2}a^6+\dots
\end{aligned}\right\}. \qquad (4)$$

The constant a is arbitrary, and $a/\sqrt{\{\frac{1}{8}(1+\delta)\}}$ denotes the initial projection from the origin along the z-axis. The series ψ is called a *triply odd series*, since only odd powers of a and odd functions of odd multiples of τ enter it. The numerical values of τ_0 and t_0 may be taken to be zero, without loss of geometric generality, but as we shall have occasion to use $\partial\psi/\partial\tau_0$ we shall keep τ_0 in evidence in (4).

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* Moulton, *Periodic Orbits*, Chap. X, §§ 167, 168.

3. *The Equations of Variation.*

We wish to show the existence of periodic solutions of equations (2) in which x and y are not both identically zero. If we integrate these equations as they stand, the only periodic solutions are

$$x \equiv y \equiv 0, \qquad z = \psi.$$

In obtaining the solution $z = \psi$, we introduced an arbitrary constant δ, by making a transformation on the independent variable t, and this arbitrary was determined by the condition that the solution should be periodic. Since the right members of (2) are functions of z and hence involve τ under trigonometric symbols, it is not practical (if, indeed, it is possible) to obtain periodic solutions by introducing another arbitrary through a transformation on the independent variable τ. Instead of this we make in the first two equations of (2) the substitution

$$m = m_0 + \lambda,$$

and consider λ as a variable parameter, while m and m_0 remain fixed. In order to obtain the solution of the physical problem under consideration, we must put $\lambda = m - m_0$ in the final results. In the last equation of (2), however, we put $m = \frac{1}{2}$. Now let

$$t - t_0 = \sqrt{\left\{\tfrac{1}{8}(1+\delta)\right\}}\,(\tau - \tau_0), \qquad z = \psi + w,$$

where δ and ψ are defined in (4), and w is a function which vanishes with λ. If we denote derivatives with respect to τ by dots, the differential equations become after these substitutions

$$\left.\begin{aligned} &\ddot{x} - 2\sqrt{\left\{\tfrac{1}{8}(1+\delta)\right\}}\,\dot{y} - \tfrac{1}{8}(1+\delta)\,x = (m_0+\lambda)\,x\,[X_0 + X_1 + X_2 + \ldots] \\ &\ddot{y} + 2\sqrt{\left\{\tfrac{1}{8}(1+\delta)\right\}}\,\dot{x} - \tfrac{1}{8}(1+\delta)\,y = (m_0+\lambda)\,y\,[Y_0 + Y_1 + Y_2 + \ldots] \\ &\ddot{w} + W_0 w = W_2 + W_3 + W_4 + \ldots \end{aligned}\right\}, \qquad (5)$$

where

$$X_0 = 4 - 6a^2(1 - 4\cos 2\tau) + \ldots,$$

$$Y_0 = -2 - 3a^2(1 + 2\cos 2\tau) + \ldots,$$

$$W_0 = 1 - 9a^2(\tfrac{1}{2} - \cos 2\tau) + a^4(\tfrac{687}{32} - 48\cos 2\tau + \tfrac{177}{8}\cos 4\tau) + \ldots,$$

$$X_1 = -96(1+\delta)\psi w, \qquad Y_1 = 24(1+\delta)\psi w,$$

$$\begin{aligned}X_2 = {} & x^2[32-48a^2(7-10\cos 2\tau)+\ldots] \\ & -y^2[48-72a^2(2-5\cos 2\tau)+\ldots] \\ & -w^2[48-216a^2(4-5\cos 2\tau)+\ldots],\end{aligned}$$

$$\begin{aligned}Y_2 = {} & -x^2[48-72a^2(2-5\cos 2\tau)+\ldots] \\ & +y^2[12-6a^2(1-10\cos 2\tau)+\ldots] \\ & +w^2[12-18a^2(7-10\cos 2\tau)+\ldots],\end{aligned}$$

$$W_2 = -x^2[24a\sin\tau+\ldots]+y^2[6a\sin\tau+\ldots]+w^2[18a\sin\tau+\ldots].$$

Before characterizing the remaining X_i, Y_i, and W_i in (5), we shall define *triply even series* as we have occasion to use such series frequently in the sequel. A triply even series is a power series in a^2 with sums of cosines of even multiples of τ in the coefficients. The highest multiple of τ in the coefficient of a^{2k} is $2k$. Such series are called triply even since they are even in a, and only even functions of even multiples of τ enter the coefficients.

The terms X_i, Y_i, W_i are homogeneous polynomials of degree i in x, y, and w, but only even powers of x and y enter, while w may enter to any power. The coefficients in X_i and Y_i are triply even, or triply odd series according as i is even or odd respectively, but the coefficients in W_i are triply even or triply odd series according as i is odd or even respectively.

The expansions in the right members of (5) converge, provided the coordinates of the infinitesimal body referred to the rotating axes satisfy the inequalities

$$-\tfrac{1}{4} < -x+x^2+y^2+z^2 < \tfrac{1}{4}, \qquad -\tfrac{1}{4} < x+x^2+y^2+z^2 < \tfrac{1}{4}.$$

If the inequality signs are replaced by signs of equality, we obtain the region of space referred to the rotating axes within which the series in the right members of (5) converge. The resulting equations represent the points $-\frac{1}{2}$, 0, 0, and $\frac{1}{2}$, 0, 0, and the spheres having centres at these points and radii $1/\sqrt{2}$. The region of convergence is then the common portion of these two spheres. When referred to the fixed axes the region is the spindle obtained by rotating the common part of the two circles with centres at $-\frac{1}{2}$, 0, 0 and $\frac{1}{2}$, 0, 0, and radii $1/\sqrt{2}$, about their common chord. The axis of the spindle coincides with the ζ-axis, and the intercepts on the ξ and η axes are both $(\sqrt{2}-1)/2$ units in length, while the intercepts on the ζ-axis are $\frac{1}{2}$ unit in length.

In order to show the existence of periodic solutions of (5) we must first obtain the complementary functions of (5), or the solutions of the differential equations

$$\left.\begin{aligned} &\ddot{x}-2\sqrt{\left\{\tfrac{1}{8}(1+\delta)\right\}}\,\dot{y}-\left[\tfrac{1}{8}(1+\delta)+m_0X_0\right]x=0\\ &\ddot{y}+2\sqrt{\left\{\tfrac{1}{8}(1+\delta)\right\}}\,\dot{x}-\left[\tfrac{1}{8}(1+\delta)+m_0\,Y_0\right]y=0\\ &\ddot{w}+W_0w=0 \end{aligned}\right\}, \tag{6}$$

which are the *equations of variation.*

4. *Solutions of the Equations of Variation.*

The *generating solutions* of the equations of variation are

$$x \equiv y \equiv 0, \qquad z = \psi.$$

When the generating solutions are not identically zero, the solutions of the equations of variation are obtained by differentiating the generating solutions partially with respect to the arbitrary constants which enter them.* The function ψ contains the arbitrary constants τ_0 and a, and therefore the two solutions of the third equation of (6) are

$$\frac{\partial\psi}{\partial\tau_0} \quad\text{and}\quad \frac{\partial\psi}{\partial a}.$$

On performing these differentiations we obtain as the general solution of the third equation of (6)

$$w = B_1\phi+B_2[\chi+A\tau\phi], \tag{7}$$

where

$$\begin{aligned} \phi &= \cos\tau+\tfrac{9}{16}a^2(\cos 3\tau-\cos\tau)+\ldots,\\ \chi &= \sin\tau+\tfrac{9}{16}a^2(\sin 3\tau+5\sin\tau)+\ldots,\\ A &= -\tfrac{9}{2}a^2+\tfrac{141}{16}a^4+\ldots, \end{aligned}$$

and B_1 and B_2 are constants of integration. The solutions ϕ and χ have been determined so that the determinant of these solutions and their first derivatives is unity.†

As the generating solutions of the first two equations of (6) are both zero, we cannot find the solutions of these differential equations by differentiating with respect to constants. These equations, however, come under the class of differential equations with periodic coefficients which have been treated in Moulton's *Periodic Orbits*, Chapter I. Their solutions are

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* Moulton, *Periodic Orbits*, §§ 32, 33 ; Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. I, Chap. 4.

† These are the same solutions as those appearing in the *Periodic Orbits*, Chap. X, § 174.

$$\left.\begin{aligned} x &= A_1 e^{\sqrt{(-1)}\beta_1\tau} u_1 + A_2 e^{-\sqrt{(-1)}\beta_1\tau} u_2 + A_3 e^{\beta_2\tau} u_3 + A_4 e^{-\beta_2\tau} u_4 \\ y &= \sqrt{(-1)}\, b_1 [A_1 e^{\sqrt{(-1)}\beta_1\tau} v_1 - A_2 e^{-\sqrt{(-1)}\beta_1\tau} v_2] + b_2 [A_3 e^{\beta_2\tau} v_3 - A_4 e^{-\beta_2\tau} v_4] \end{aligned}\right\}. \tag{8}$$

The undefined terms in (8) have the following properties. The

$$A_i \quad (i = 1, 2, 3, 4)$$

are the constants of integration. The

$$\beta_i \quad (i = 1, 2)$$

are power series in a^2 of the form

$$\beta_i = \sum_{j=0}^{\infty} \beta_i^{(2j)} a^{2j}, \tag{9}$$

where the coefficients $\beta_i^{(2j)}$ are real constants. The values of $-(\beta_1^{(0)})^2$ and $(\beta_2^{(0)})^2$ are the roots of the quadratic

$$64\gamma^2 - 16(8m_0 - 1)\gamma + 1 + 16m_0 - 512m_0^2 = 0.$$

When $m_0 = \frac{1}{2}$, then $\beta_1^{(0)}$ and $\beta_2^{(0)}$ have the values $\sqrt{(\sqrt{2} - \frac{3}{8})}$ and $\sqrt{(\sqrt{2} + \frac{3}{8})}$ respectively. These values are irrational. Since $m_0 + \lambda = \frac{1}{2}$, and since λ must be taken small numerically, so that certain series appearing in the sequel shall converge, the value of m_0 will be in the neighbourhood of $\frac{1}{2}$, and therefore the values of $\beta_1^{(0)}$ and $\beta_2^{(0)}$ will be irrational in general. The remaining $\beta_i^{(2j)}$ $(j > 0)$ in (9) are determined by the conditions that u_i and v_i $(i = 1, 2, 3, 4)$ shall be periodic with the period 2π in τ. The u_i and v_i are power series in a^2 with sums of sines and cosines of even multiples of τ in the coefficients. The highest multiple of τ in the coefficient of a^{2k} is $2k$. In u_1, u_2, v_1 and v_2 the coefficients of the sines are purely imaginary, while the coefficients of the cosines are real. In u_3, u_4, v_3 and v_4 all the coefficients are real. Further

$$u_1\{\sqrt{(-1)}\} = u_2\{-\sqrt{(-1)}\}, \quad v_1\{\sqrt{(-1)}\} = v_2\{-\sqrt{(-1)}\},$$

$$u_3(\tau) = u_4(-\tau), \quad v_3(\tau) = v_4(-\tau).$$

Since the u_i are multiplied by the arbitrary constants A_i, these solutions have been determined so that their initial values are unity. If $\beta_1^{(0)}$ and $\beta_2^{(0)}$ were integers, then u_i and v_i $(i = 1, 2)$ would contain additional terms in $e^{\pm j\sqrt{(-1)}\beta_1^{(0)}\tau}$, and u_i and v_i ($i = 3, 4$) would contain additional terms in $e^{\pm j\beta_2^{(0)}\tau}$, j being an integer. The b_1 and b_2 are real constants of the form

$$b_i = \sum_{j=0}^{\infty} b_i^{(2j)} a^{2j} \quad (i = 1, 2),$$

where the $b_i^{(2j)}$ are determined by the conditions that

$$v_i(0) = 1 \quad (i = 1, 2, 3, 4).$$

The computation shows that

$$b_1^{(0)} = -\sqrt{2}\,[(\beta_1^{(0)})^2+4m_0+\tfrac{1}{8}]/\beta_1^{(0)}, \qquad b_2^{(0)} = \sqrt{2}[(\beta_2^{(0)})^2-4m_0-\tfrac{1}{8}]/\beta_2^{(0)}.$$

The determinant of the solutions (8) and their first derivatives is a constant,* and its value at $\tau = 0$ is

$$\Delta = 4\sqrt{(-1)}\,[\beta_1^{(0)}\beta_2^{(0)}\{(b_1^{(0)})^2-(b_2^{(0)})^2\}+b_1^{(0)}b_2^{(0)}\{(\beta_2^{(0)})^2-(\beta_1^{(0)})^2\}]+\text{terms in } a^2. \tag{10}$$

For $m_0 = \frac{1}{2}$ the value of Δ is

$$10\sqrt{(-119)}+\text{terms in } a^2,$$

which is different from zero for $|a|$ sufficiently small. Since Δ is a continuous function of m_0 its value will remain distinct from zero when m_0 is in the neighbourhood of $\frac{1}{2}$. Hence the solutions in (8) constitute a fundamental set.

5. *Determination of the Period by a Necessary Condition for a Periodic Solution.*

If periodic solutions of (5) exist, let their period in τ be denoted by $2P$. Since the period of the generating solutions is 2π in τ, then P must have the value

$$P = \nu\pi, \tag{11}$$

where ν is an integer.

If we take the initial conditions

$$x(0) = \alpha_1, \quad \dot{x}(0) = 0, \quad y(0) = 0, \quad \dot{y}(0) = \alpha_2, \quad w(0) = 0, \quad \dot{w}(0) = \alpha_3, \tag{12}$$

that is, if μ is projected perpendicularly from the x-axis at the origin of time, then it can be shown from the differential equations (5) that x is even in τ, and y and w are odd in τ. Sufficient conditions therefore that x, y, and w shall be periodic with the period $2P$, are

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* Moulton's *Periodic Orbits*, § 18.

$$\dot{x} = y = w = 0 \quad \text{at} \quad \tau = P, \tag{13}$$

or the infinitesimal body must again cross the x axis perpendicularly at the half period.

In order to determine the period we integrate equations (5) as power series in λ and a_i $(i = 1, 2, 3)$, and impose the conditions (13) on the solutions. There are no linear terms in λ alone. The solutions of the differential equations yielding the terms in a_i are the same as (7) and (8). When the conditions (12) are imposed upon these solutions it is found that $A_1 = A_2$, $A_3 = A_4$, $B_1 = 0$, and that A_1, A_3, B_2 are linear functions of a_i which vanish with a_i. It is more convenient to integrate (5) as power series in λ, A_1, A_3 and B_2, and this is permissible since A_1, A_3 and B_2 are linear in a_i and vanish with a_i. Hence the solutions of (5) which are linear in A_1, A_3, B_2, and which satisfy the symmetrical initial conditions

$$\dot{x} = y = w = 0 \quad \text{at} \quad \tau = 0, \tag{14}$$

are
$$\left.\begin{aligned} x_{10} &= A_1\left[e^{\sqrt{(-1)}\,\beta_1\tau}u_1 + e^{-\sqrt{(-1)}\,\beta_1\tau}u_2\right] + A_3\left[e^{\beta_2\tau}u_3 + e^{-\beta_2\tau}u_4\right] \\ y_{10} &= \sqrt{(-1)}\,b_1 A_1\left[e^{\sqrt{(-1)}\,\beta_1\tau}v_1 - e^{-\sqrt{(-1)}\,\beta_1\tau}v_2\right] \\ &\qquad\qquad + b_2 A_3\left[e^{\beta_2\tau}v_3 - e^{-\beta_2\tau}v_4\right] \\ w_{10} &= B_2[\chi + A\tau\phi] \end{aligned}\right\}. \tag{15}$$

If we take the terms of higher degree in λ, A_1, A_3, and B_2 in (5), we obtain solutions of the form

$$\left.\begin{aligned} x &= x_{10} + \text{terms in } \lambda A_1,\ \lambda A_3,\ A_1B_2,\ A_3B_2,\ \ldots \\ y &= y_{10} + \qquad ,, \qquad ,, \qquad ,, \\ w &= w_{10} + \text{quadratic terms in } A_1,\ A_3,\ B_2, \text{ and terms in } \lambda^2A_1^2,\ \lambda^2A_3^2,\ \ldots \end{aligned}\right\}, \tag{16}$$

in which some of the coefficients involve τ outside of trigonometric symbols, or multiplied by the exponentials $e^{\pm\sqrt{(-1)}\,\beta_1\tau}$ and $e^{\pm\beta_2\tau}$. The terms in (16) vanish with A_1, A_3, and B_2, but not with λ. Now we impose the conditions (13) on (16). To simplify the resulting equations, we observe that

$$u_i(P) = v_i(P) = u_i(0) = v_i(0) = 1,$$

$$\dot{u}_i(P) = \dot{u}_i(0), \quad \dot{v}_i(P) = \dot{v}_i(0),$$

since u_i and v_i have even multiples of τ under trigonometric symbols.

Hence from (13) and (16), we obtain

$$\left.\begin{aligned}0 = A_1[\sqrt{(-1)}\beta_1+\dot{u}_1(0)][e^{\sqrt{(-1)}\beta_1 P}-e^{-\sqrt{(-1)}\beta_1 P}]&\\ +A_3[\beta_2+\dot{u}_3(0)][e^{\beta_2 P}-e^{-\beta_2 P}]&\\ +\text{terms in } \lambda A_1,\ \lambda A_3,\ A_1B_2,\ A_3B_2,\ \dots&\\ 0 = \sqrt{(-1)}\, b_1A_1[e^{\sqrt{(-1)}\beta_1 P}-e^{-\sqrt{(-1)}\beta_1 P}]&\\ +b_2A_3[e^{\beta_2 P}-e^{-\beta_2 P}]&\\ +\text{terms in } \lambda A_1,\ \lambda A_3,\ A_1B_2,\ A_3B_2,\ \dots&\\ 0 = B_2AP\phi(P)+\text{quadratic terms in } A_1,\ A_3,\ B_2 \text{ and}&\\ \text{terms in } \lambda^2A_1^2,\ \lambda^2A_3^2,\ \dots&\end{aligned}\right\}. \tag{17}$$

The coefficient of B_2 in the last equation of (17) is different from zero for a not zero, but sufficiently small numerically, and hence this equation can be solved for B_2 as a power series in A_1^2, A_3^2, $\lambda^2A_1^2$, and $\lambda^2A_3^2$ which vanishes with A_1 and A_3, but not with λ. Since A contains a^2 as a factor, the coefficients of the series for B_2 will be power series in a^2 multiplied by $1/a^2$. When this series for B_2 is substituted in the first two equations of (17), we obtain two equations in A_1, A_3 and λ which vanish with A_1 and A_3 but not with λ. The determinant of the linear terms in A_1 and A_3 in these two equations is

$$[e^{\sqrt{(-1)}\beta_1 P}-e^{-\sqrt{(-1)}\beta_1 P}][e^{\beta_2 P}-e^{-\beta_2 P}][b_2\{\sqrt{(-1)}\beta_1+\dot{u}_1(0)\} \\ -\sqrt{(-1)}\, b_1\{\beta_2+\dot{u}_3(0)\}]. \tag{18}$$

The last factor has the form

$$\sqrt{(-1)}\,(b_2^{(0)}\beta_1^{(0)}-b_1^{(0)}\beta_2^{(0)})+\text{terms in } a^2,$$

and is different from zero for a^2 sufficiently small. Hence the determinant (18) can vanish only with either of the first two factors. If it does not vanish, then the first two equations of (17), when B_2 has been eliminated, can be solved uniquely for A_1 and A_3 as power series in λ. But since these equations are satisfied by $A_1 = A_3 = 0$, the only solutions for A_1 and A_3 are $A_1 = A_3 = 0$, unless the determinant (18) is zero. The second factor of (18) cannot vanish if the period $2P$ is real, and we desire solutions having a real period. The first factor is zero, provided $P = N\pi/\beta_1$, where N is an integer. Thus, if periodic solutions exist which are not identically zero, they must have the period $N\pi/\beta_1$. From (11) it follows that $N = \nu\beta_1$. If β_1 is a rational function, ν can be chosen so that N will be an integer relatively prime to ν. Inasmuch as β_1 is a continuous function of a^2, it can be made a rational fraction by a proper choice of a less numerically than the smallest value of $|a|$ which insures the convergence of the power series depending upon a.

6. *Existence of Symmetrical Periodic Orbits.*

In order to establish the existence of periodic solutions of (5) we integrate (5) as power series in λ, A_1, A_3 and B_2 as in § 5, and it is necessary to obtain the terms in λA_1, λA_3 and $A_1^{\frac{3}{1}}$. The linear terms in A_1, A_2, and B_2 are the same as (15). If we denote the quadratic terms in λA_1 and λA_3 by x_{11} and y_{11}, then the differential equations defining x_{11} and y_{11} are

$$\left.\begin{aligned} \ddot{x}_{11}-2\sqrt{\{\tfrac{1}{8}(1+\delta)\}}\,\dot{y}_{11}-x_{11}[\tfrac{1}{8}(1+\delta)+m_0X_0] \\ = X_{11} = x_{10}[-96m_0(1+\delta)\psi w_{10}+\lambda X_0] \\ \ddot{y}_{11}+2\sqrt{\{\tfrac{1}{8}(1+\delta)\}}\,\dot{x}_{11}-y_{11}[\tfrac{1}{8}(1+\delta)+m_0Y_0] \\ = Y_{11} = y_{10}[24m_0(1+\delta)\psi w_{10}+\lambda Y_0] \end{aligned}\right\}. \quad (19)$$

The complementary functions of these equations are the same as the solutions of the equations of variation. If we denote them by

$$\left.\begin{aligned} x_{11} &= a_1e^{\sqrt{(-1)}\,\beta_1\tau}u_1+a_2e^{-\sqrt{(-1)}\,\beta_1\tau}u_2+a_3e^{\beta_2\tau}u_3+a_4e^{-\beta_2\tau}u_4 \\ y_{11} &= \sqrt{(-1)}\,b_1[a_1e^{\sqrt{(-1)}\,\beta_1\tau}v_1-a_2e^{-\sqrt{(-1)}\,\beta_1\tau}v_2]+b_2[a_3e^{\beta_2\tau}v_3-a_4e^{-\beta_2\tau}v_4] \end{aligned}\right\}, \quad (20)$$

then the particular integrals are obtained by varying the parameters

$$a_i \quad (i = 1, 2, 3, 4).$$

By this method we have

$$\left.\begin{aligned} &\dot{a}_1e^{\sqrt{(-1)}\beta_1\tau}u_1+\dot{a}_2e^{-\sqrt{(-1)}\,\beta_1\tau}u_2+a_3e^{\beta_2\tau}u_3+\dot{a}_4e^{-\beta_2\tau}u_4 = 0 \\ &\dot{a}_1\{\sqrt{(-1)}\beta_1u_1+\dot{u}_1\}\,e^{\sqrt{(-1)}\beta_1\tau}+\dot{a}_2\{-\sqrt{(-1)}\beta_1u_2+\dot{u}_2)e^{-\sqrt{(-1)}\,\beta_1\tau} \\ &\qquad +\dot{a}_3(\beta_2u_3+\dot{u}_3)\,e^{\beta_2\tau}+\dot{a}_4(-\beta_2u_4+\dot{u}_4)\,e^{-\beta_2\tau} = X_{11} \\ &\sqrt{(-1)}\,b_1[\dot{a}_1e^{\sqrt{(-1)}\beta_1\tau}v_1-\dot{a}_2e^{-\sqrt{(-1)}\beta_1\tau}v_2]+b_2[\dot{a}_3e^{\beta_2\tau}v_3-\dot{a}_4e^{-\beta_2\tau}v_4] \qquad = 0 \\ &\sqrt{(-1)}\,b_1[\dot{a}_1\{\sqrt{(-1)}\beta_1v_1+\dot{v}_1\}\,e^{\sqrt{(-1)}\beta_1\tau}-\dot{a}_2\{-\sqrt{(-1)}\beta_1v_2+\dot{v}_2\}\,e^{-\sqrt{(-1)}\beta_1\tau}] \\ &\qquad +b_2[\dot{a}_3(\beta_2v_3+\dot{v}_3)e^{\beta_2\tau}-\dot{a}_4(-\beta_2v_4+\dot{v}_4)e^{-\beta_2\tau}] = Y_{11} \end{aligned}\right\}. \quad (21)$$

The determinant of the coefficients of $\dot{a}_i$ is Δ, the same as in (10), and is different from zero for $|a|$ sufficiently small. Hence the equations (21) can be solved for $\dot{a}_i$. Since X_{11} and Y_{11} carry the exponentials $e^{\pm\sqrt{(-1)}\beta_1\tau}$

and $e^{\pm\beta_2\tau}$, it follows from the nature of u_i and v_i ($i = 1, 2, 3, 4$) that the solutions for $\dot{a}_i$ will involve constant terms. We shall be interested in these constant terms as they yield terms in τ when the integrations are performed for the a_i. It is found that the constant terms in the equations for $\dot{a}_1$ and $\dot{a}_2$ differ only in sign; since they carry the factors λA_1 or $A_1 B_2$, let us denote them by

$$\frac{A_1}{\Delta}[\lambda P_1 + B_2 Q_1],$$

where P_1 and Q_1 are power series in a^2 with constant coefficients. Similarly the constants in the equations for $\dot{a}_3$ and $\dot{a}_4$ differ only in sign, and they have the form

$$\frac{\sqrt{(-1)}\,A_3}{\Delta}[\lambda P_2 + B_2 Q_2],$$

where P_2 and Q_2 are power series in a^2 with real constant coefficients. Upon performing the integrations for a_i, we obtain, for the terms in τ,

$$\left.\begin{aligned} a_1 = -a_2 &= \frac{A_1}{\Delta}[\lambda P_1 + B_2 Q_1]\tau + \ldots \\ a_3 = -a_4 &= \frac{\sqrt{(-1)}\,A_3}{\Delta}[\lambda P_2 + B_2 Q_2]\tau + \ldots \end{aligned}\right\} . \tag{22}$$

The additional terms in these equations carry the exponentials $e^{\pm\sqrt{(-1)}\beta_1}$ or $e^{\pm\beta_2\tau}$. When these equations are substituted in (20), we obtain

$$\left.\begin{aligned} x_{11} = {} & \frac{A_1}{\Delta}\tau[\lambda P_1 + B_2 Q_1][e^{\sqrt{(-1)}\beta_1\tau}u_1 - e^{-\sqrt{(-1)}\beta_1\tau}u_2] \\ & + \frac{\sqrt{(-1)}\,A_3\tau}{\Delta}[\lambda P_2 + B_2 Q_2][e^{\beta_2\tau}u_3 - e^{-\beta_2\tau}u_4] + \ldots \\ y_{11} = {} & \frac{\sqrt{(-1)}\,b_1 A_1\tau}{\Delta}[\lambda P_1 + B_2 Q_1][e^{\sqrt{(-1)}\beta_1\tau}v_1 + e^{-\sqrt{(-1)}\beta_1\tau}v_2] \\ & + \frac{\sqrt{(-1)}\,b_2 A_3\tau}{\Delta}[\lambda P_2 + B_2 Q_2][e^{\beta_2\tau}v_3 + e^{-\beta_2\tau}v_4] + \ldots \end{aligned}\right\} . \tag{23}$$

Let us denote the terms of the third degree in A_1 and A_3 by x_{30} and y_{30}. The solutions for these terms are found in the same way as (23) were obtained. As we need only one solution, and only the terms in A_1^3 or A_2^3, we shall choose the solution x_{30} and that part of it which carries A_1^3 as a factor. This solution is

$$x_{30} = \frac{A_1^3}{\Delta}P_3\tau[e^{\sqrt{(-1)}\beta_1\tau}u_1 - e^{-\sqrt{(-1)}\beta_1\tau}u_2] + \ldots,$$

where P_3 is a power series in a^2 with real constant coefficients.

Now let us impose the periodicity conditions (13) on the solutions

$$x = x_{10}+x_{11}+x_{30}+\dots,$$

$$y = y_{10}+y_{11}+\dots,$$

$$w = w_{10}+\dots .$$

We obtain the equations

$$\left.\begin{aligned}
0 = {} & A_3[\beta_2+\dot{u}_3(0)][e^{\beta_2 P}-e^{-\beta_2 P}] \\
& +\frac{A_1 P}{\Delta}[\lambda P_1+B_2 Q_1][\sqrt{(-1)}\beta_1+\dot{u}_1(0)][e^{\sqrt{(-1)}\beta_1 P}+e^{-\sqrt{(-1)}\beta_1 P}] \\
& +\frac{\sqrt{(-1)}A_3 P}{\Delta}[\beta_2+u(0)][e^{\beta_2 P}+e^{-\beta_2 P}] \\
& +\frac{A_1^3 P P_3}{\Delta}[\sqrt{(-1)}\beta_1+\dot{u}_1(0)][e^{\sqrt{(-1)}\beta_1 P}+e^{-\sqrt{(-1)}\beta_1 P}]+\dots \\
0 = {} & b_2 A_3[e^{\beta_2 P}-e^{-\beta_2 P}] \\
& +\frac{\sqrt{(-1)}b_1}{\Delta}A_1 P[\lambda P_1+B_2 Q_1][e^{\sqrt{(-1)}\beta_1 P}+e^{-\sqrt{(-1)}\beta_1 P}] \\
& +\frac{\sqrt{(-1)}b_2}{\Delta}A_3 P[\lambda P_2+B_2 Q_2][e^{\beta_2 P}+e^{-\beta_2 P}]+\dots \\
0 = {} & AB_2 P\phi(P)+\text{terms in } A_1^2,\ A_3^2,\ \dots
\end{aligned}\right\} . \quad (24)$$

The last equation of (24) can be solved for B_2 as a power series in A_1^2 and A_3^2 in which the lowest power of λ in the coefficients is λ^2. Let us denote this series for B_2 by (24*c*). It vanishes with A_1 and A_3, but not with λ alone. When (24*c*) is substituted in the second equation of (24), we obtain an equation which vanishes with A_1 and A_3, but not with λ. Since the coefficient of A_3 is different from zero in the equation thus obtained, it can be solved for A_3 as a power series in λ and A_1 in which the terms of lowest degree in λ and A_1 are λA_1 and A_1^3. Let this series for A_3 be denoted by (24*b*). Now, when B_2 and A_3 are eliminated from the first equation of (24) by (24*c*) and (24*b*), we obtain an equation (24*a*) in A_1 and λ of which A_1 is a factor. Since we desire the solutions which are not identically zero in x and y, we must consider A_1 as different from zero. Hence we can divide out A_1 in (24*a*), and we obtain an equation in A_1 and λ of which the terms of lowest degree are A_1^2 and λ. The coefficient of A_1^2, when $m_0 = \frac{1}{2}$, is approximately

$$\frac{1}{\Delta}\left[9455 + \text{terms in } a^2\right],$$

and is different from zero for a^2 sufficiently small. When m_0 is in the vicinity of $\frac{1}{2}$ this coefficient will remain different from zero, and hence, by the theory of implicit functions, equation (24*a*) can be solved for A_1 as a power series in $\pm\lambda^{\frac{1}{2}}$ vanishing with $\lambda^{\frac{1}{2}}$. From (24*b*) and (24*c*) it follows that A_3 and B_2 are also power series of the same form. Therefore, if the initial values of x, y, and w are power series in $\lambda^{\frac{1}{2}}$, the solutions themselves are power series of the same form. In the practical construction of the solutions it may be shown that x and y are power series in odd powers of $\lambda^{\frac{1}{2}}$ and that w is a power series in λ. This can be shown more directly from the differential equations as follows.

The first two equations of (5) are odd in x and y, the last equation is even in x and y and all three equations are even in $\lambda^{\frac{1}{2}}$. Let the solutions for x, y and w be denoted by

$$x = f_1(\lambda^{\frac{1}{2}}), \qquad y = f_2(\lambda^{\frac{1}{2}}), \qquad w = f_3(\lambda^{\frac{1}{2}}).$$

If we change the sign of $\lambda^{\frac{1}{2}}$ in (5) the differential equations remain unchanged, and hence

$$x = f_1(-\lambda^{\frac{1}{2}}), \qquad y = f_2(-\lambda^{\frac{1}{2}}), \qquad w = f_3(-\lambda^{\frac{1}{2}}), \tag{25}$$

are also solutions. If the signs of x and y, and the signs of a_1 and a_2 in (12) are changed, then the solutions for $-x$ and $-y$ are the same as for $+x$ and $+y$. Hence

$$-x = f_1(\lambda^{\frac{1}{2}}), \qquad -y = f_2(\lambda^{\frac{1}{2}}). \tag{26}$$

Therefore, from (25) and (26), we have

$$f_1(-\lambda^{\frac{1}{2}}) = -f_1(\lambda^{\frac{1}{2}}), \qquad f_2(-\lambda^{\frac{1}{2}}) = -f_2(\lambda^{\frac{1}{2}}),$$

and it follows that x and y are odd in $\lambda^{\frac{1}{2}}$. Since w is a function of x^2 and y^2, it follows that w is even in $\lambda^{\frac{1}{2}}$. Hence the periodic solutions of (5) have the form

$$x = \sum_{j=0}^{\infty} x_{2j+1}\lambda^{\frac{1}{2}(2j+1)}, \qquad y = \sum_{j=0}^{\infty} y_{2j+1}\lambda^{\frac{1}{2}(2j+1)}, \qquad w = \sum_{j=1}^{\infty} w_{2j}\lambda^{\frac{1}{2}(2j)}, \tag{27}$$

where each x_{2j+1}, y_{2j+1}, and w_{2j} is separately periodic with the period $2P$.

The question might be raised as to the convergence of the solutions in

(27). In Poincaré's extension of Cauchy's theorem,* the period T is chosen arbitrarily in advance, and the solutions converge for $0 \leqslant t \leqslant T$, provided a certain parameter which enters the differential equations is sufficiently small numerically. In the problem under consideration the period depends upon m_0, which in turn depends upon λ, and thus the period is not chosen arbitrarily in advance. Hence Poincaré's theorem does not apply directly to the solutions (27). It has been shown, however, by Moulton† that the convergence of solutions, such as (27), can be controlled by λ alone even if the period is not chosen arbitrarily in advance.

7. *Construction of the Symmetrical Periodic Orbits.*

Let us substitute (27) in (5) and equate the coefficients of the various powers of $\lambda^{\frac{1}{2}}$. We thus obtain a series of differential equations which can be integrated step by step, and the constants of integration are to be determined so that the solutions shall be periodic and satisfy the symmetrical initial conditions (14). When these initial conditions are imposed on (27), we obtain

$$\left.\begin{array}{ll} \dot{x}_{2j+1}(0) = 0, \quad y_{2j+1}(0) = 0 & (j = 0, \ldots, \infty) \\ w_{2j}(0) = 0 & (j = 1, \ldots, \infty) \end{array}\right\}. \tag{28}$$

The differential equations defining the terms in x_1 and y_1 are the same as the equations of variation, and the periodic solutions which satisfy (28) are

$$\left.\begin{array}{l} x_1 = A_1^{(1)}\left[e^{\sqrt{(-1)}\beta_1\tau}u_1 + e^{-\sqrt{(-1)}\beta_1\tau}u_2\right] \\ y_1 = \sqrt{(-1)}\, b_1 A_1^{(1)}\left[e^{\sqrt{(-1)}\beta_1\tau}v_1 - e^{-\sqrt{(-1)}\beta_1\tau}v_2\right] \end{array}\right\}, \tag{29}$$

where $A_1^{(1)}$ remains undetermined at this step.

The differential equation for the terms in λ is

$$\ddot{w}_2 + W_0 w_2 = W^{(2)} = x_1^2[-24a \sin\tau + \ldots] + y_1^2[-6a\sin\tau + \ldots]. \tag{30}$$

When the solutions (29) are substituted in (30), the right member $W^{(2)}$ has the form

$$W^{(2)} = (A_1^{(1)})^2[S_0^{(2)} + S_2^{(2)}], \tag{31}$$

where $S_0^{(2)}$ is a triply odd series. As series similar to $S_2^{(2)}$ occur in the construction of the solutions of the w-equations, we shall define the function $S_{2i}^{(2k)}$, where $2k$ denotes the step of the integration. The function $S_{2i}^{(2k)}$ is homogeneous and of degree $2i$ in $e^{\sqrt{(-1)}\beta_1\tau}$ and $e^{-\sqrt{(-1)}\beta_1\tau}$, but does not include the terms which are independent of the exponentials. The coefficients of these exponentials are power series in odd powers of a with sums of sines and $\sqrt{(-1)}$ times cosines of odd multiples of τ in the coefficients. The highest multiple of τ in the coefficient of a^{2k+1} is $2k+1$. The coefficients of $e^{2h\sqrt{(-1)}\beta_1\tau}$ and $e^{-2h\sqrt{(-1)}\beta_1\tau}$ $(h = 1, \ldots, i)$, differ only in the sign of $\sqrt{(-1)}$. If the exponentials are expressed in the form

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* *Mécanique Céleste*, Vol. I, Chap. II, p. 58.

† *Periodic Orbits*, § 115.

$$\cos 2h\beta_1\tau \pm \sqrt{(-1)} \sin 2h\beta_1\tau,$$

then
$$S_{2i}^{(2k)} = \sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{h=1}^{i} S_{j,\,l,\,h}^{(2k)}\, a^{2j+1} \sin\{2h\beta_1 \pm (2l+1)\}\tau, \tag{32}$$

where $S_{j,\,l,\,h}^{(2k)}$ are real constants. If $2h\beta_1$ is an integer for some value of $h = 1, \ldots, i$, then some of the multiples of τ in (32) are integral. In order that $2h\beta_1$ may be an integer, we must have $2h = \nu$, or $h = \nu$. If $2h = \nu$, then $2h\beta_1 = N$, an odd integer, since ν and N are relatively prime. Hence, in this case, the integral multiples of τ in (32) are even integers. If $h = \nu$, then $2h\beta_1 = 2N$, an even integer, and the integral multiples of τ in (32) are odd integers.

We shall now find the periodic solution of (30) which satisfies (28). The complementary function of (30) is

$$w_2 = n_1^{(2)}\phi + n_2^{(2)}[\chi + A\tau\phi], \tag{33}$$

where ϕ, χ, and A are defined in (7). The $n_1^{(2)}$ and $n_2^{(2)}$ are constants, but regarding them as functions of τ, according to the method of the variation of parameters, we have

$$\left.\begin{aligned} \dot{n}_1^{(2)}\phi + \dot{n}_2^{(2)}[\chi + A\tau\phi] &= 0 \\ \dot{n}_1^{(2)}\dot{\phi} + \dot{n}_2^{(2)}[\chi + A(\tau\dot{\phi} + \phi)] &= W^{(2)} \end{aligned}\right\}, \tag{34}$$

and since the determinant of $n_1^{(2)}$ and $n_2^{(2)}$ is unity, it follows that

$$\dot{n}_1^{(2)} = -[\chi + A\tau\phi]W^{(2)}, \qquad \dot{n}_2^{(2)} = \phi W^{(2)}. \tag{35}$$

Since both χ and $W^{(2)}$ contain sines of odd multiples of τ not multiplied by the exponentials $e^{\pm 2\sqrt{(-1)}\beta_1\tau}$, the product $\chi W^{(2)}$ will contain constant terms which will yield terms in τ when the integrations are performed. The constant part of $\chi W^{(2)}$ is obtained from $\chi S_0^{(2)}$ alone, unless $\nu = 1$. The function $S_2^{(2)}$ contains fractional multiples of τ if $2\beta_1$ is a fraction, and even multiples of τ if $\nu = 2$. If $\nu = 1$, then $S_2^{(2)}$ will contain odd multiples of τ. In any case, the constant part of $\chi W^{(2)}$ will have the same form $ap^{(2)}$, where $p^{(2)}$ is a power series in a^2 with real coefficients, whether constant terms enter through $S_2^{(2)}$ or not.

When equations (35) are integrated and the results are substituted in (33), the general solution of (30) takes the form

$$w_2 = B_1^{(2)}\,\phi + B_2^{(2)}[\chi + A\tau\phi] + (A_1^{(2)})^2[\bar{S}_0^{(2)} + \bar{S}_2^{(2)} - ap^{(2)}\tau\phi], \tag{36}$$

where $B_i^{(2)}$ $(i = 1, 2)$ are the constants of integration; $\bar{S}_0^{(2)}$ is a triply odd series, and $\bar{S}_2^{(2)}$ is a function similar in form to $S_2^{(2)}$. If (36) is to be periodic, then

$$B_2^{(2)} = \frac{a}{A}\,p^{(2)}(A_1^{(1)})^2 = \frac{1}{a}\,q^{(2)}\,(A_1^{(1)})^2,$$

where $q^{(2)}$ is a power series in a^2 with the constant term $[16 - 4(b_1^0)^2]/3$. The constant $B_1^{(2)}$ must be zero to satisfy (28), and the desired solution of (30) becomes

$$w_2 = (A_1^{(1)})^2\left[\frac{1}{a^2}\,S^{(2)} + \bar{S}_2^{(2)}\right], \tag{37}$$

where $S^{(2)}$ is a triply odd power series. The constant $A_1^{(1)}$ remains undetermined at this step.

In order to determine $A_1^{(1)}$ we must consider the terms in $\lambda^{\frac{3}{2}}$. The differential equations for these terms are

$$\left.\begin{aligned} \ddot{x}_3 - 2\sqrt{\{\tfrac{1}{8}(1+\delta)\}}\,\dot{y}_3 - x_3[\tfrac{1}{8}(1+\delta) + m_0 X_0] &= X^{(3)} \\ \ddot{y}_3 + 2\sqrt{\{\tfrac{1}{8}(1+\delta)\}}\,\dot{x}_3 - y_3[\tfrac{1}{8}(1+\delta) + m_0 Y_0] &= Y^{(3)} \end{aligned}\right\}, \tag{38}$$

where
$$X^{(3)} = A_1^{(1)}\rho_3^{(1)} + (A_1^{(1)})^3\,\rho_3^{(3)}, \tag{39}$$

and $\rho_3^{(2i+1)}$ $(i = 0, 1)$, is a homogeneous polynomial of degree $2i+1$ in the exponentials $e^{\sqrt{(-1)}\,\beta_1\tau}$ and $e^{-\sqrt{(-1)}\beta_1\tau}$. The coefficients are power series similar in form to u_1 and u_2, and the coefficients of $[e^{\sqrt{(-1)}\,\beta_1\tau}]^k\,[e^{-\sqrt{(-1)}\,\beta_1\tau}]^j$ and $[e^{\sqrt{(-1)}\,\beta_1\tau}]^j\,[e^{-\sqrt{(-1)}\beta_1\tau}]^k$, $j+k = 2i+1$, differ only in the sign of $\sqrt{(-1)}$. Similarly,

$$Y^{(3)} = \sqrt{(-1)}\,[A_1^{(1)}\sigma_3^{(1)} + (A_1^{(1)})^3\,\sigma_3^{(3)}], \tag{40}$$

where $\sigma_3^{(2i+1)}$ $(i = 0, 1)$ is similar in form to $\rho_3^{(2i+1)}$, except that it is the coefficients of $[e^{\sqrt{(-1)}\,\beta_1\tau}]^k\,[-e^{-\sqrt{(-1)}\,\beta_1\tau}]^j$ and $[e^{\sqrt{(-1)}\,\beta_1\tau}]^j\,[-e^{-\sqrt{(-1)}\,\beta_1\tau}]^k$, $j+k = 2i+1$, which differ only in sign of $\sqrt{(-1)}$.

The complementary functions of (38) are the same as (20). The terms in $e^{\pm\sqrt{(-1)}\,\beta_1\tau}$ in $X^{(3)}$ and $Y^{(3)}$ give rise to non-periodic terms in the particular integrals since they are multiplied by power series similar to u_1 and u_2. The particular integrals are found in the same way as the solutions (23) were obtained. The complete solutions are

$$\left.\begin{aligned} x_3 &= A_1^{(3)} e^{\sqrt{(-1)}\beta_1\tau} u_1 + A_2^{(3)} e^{-\sqrt{(-1)}\beta_1\tau} u_2 + A_3^{(3)} e^{\beta_2\tau} u_3 + A_4^{(3)} e^{-\beta_2\tau} u_4 \\ &\quad + A_1^{(1)} \tau [P_1^{(3)} + (A_1^{(1)})^2 P_2^{(3)}][e^{\sqrt{(-1)}\beta_1\tau} u_1 - e^{-\sqrt{(-1)}\beta_1\tau} u_2] \\ &\quad + A_1^{(1)} [\bar\rho_3^{(1)} + (A_1^{(1)})^2 \bar\rho_3^{(3)}] \\ y_3 &= \sqrt{(-1)}\, b_1 [A_1^{(3)} e^{\sqrt{(-1)}\beta_1\tau} v_1 - A_2^{(3)} e^{-\sqrt{(-1)}\beta_1\tau} v_2] \\ &\quad + b_2 [A_3^{(3)} e^{\beta_2\tau} v_3 - A_4^{(3)} e^{-\beta_2\tau} v_4] \\ &\quad + \sqrt{(-1)}\, b_1 A_1^{(1)} \tau [P_1^{(3)} + (A_1^{(1)})^2 P_2^{(3)}][e^{\sqrt{(-1)}\beta_1\tau} v_1 + e^{-\sqrt{(-1)}\beta_1\tau} v_2] \\ &\quad + \sqrt{(-1)}\, A_1^{(1)} [\bar\sigma_3^{(1)} + (A_1^{(1)})^2 \bar\sigma_3^{(3)}] \end{aligned}\right\}, \tag{41}$$

where $A_i^{(3)}$ $(i = 1, 2, 3, 4)$ are the constants of integration; $P_1^{(3)}$ and $P_2^{(3)}$ are power series in a^2 with constant coefficients; and

$$\bar\rho_3^{(2i+1)} \quad \text{and} \quad \bar\sigma_3^{(2i+1)} \quad (i = 0, 1)$$

are functions of the same form as $\rho_3^{(2i+1)}$ and $\sigma_3^{(2i+1)}$ respectively. In order that x_3 and y_3 may be periodic, the constants $A_3^{(3)}$ and $A_4^{(3)}$ must be zero since the terms in $e^{\pm\beta_2\tau}$ do not have the period $2P$; also the coefficients of the terms in τ must be zero. Hence

$$P_1^{(3)} + (A_1^{(1)})^2 P_2^{(3)} = 0.$$

Now, for $m_0 = \frac{1}{2}$, the constant terms of $\Delta P_1^{(3)}$ and $\Delta P_2^{(3)}$ are approximately -111 and 9455 respectively. When m_0 is in the neighbourhood of $\frac{1}{2}$, the constant terms of $P_1^{(3)}$ and $P_2^{(3)}$ will remain different from zero and opposite in signs, provided a^2 is sufficiently small. Therefore

$$A_1^{(1)} = \sqrt{(-P_1^{(3)}/P_2^{(3)})} = \pm r, \tag{42}$$

where r is a power series in a^2 with real constant coefficients. When (42) is substituted in (41) and the conditions $\dot x_3(0) = y_3(0) = 0$ are imposed, then, since

$$\dot{\bar\rho}_3^{(2i+1)}(0) = \bar\sigma_3^{(2i+1)}(0) = 0,$$

the desired solutions become

$$\left.\begin{aligned} x_3 &= A_1^{(3)} [e^{\sqrt{(-1)}\beta_1\tau} u_1 + e^{-\sqrt{(-1)}\beta_1\tau} u_2] + A_1^{(1)} [\bar\rho_3^{(1)} + r^2 \bar\rho_3^{(3)}] \\ y_3 &= \sqrt{(-1)}\, b_1 A_1^{(3)} [e^{\sqrt{(-1)}\beta_1\tau} v_1 - e^{-\sqrt{(-1)}\beta_1\tau} v_2] + \sqrt{(-1)}\, A_1^{(1)} [\bar\sigma_3^{(1)} + r^2 \bar\sigma_3^{(3)}] \end{aligned}\right\}. \tag{43}$$

The constant $A_1^{(3)}$ remains undetermined at this step. There are two solutions in (43) and also in (29) obtained by taking the two values for $A_1^{(1)}$. The solution for w_2 in (37) is unique since w_2 carries $(A_1^{(1)})^2$ as a factor.

In order to determine $A_1^{(3)}$ it will be necessary to consider the terms in λ^2 and $\lambda^{\frac{5}{2}}$. There is but one differential equation for the terms in λ^2, and it has the form

$$\ddot{w}_4 + W_0 w_4 = W^{(4)} = A_1^{(1)} A_1^{(3)} [S_0^{(4)} + S_2^{(4)}] + \frac{1}{a^2}\, \omega_0^{(4)} + \omega_4^{(4)}, \tag{44}$$

where $S_0^{(4)}$ and $\omega_0^{(4)}$ are triply odd power series, and $S_2^{(4)}$ and $\omega_4^{(4)}$ are functions similar to $S_{2i}^{(2k)}$ defined under (31). The undetermined constant $A_1^{(3)}$ is written explicitly in so far as it occurs. Where $A_1^{(1)}$ enters to even degrees it is unique, and is absorbed by the functions of which it is a factor. Where it enters to odd degrees it is written explicitly.

The general solution of (44) is derived in the same way as (36) was obtained, and it has the form

$$w_4 = B_1^{(4)} \phi + B_2^{(4)} [\chi + A \tau \phi] + \frac{1}{a^2}\, \bar{S}_0^{(4)} + \bar{S}_4^{(4)} - \left[a A_1^{(1)} A_1^{(3)} p_1^{(4)} + \frac{1}{a}\, p_2^{(4)} \right] \tau \phi, \tag{45}$$

where $\bar{S}_0^{(4)}$ is a triply odd series, and $\bar{S}_4^{(4)}$ is a function similar to $S_{2i}^{(2k)}$ defined under (31). The functions $p_i^{(4)}$ $(i = 1, 2)$ are power series in a^2 with real constant coefficients and are derived from the integration of the constant terms in $\chi W^{(4)}$. If $\nu = 1$, then some of the multiples of τ in $S_2^{(4)}$ and $\omega_4^{(4)}$ are odd integers, and constant terms will be obtained when χ is multiplied by these functions. If $\nu \neq 1$, then constant terms will arise only from $\chi S_0^{(4)}$ and $\chi \omega_4^{(4)}$. The solution (45) can be made to satisfy the initial and periodicity conditions by choosing

$$B_1^{(4)} = 0, \quad B_2^{(4)} = \frac{1}{A} \left[a A_1^{(1)} A_1^{(3)} p_1^{(4)} + \frac{1}{a}\, p_2^{(4)} \right] = \frac{1}{a}\, A_1^{(1)} A_1^{(3)} q_1^{(4)} + \frac{1}{a^3}\, q_2^{(4)},$$

where $q_i^{(4)}$ $(i = 1, 2)$ are power series in a^2 with constant coefficients. When these values of the constants of integration are substituted in (45), the desired solution becomes

$$w_4 = \left[\frac{1}{a}\, A_1^{(1)} A_1^{(3)} q_1^{(4)} + \frac{1}{a^3}\, q_2^{(4)} \right] \chi + \frac{1}{a^2}\, \bar{S}_0^{(4)} + \bar{S}_4^{(4)}.$$

The constant $A_1^{(3)}$ remains arbitrary at this step also, and the terms in $\lambda^{\frac{5}{2}}$ must be considered.

The differential equation for the terms in $\lambda^{\frac{5}{2}}$ are

$$\left.\begin{array}{l} \ddot{x}_5-2\sqrt{\{\frac{1}{8}(1+\delta)\}}\,\dot{y}_5-x_5[\frac{1}{8}(1+\delta)+m_0X_0]=X^{(5)} \\ \ddot{y}_5+2\sqrt{\{\frac{1}{8}(1+\delta)\}}\,\dot{x}_5-y_5[\frac{1}{8}(1+\delta)+m_0Y_0]=Y^{(5)} \end{array}\right\}, \tag{46}$$

where

$$\left.\begin{array}{l} X^{(5)}=A_1^{(3)}\rho_5^{(3)}+A_1^{(1)}\dfrac{1}{a^2}\,\rho_5^{(5)} \\ Y^{(5)}=\sqrt{(-1)}\,A_1^{(3)}\sigma_5^{(3)}+\sqrt{(-1)}\,A_1^{(1)}\dfrac{1}{a^2}\,\sigma_5^{(5)} \end{array}\right\}. \tag{47}$$

The functions $\rho_5^{(2i+1)}$ and $\sigma_5^{(2i+1)}$ are similar in form to $\rho_3^{(2i+1)}$ and $\sigma_3^{(2i+1)}$ respectively. Where $A_1^{(1)}$ occurs to even degrees it is absorbed by the functions of which it is a factor. The undetermined constant $A_1^{(3)}$ is written explicitly in so far as it occurs.

The complementary functions of (46) are the same as (20), and the particular integrals are obtained in the same way as the solutions (23) were obtained. The terms in $e^{\pm\sqrt{(-1)}\beta_1\tau}$ in $X^{(5)}$ and $Y^{(5)}$ give non-periodic terms in the solutions. The general solutions have the form

$$\left.\begin{array}{l} x_5=A_1^{(5)}e^{\sqrt{(-1)}\beta_1\tau}u_1+A_2^{(5)}e^{-\sqrt{(-1)}\beta_1\tau}u_2+A_3^{(5)}e^{\beta_2\tau}u_3+A_4^{(5)}e^{-\beta_2\tau}u_4 \\ \qquad +\tau\left[A_1^{(3)}P_1^{(5)}+\dfrac{1}{a^2}A_1^{(1)}P_2^{(5)}\right][e^{\sqrt{(-1)}\beta_1\tau}u_1-e^{-\sqrt{(-1)}\beta_1\tau}u_2]+\dfrac{\eta}{a^2}\bar{\rho}_5^{(5)} \\ y_5=\sqrt{(-1)}\,b_1[A_1^{(5)}e^{\sqrt{(-1)}\beta_1\tau}v_1-A_2^{(5)}e^{-\sqrt{(-1)}\beta_1\tau}v_2] \\ \qquad +b_2[A_3^{(5)}e^{\beta_2\tau}v_3-A_4^{(5)}e^{-\beta_2\tau}v_4] \\ \qquad +\sqrt{(-1)}\,b_1\tau\left[A_1^{(3)}P_1^{(5)}+\dfrac{1}{a^2}A_1^{(1)}P_2^{(5)}\right][e^{\sqrt{(-1)}\beta_1\tau}v_1+e^{-\sqrt{(-1)}\beta_1\tau}v_2] \\ \qquad +\dfrac{\sqrt{(-1)}\,\eta}{a^2}\bar{\sigma}_5^{(5)} \end{array}\right\}. \tag{48}$$

The undefined terms in (48) have the following properties:

$$A_i^{(5)} \quad (i=1,\,2,\,3,\,4)$$

are the constants of integration; $P_i^{(5)}$ $(i=1,\,2)$ are power series in a^2 with real constant coefficients; $\bar{\rho}_5^{(5)}$ and $\bar{\sigma}_5^{(5)}$ are functions similar to $\rho_5^{(5)}$ and $\sigma_5^{(5)}$ respectively; η denotes that the constants $A_1^{(2j+1)}$, which have been determined, enter the function of which η is a factor to odd degrees considered together.

In order that the solutions (48) shall be periodic, the constants of integration must have the values

$$A_3^{(5)} = A_4^{(5)} = 0, \qquad A_1^{(3)} = -A_1^{(1)} P_2^{(5)}/a^2 P_1^{(5)} = A_1^{(1)} r^{(3)}/a^2, \tag{49}$$

where $r^{(3)}$ is a power series in a^2 with constant coefficients. There are two values of $A_1^{(3)}$ differing only in sign according as the + or − sign is chosen in (42). The solution for w_4 is independent of the double sign in (42), and has the form

$$w_4 = \frac{1}{a^4} S^{(4)} + \frac{1}{a^2} [\bar{S}_0^{(4)} + \bar{S}_4^{(4)}], \tag{50}$$

where $S^{(4)}$ is a triply odd series.

If we impose the initial conditions (28) upon (48), then $A_1^{(5)} = A_2^{(5)}$, since $\dot{\bar{\rho}}_5^{(5)}(0) = \sigma_5^{(5)}(0) = 0$. Hence the symmetrical periodic solutions of (46) are

$$\left.\begin{aligned} x_5 &= A_1^{(5)} [e^{\sqrt{(-1)}\beta_1\tau} u_1 + e^{-\sqrt{(-1)}\beta_1\tau} u_2] + \frac{\eta}{a^2} \bar{\rho}_5^{(5)} \\ y_5 &= \sqrt{(-1)}\, b_1 A_1^{(5)} [e^{\sqrt{(-1)}\beta_1\tau} v_1 - e^{-\sqrt{(-1)}\beta_1\tau} v_2] + \frac{\sqrt{(-1)}\,\eta}{a^2} \bar{\sigma}_5^{(5)} \end{aligned}\right\} . \tag{51}$$

The constant $A_1^{(5)}$ remains arbitrary at this step, but the remaining parts of the solutions differ only in the sign of η according as the + or − sign is taken in (42).

The remaining steps of the integration can be carried on in precisely the same way, as non-periodic terms arise in the subsequent steps of the integration only in the same way as in those already considered. The constants of integration are determined by the initial and periodicity conditions, and they are unique for one value of $A_1^{(1)}$. The constants arising from the differential equation in w_{2j} are determined at the stage in which they appear, but the solutions for x_{2j+1} and y_{2j+1} contain the arbitrary $A_1^{(2j+1)}$ which is not determined until the solutions at the step $2j+3$ are considered.

The presence of the terms in $1/a^2$ and $1/a^4$ in the solutions already considered introduces some slight change in the remaining steps, but only in so far as the factors $1/a^{2i}$ are concerned. At the step $2j+1$ the solutions are

$$\left.\begin{aligned} x_{2j+1} &= A_1^{(2j+1)} [e^{\sqrt{(-1)}\beta_1\tau} u_1 + e^{-\sqrt{(-1)}\beta_1\tau} u_2] + \frac{\eta}{a^{2j-2}} \bar{\rho}_{2j+1}^{(2j+1)} \\ y_{2j+1} &= \sqrt{(-1)}\, b_1 A_1^{(2j+1)} [e^{\sqrt{(-1)}\beta_1\tau} v_1 - e^{-\sqrt{(-1)}\beta_1\tau} v_2] + \frac{\sqrt{(-1)}\,\eta}{a^{2j-2}} \bar{\sigma}_{2j+1}^{(2j+1)} \end{aligned}\right\}, \tag{52}$$

where the undefined terms in (53) have the same significance as the corresponding terms in (52). At the step $2j$ the solution is

$$w_{2j} = \frac{1}{a^{2j}} S^{(2j)} + \frac{1}{a^{2j-2}} [\bar{S}_0^{(2j)} + \bar{S}_{2j}^{(2j)}], \tag{53}$$

where $S^{(2j)}$, $\bar{S}_0^{(2j)}$ are triply odd power series, and the $\bar{S}_{2j}^{(2j)}$ is similar in form to $S_{2j}^{(2k)}$ defined under (31).

Thus the constants of integration can be determined by the periodicity and symmetrical initial conditions. There are two values for each $A_1^{(2j+1)}$ differing only in sign, but each $B_2^{(2k)}$ is unique. Hence there are two sets of periodic solutions for x and y in (5), and the solutions of the one set differ from the corresponding solutions of the other set only in sign. There is but one periodic solution for w. Since x and y are odd in $\lambda^{\frac{1}{2}}$, the two sets of solutions are obtained by taking the $+$ and $-$ signs before $\lambda^{\frac{1}{2}}$.

Let us denote these two sets of solutions by

$$x = x(+\lambda^{\frac{1}{2}}, \tau), \quad y = y(+\lambda^{\frac{1}{2}}, \tau), \quad w = w(\lambda, \tau),$$

$$x = x(-\lambda^{\frac{1}{2}}, \tau), \quad y = y(-\lambda^{\frac{1}{2}}, \tau), \quad w = w(\lambda, \tau).$$

If ν is even then

$$x(+\lambda^{\frac{1}{2}}, \tau+P) = -x(+\lambda^{\frac{1}{2}}, \tau) = x(-\lambda^{\frac{1}{2}}, \tau),$$

$$y(+\lambda^{\frac{1}{2}}, \tau+P) = -y(+\lambda^{\frac{1}{2}}, \tau) = y(-\lambda^{\frac{1}{2}}, \tau),$$

$$w(+\lambda^{\frac{1}{2}}, \tau+P) = w(+\lambda^{\frac{1}{2}}, \tau) = w(-\lambda^{\frac{1}{2}}, \tau).$$

Hence, if ν is even, the orbits represented by $+\lambda^{\frac{1}{2}}$ and $-\lambda^{\frac{1}{2}}$ are the same geometrically, but in the one the infinitesimal body is half a period ahead of its position in the other.

If both ν and N are odd, then

$$x(+\lambda^{\frac{1}{2}}, \tau+P) = -x(+\lambda^{\frac{1}{2}}, \tau) = x(-\lambda^{\frac{1}{2}}, \tau),$$

$$y(+\lambda^{\frac{1}{2}}, \tau+P) = -y(+\lambda^{\frac{1}{2}}, \tau) = y(-\lambda^{\frac{1}{2}}, \tau),$$

$$w(+\lambda^{\frac{1}{2}}, \tau+P) = -w(+\lambda^{\frac{1}{2}}, \tau) = -w(-\lambda^{\frac{1}{2}}, \tau).$$

In this case the orbits represented by $+\lambda^{\frac{1}{2}}$ and $-\lambda^{\frac{1}{2}}$ are not, in general, the same geometrically, but the one is the reflection of the other with respect to the xy-plane. In the one orbit the infinitesimal is half a period ahead of its position in the other.

If ν is odd and N is even, then

$$\begin{aligned} x(+\lambda^{\frac{1}{2}}, \tau+P) &= \quad x(+\lambda^{\frac{1}{2}}, \tau) = -x(-\lambda^{\frac{1}{2}}, \tau), \\ y(+\lambda^{\frac{1}{2}}, \tau+P) &= \quad y(+\lambda^{\frac{1}{2}}, \tau) = -y(-\lambda^{\frac{1}{2}}, \tau), \\ w(+\lambda^{\frac{1}{2}}, \tau+P) &= -w(+\lambda^{\frac{1}{2}}, \tau) = -w(-\lambda^{\frac{1}{2}}, \tau). \end{aligned}$$

In this case also the orbits represented by $+\lambda^{\frac{1}{2}}$ and $-\lambda^{\frac{1}{2}}$ are, in general, distinct geometrically. They are symmetrical in the octants, symmetrically situated with respect to the origin. In the one orbit the infinitesimal is half a period ahead of its position in the other.

THE CONDITION THAT A QUINTIC EQUATION SHOULD BE SOLUBLE BY RADICALS

By W. E. H. Berwick.

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1. It is well known that the roots $\alpha, \beta, \ldots, \eta$, of an irreducible equation

$$ax^n+bx^{n-1}+\ldots+h = 0,$$

can be expressed in terms of the coefficients in a finite number of steps by the four elementary operations of arithmetic, addition, subtraction, multiplication and division, combined with a finite number of operations of root extraction, when, and only when, the group of the equation is soluble. The group of an irreducible equation being always transitive, that of an irreducible quintic is a transitive substitution group of degree 5. Such groups are five in number G_{120}, G_{60}, G_{20}, G_{10}, G_5 (the order being equal to the suffix in each case), and of these the last three are soluble. The group G_{20} contains the twenty substitutions

$$U^m V^n, \quad m = 1, 2, 3, 4, 5; \quad n = 1, 2, 3, 4, \tag{1}$$

where $\qquad U = (\alpha\beta\gamma\delta\epsilon), \quad V = (\alpha\beta\delta\gamma)(\epsilon),$

so that $\qquad U^5 = V^4 = 1, \quad UV = VU^2.$

G_{10}, a self-conjugate sub-group of G_{20}, containing

$$U^m W^n, \quad m = 1, 2, 3, 4, 5; \quad n = 1, 2, \tag{2}$$

where $\qquad W = V^2 = (\alpha\delta)(\beta\gamma)(\epsilon)$

is defined by $\qquad U^5 = W^2 = 1, \quad UW = WU^4;$

whereas G_5 consists of the single cycle

$$U, U^2, U^3, U^4, U^5 \equiv 1. \tag{3}$$

In order that the quintic may be soluble by radicals in a field of rationality

R, which contains a, b, c, d, e, f, it is necessary and sufficient that some rational function of the roots $\alpha, \beta, \gamma, \delta, \epsilon$ which is unaltered in form by the substitutions of G_{20}, but by no others, should have a value rational in R.

The quintic can be reduced to the standard (Bring-Jerrard) form

$$y^5+uy+v=0,$$

with the help of soluble equations, but, in general, the coefficients u, v belong to a field R' of relative degree 6 with regard to R. It is only for a certain limited class of quintics that u, v are rational in the original field R. Adopting this standard form, Runge ("Ueber die auflösbaren Gleichungen von der Form $x^5+ux+v=0$," *Acta Mathematica*, t. VII, S. 173–186) takes the function

$$\psi = \tfrac{1}{4}(\alpha\beta+\beta\gamma+\gamma\delta+\delta\epsilon+\epsilon\alpha-\alpha\gamma-\gamma\epsilon-\epsilon\beta-\beta\delta-\delta\alpha)^2,$$

and shows that ψ is a root of the sextic

$$\Psi(\psi) \equiv (\psi-u)^4(\psi^2-6\psi u+25u^2)-5^5v^4\psi = 0.$$

When this equation has a rational root in R', the quintic can be solved by radicals in R' and consequently in R. The object of this paper is to obtain a sextic equation $\Phi(\phi)=0$, satisfied by a function belonging to G_{20}, but with rational coefficients in R. Since two rational functions of the roots which belong to the same group are rational functions of each other, the equation obtained will be connected with Runge's, by a transformation of the type

$$\phi = y_0+y_1\psi+y_2\psi^2+y_3\psi^3+y_4\psi^4+y_5\psi^5,$$

or

$$\psi = z_0+z_1\phi+z_2\phi^2+z_3\phi^3+z_4\phi^4+z_5\phi^5,$$

where y_i, z_i are rational in R'. The discovery of roots of $\Phi(\phi)=0$ rational in R is, however, a simpler process than that of roots of $\Psi(\psi)=0$ rational in R'. In fact, when a, b, c, d, e, f are ordinary rational numbers and $R=1$, R' is in general a soluble sextic field, and the testing of a sextic equation whose coefficients belong to a sextic field for roots rational in that field, though possible in a finite number of steps, is a somewhat lengthy operation.

2. It is readily seen that the function

$$\phi = a^4\left[(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\epsilon)^2(\epsilon-\alpha)^2 \right.$$
$$\left. +(\alpha-\gamma)^2(\gamma-\epsilon)^2(\epsilon-\beta)^2(\beta-\delta)^2(\delta-\alpha)^2\right]$$

belongs to G_{20}. This number ϕ is one of a set of six conjugates which may be obtained by applying to it the six substitutions

$$1,\ (\alpha\beta\gamma),\ (\alpha\gamma\beta),\ (\beta\gamma)(\alpha),\ (\gamma\alpha)(\beta),\ (\alpha\beta)(\gamma),$$

δ, ϵ being unaltered, and, of course, the group of each of the conjugate functions is a sub-group of the symmetric group G_{120}, conjugate to G_{20}.

Now a symmetric function of the six ϕ's is an invariant of the quintic form

$$F(x,y) \equiv ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5,$$

and the six elementary symmetric functions are rational integral invariants. We may therefore assume that ϕ is a root of a sextic equation

$$\Phi(\phi) \equiv \phi^6+A_1\phi^5+A_2\phi^4+A_3\phi^3+A_4\phi^2+A_5\phi+A_6=0,$$

where A_1, A_2, ..., A_6 are rational integral invariants of respective degrees 4, 8, ..., 24.

3. Using the notation of Salmon's *Higher Algebra*, the quintic form $F(x, y)$ has four irreducible invariants J, K, L, I of respective degrees 4, 8, 12, 18. When the coefficients of $F(x, y)$ are numerical, the values of the first three of these are most easily calculated in the following way.

First, taking the quadratic covariant

$$\begin{aligned} S &\equiv \tfrac{1}{100}\big[(20ae-8bd+3c^2)\,x^2+(100af-12be+2cd)\,xy \\ &\qquad\qquad +(20bf-8ce+3d^2)\,y^2\big] \\ &\equiv Ax^2+Bxy+Cy^2, \text{ say,} \end{aligned}$$

we have
$$J = B^2-4AC.$$

Next, taking the canonizant

$$\begin{aligned} T &\equiv \begin{vmatrix} 10ax+2by, & 2bx+cy, & cx+dy \\ 2bx+cy, & cx+dy, & dx+2ey \\ cx+dy, & dx+2ey, & 2ex+10fy \end{vmatrix} \div 10^3 \\ &\equiv Dx^3+Ex^2y+Fxy^2+Gy^3, \end{aligned}$$

K, the lineo-linear invariant of S and the Hessian of T, and L, the discriminant of T, are given by

$$K = 2A(3EG-F^2)-B(9DG-EF)+2C(3FD-E^2),$$
$$L = \tfrac{1}{3}\big[4(3EG-F^2)(3FD-E^2)-(9DG-EF)^2\big].$$

4. We may now assume the rational equation satisfied by ϕ to be

$$\begin{aligned}\Phi(\phi) \equiv \phi^6 &+ x_1 J\phi^5 + (x_2 J^2 + x_3 K)\,\phi^4 \\ &+ (x_4 J^3 + x_5 JK + x_6 L)\,\phi^3 + (x_7 J^4 + x_8 J^2 K + x_9 K^2 + x_{10} JL)\,\phi^2 \\ &+ (x_{11} J^5 + x_{12} J^3 K + x_{13} JK^2 + x_{14} J^2 L + x_{15} KL)\,\phi \\ &+ x_{16} J^6 + x_{17} J^4 K + x_{18} J^2 K^2 + x_{19} K^3 + x_{20} J^3 L + x_{21} JKL + x_{22} L^2 = 0,\end{aligned}$$

where $x_1, x_2, \ldots, x_{22}$ are numerical. These numerical coefficients can be found by taking a sufficient number of special quintics, whether irreducible or not, and comparing the actual coefficients of the ϕ-equation with those involving the x's.

First, taking $$x^5 - n = 0,$$

we have $(A, B, C) = (0, -n, 0)$, $(D, E, F, G) = (0, 0, 0, 0)$,

$$J = n^2, \quad K = L = 0,$$

$$\phi_1 = -625n^2, \quad \phi_2 = \phi_3 = \phi_4 = \phi_5 = \phi_6 = -125n^2, \tag{4}$$

so that $(\phi + 625n^2)(\phi + 125n^2)^5$

$$\equiv \phi^6 + x_1 n^2 \phi^5 + x_2 n^4 \phi^4 + x_4 n^6 \phi^3 + x_7 n^8 \phi^2 + x_{11} n^{10} \phi + n_{16} n^{12},$$

giving $$x_1 = 2.5^4, \quad x_2 = 7.5^7, \quad x_4 = 2^2.3.5^{10}, \quad x_7 = 5^{13}.11,$$

$$x_{11} = 2.5^{15}.13, \quad x_{16} = 5^{19}.$$

Again, taking $$x^5 - x = 0,$$

we have $$J = L = 0, \quad K = 2.5^{-5},$$

$$\phi_1 = \phi_4 = 40i, \quad \phi_2 = \phi_6 = -40i, \quad \phi_3 = \phi_5 = 0,$$

so that

$$\phi^6 + 2^7.5^2.\phi^4 + 2^{12}.5^4\phi^2 \equiv \phi^6 + 2.5^{-5} x_3 \phi^4 + 2^2.5^{-10} x_9 \phi^2 + 2^3.5^{-15} x_{19},$$

whence $$x_3 = 2^6.5^7, \quad x_9 = 2^{10}.5^{14}, \quad x_{19} = 0.$$

Taking further

$(\alpha, \beta, \gamma, \delta, \epsilon) = (1, -1, 3, -3, 0)$, $(1, 1, -1, -1, 0)$, $(1, 2, -2, 0, 0)$, $(1, 3, -3, 0, 0)$, $(1, -1, \rho, \rho^2, 0)$, $[\rho^2 + \rho + 1 = 0]$, the remaining coefficients are found without difficulty—beyond that of the calculation. A complete verification is given by

$$(\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon) = (1,\ -1,\ 2,\ -2,\ 0).$$

The final result is

$$\begin{aligned}\Phi(\phi) \equiv \phi^6 &+ 2.5^4 J\phi^5 + (5^7.7J^2 + 2^6.5^7 K)\phi^4 \\ &+ (2^2.3.5^{10}J^3 + 2^6.3.5^{10}JK - 2^{11}.5^{10}L)\phi^3 \\ &+ (5^{13}.11J^4 + 2^6.3.5^{13}J^2K + 2^{10}.5^{14}K^2 - 2^{11}.5^{14}JL)\phi^2 \\ &+ (2.5^{15}.13J^5 + 2^6.5^{16}J^3K + 2^{12}.5^{15}.11JK^2 \\ &\qquad - 2^{10}.5^{15}.59J^2L - 2^{16}.5^{15}.7KL)\phi \\ &+ 5^{19}J^6 + 2^{12}.5^{19}J^2K^2 - 2^{12}.5^{19}J^3L \\ &- 2^{17}.5^{19}JKL + 2^{20}.5^{20}L^2 = 0.\end{aligned}$$

A simplification can be made by writing

$$5^3 J = j, \quad 2^5.5^6 K = k, \quad -2^{10}.5^9 L = l;$$

the equation then becomes

$$\begin{aligned}\Phi(\phi) \equiv \phi^6 &+ 10j\phi^5 + (35j^2 + 10k)\,\phi^4 + (60j^3 + 30jk + 10l)\,\phi^3 \\ &+ (55j^4 + 30j^2k + 25k^2 + 50jl)\,\phi^2 \\ &+ (26j^5 + 10j^3k + 44jk^2 + 59j^2l + 14kl)\,\phi \\ &+ 5j^6 + 20j^2k^2 + 20j^3l + 20jkl + 25l^2 = 0.\end{aligned}$$

5. We now assume that the quintic equation

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0,$$

is irreducible in R.

When the sextic has no rational root, the group is G_{120} or G_{60} according as the discriminant

$$\Delta \equiv a^8 \Pi(\alpha-\beta)^2 \equiv 5^5(J^2 - 2^7 K),$$

is not or is a rational square.

The quintic is soluble by radicals in R when, and only when, $\Phi(\phi) = 0$ has a rational root, and the group is then G_{20}, G_{10}, or G_5.

Should the sextic have two different rational roots it must reduce to the form

$$\Phi(\phi) \equiv (\phi-\phi_1)(\phi-\phi_2)^5 = 0,$$

as in (4) above. For taking the group G to consist of the substitutions enumerated in (1), (2), or (3) above, according as it is of order 20, 10, 5, the function

$$\phi_1 \equiv a^4\left[(a-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\epsilon)^2(\epsilon-a)^2 \right.$$
$$\left. +(a-\gamma)^2(\gamma-\epsilon)^2(\epsilon-\beta)^2(\beta-\delta)^2(\delta-a)^2\right]$$

is unaltered in form by the substitutions of G and therefore has a rational value. If also the value of $\phi_2 \equiv (a\beta\gamma)(\delta)(\epsilon)\phi_1$ is rational, this rational value must be unaltered by the substitutions of G. Among these, however, there is at least one which changes ϕ_2 into each of the other ϕ's, ϕ_3, ϕ_4, ϕ_5, and ϕ_6. Hence $\phi_2 = \phi_3 = \phi_4 = \phi_5 = \phi_6$ as stated.

The sextic cannot have an irreducible quadratic or cubic or quartic factor, for then a symmetric function of two or three of the ϕ's would be rational and the group intransitive.

6. The group of the function

$$\chi \equiv a^2(a-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\epsilon)^2(\epsilon-a)^2$$

is G_{10}, whereas that of

$$\sqrt{\chi} \equiv a(a-\beta)(\beta-\gamma)(\gamma-\delta)(\delta-\epsilon)(\epsilon-a)$$

is G_5, the only transitive Abelian group of the fifth degree.

Now, if ϕ be a rational root of the sextic, χ is a root of the rational quadratic

$$\chi^2-\phi\chi+5^7(J^2-2^7K) = 0.$$

When this quadratic (or each of them when there are two) has no rational root, the group of the quintic is G_{20}. But when a value of χ is rational the group is G_{10}, unless one of the roots happens to be a rational square when the group reduces to G_5 and the quintic is normal and cyclical. In any case the group reduces to G_{10} when R is extended by adjunction of χ, and a further adjunction of $\sqrt{\chi}$ reduces it to G_5. When $R = 1$ and the group is G_5, the quintic is cyclotomic by a theorem due to Kronecker (Hilbert, *Zahlbericht*, § 104).

There is an extensive range of literature on soluble quintics. Many references (prior to 1886) are given in the series of papers by Young and

McClintock, *American Journal*, 6, 7, and 8. A method of obtaining radical expressions for the roots, due substantially to Lagrange, but fully developed by Kronecker, is given in Weber's *Algebra* (Bd. I, § 191 *seq.*). Complete material for writing down the actual radicals in the case of the standard quintic

$$y^5+uy+v=0$$

(when soluble) appears in Prof. Mathews' Cambridge Tract, *Algebraic Equations*. To the latter named gentleman I am indebted for some suggestions as to the method of presenting the results contained in this paper. One of the Society's referees has also kindly pointed out corrections in two of the references given in the paper, which were inaccessible to me at the time of writing.

ON THE SOLUTION OF NUMERICAL FUNCTIONAL EQUATIONS

Illustrated by an Account of a Popular Puzzle and of its Solution.

By Eric H. Neville.

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The object of this paper is to introduce a method of solving simultaneous numerical functional equations which is original and is thought to be new. Any process of solution determines the value of each variable as the sum of a series whose terms are calculated in succession, but the practical value of a process depends not merely on the rate at which the several series converge, but also on the labour involved in evaluating the terms; in the method proposed this labour is almost beyond comparison less than in the classical method.

The problem is introduced and illustrated by a particular case, arising out of a showman's puzzle that has interested a large number of mathematicians, and when the general method has been explained and its application to two special problems of importance indicated, the equations arising from the puzzle are solved and some interesting consequences of the solution described.

I.

A familiar figure at fairs and shows is the sportsman with a cloth on which a large circle is painted and five smaller equal circular discs of thin metal, who offers the holiday-maker some considerable reward if he can lay the five discs on the cloth in such a way that no part of the large circle can be seen, the experimenter of course paying for each attempt. At a time when work of a more serious kind was for a few days impossible to me, I welcomed the problem of calculating the best arrangement of the discs, and the least value of the ratio of their radius to that of the painted circle, and the results may interest others. The equations on which the solution of the problem depends were much easier to find, and much simpler in form, than I had expected, but the numerical resolution of these equations with sufficient accuracy seemed likely to prove intolerably tedious, until an original method of dealing with them was devised: what I am anxious to put into the hands of mathematicians is this method.

In deciding on the general features of the most efficient arrangement of the discs, I was helped by having a specimen of the apparatus actually used (belonging to Mr. J. H. Grace, whose kindness in lending it both then, and on the occasion of the reading of this paper to the Society, I gladly acknowledge); it is only to be expected that no great margin is left to the inaccurate speculator, and certain types of arrangement were seen unmistakably to be ineffective; such was, for example, the arrangement symmetrical about each of five diameters, the small circles all passing through the centre of the large circle. It is taken for granted that there is symmetry about one line, a common diameter of the large circle and of one of the small circles. If K is the centre of the large circle, D the centre of this small circle, B the end of the diameter DK of the large circle which is not covered by the small circle, C the point in which the small circle cuts DB, and G, H the points in which the small circle cuts the large circle, the arrangements between which decision must be made can be enumerated. Two circles must pass through B, and intersect in a point L in DB, which may be identical with C, or may be a distinct point in CB; let one of these circles cut the arc BG of the large circle in E, the other cut the arc BH in F. Of the remaining circles one covers E and G, the other covers F and H. If L is distinct from C, the circle covering E and G covers also L and C, and either passes through three of the four points E, G, L, C or has the line joining two of them for a diameter. If L coincides with C, the circle BCE cuts the circle whose centre is D in a point M distinct from C, and the circle covering E and G either is the circle through E, G, and M, or has one of the lines GM, ME, EG for its diameter. It would be possible to apply calculation to each case, but actual trial is sufficient to convince that the only arrangement which allows success with the apparatus used is of the last type; what remains for calculation is the discovery of the smallest ratio of the common radius of the discs to the radius of the painted circle which allows this most effective arrangement to succeed, and the determination of the corresponding position of the point we have denoted by C.

Let b be the radius of the large circle, a that of the small circles, let c be the distance KD, and let the angles BKE, DKG, BCE, KDG be 2θ, ϕ, ψ, $\pi-\chi$. It is a/b that it is our chief object to find, but $(a-c)/b$

derives considerable interest from being extremely small. The equations

$$a \sin \psi = b \sin \theta, \qquad (a-c) \sin \psi = b \sin (\psi - 2\theta),$$

$$c = b \cos \phi - a \cos \chi, \qquad a \sin \chi = b \sin \phi,$$

implicitly determine θ, ϕ, ψ, χ in terms of c. The best arrangement of the first three discs, a, b being supposed constant, is that in which c has the value which makes EG, FH as small as possible, that is, is found by making $2\theta + \phi$ a maximum subject to these equations; the condition for this is

$$\sin 2\psi \sin (\chi - \phi) = \cos \chi \{1 - \cos 2(\psi - \theta)\},$$

and if, when the position is such that the five equations involving c, θ, ϕ, ψ, χ are all satisfied, EG and FH are less than $2a$, the last two discs can be set down completely to cover what is left exposed of the large circle. The least value of a/b which allows the covering is that in which EG, FH, found as before, are equal to $2a$, and this value of a/b is found by adding to the five equations already written down, the equation

$$a = b \cos (\theta + \tfrac{1}{2}\phi).$$

For purposes of calculation, the ratios $a : b : c$ are eliminated from the equations, and the resulting equations are taken in a form involving sums and differences, not products and quotients, of circular functions; the equations are

$$2 \sin \theta - \sin (\theta + \tfrac{1}{2}\phi + \psi) - \sin (\psi - \theta - \tfrac{1}{2}\phi) = 0,$$

$$2 \sin \phi - \sin (\theta + \tfrac{1}{2}\phi + \chi) - \sin (\chi - \theta - \tfrac{1}{2}\phi) = 0,$$

$$2 \sin \theta + \sin (\chi + \theta) - \sin (\chi - \theta) - \sin (\psi + \phi) \\ - \sin (\psi - \phi) - 2 \sin (\psi - 2\theta) = 0,$$

$$\cos (2\psi - \chi + \phi) - \cos (2\psi + \chi - \phi) - 2 \cos \chi \\ + \cos (2\psi + \chi - 2\theta) + \cos (2\psi - \chi - 2\theta) = 0,$$

and what we require is a numerical solution, which can be based on the fact that a crude approximation is given by the values 36°, 36°, 72°, 72° for θ, ϕ, ψ, χ.

It is evident that these particular equations are algebraic and even rational in the tangents of the angles $\tfrac{1}{2}\theta$, $\tfrac{1}{4}\phi$, $\tfrac{1}{2}\psi$, $\tfrac{1}{2}\chi$, so that an algebraic equation could be found for $\cos (\theta + \tfrac{1}{2}\phi)$, which is the number of greatest interest; to calculate by Horner's, or any other of the familiar methods, the root to which cos 54° is a rough approximation is in theory simplicity itself. The briefest effort will convince the reader that it is impracticable to solve the problem on these lines. The alternative is to regard the equations as functional, rather than algebraic, and to accept suggestions from the differential calculus: to a consideration of numerical functional equations we now turn.

II.

Of all the common problems of mathematics whose solutions in theory are both simple and complete, probably in no other is the application of the theoretical solution to a numerical case as tedious, and in no other does this application find the mathematician who is not an accountant at as serious a disadvantage, as in the problem of solving a set of numerical functional equations. Let us outline the classical process in the case of three equations

$$F(x, y, z) = 0, \quad G(x, y, z) = 0, \quad H(x, y, z) = 0,$$

of which it is known that an approximate solution is given by

$$x = a_1, \quad y = b_1, \quad z = c_1.$$

If the corresponding accurate solution is

$$x = a_1 + x_1, \quad y = b_1 + y_1, \quad z = c_1 + z_1,$$

an approximation to the values of x_1, y_1, z_1 is given by

$$x_1 = a_2', \quad y_1 = b_2', \quad z_1 = c_2',$$

where

$$a_2' F_x(a_1, b_1, c_1) + b_2' F_y(a_1, b_1, c_1) + c_2' F_z(a_1, b_1, c_1) = -F(a_1, b_1, c_1),$$

$$a_2' G_x(a_1, b_1, c_1) + b_2' G_y(a_1, b_1, c_1) + c_2' G_z(a_1, b_1, c_1) = -G(a_1, b_1, c_1),$$

$$a_2' H_x(a_1, b_1, c_1) + b_2' H_y(a_1, b_1, c_1) + c_2' H_z(a_1, b_1, c_1) = -H(a_1, b_1, c_1);$$

the second approximation to the values of x, y, z is given by

$$x = r_2', \quad y = s_2', \quad z = t_2',$$

where

$$r_2' = a_1 + a_2', \quad s_2' = b_1 + b_2', \quad t_2' = c_1 + c_2'.$$

To obtain a third approximation to the values of x, y, z the process is repeated, terms a_3', b_3', c_3' being obtained from three equations of which the first is

$$a_3' F_x(r_2', s_2', t_2') + b_3' F_y(r_2', s_2', t_2') + c_3' F_z(r_2', s_2', t_2') = -F(r_2', s_2', t_2'),$$

and these terms being added to those already known. We may say that the exact roots r, s, t are the sums of series $\Sigma a_m'$, $\Sigma b_m'$, $\Sigma c_m'$ whose terms are calculated in successive triads from three equations such as

$$a'_{m+1}F_x(r'_m,\ s'_m,\ t'_m)+b'_{m+1}F_y(r'_m,\ s'_m,\ t'_m)+c'_{m+1}F_z(r'_m,\ s'_m,\ t'_m) = -F(r'_m,\ s'_m,\ t'_m),$$

r'_m, s'_m, t'_m being the sums to m terms of the series whose m-th terms are a'_m, b'_m, c'_m and the first terms a'_1, b'_1, c'_1 being the given first approximations a_1, b_1, c_1.

This process is open to a twofold criticism. The solution of simultaneous linear algebraic equations is by no means the attractive process in practice that it is in theory: if there are only a few equations and the coefficients are small integers the solution is not prohibitively troublesome, especially if a multiplication table or a mechanical multiplier is accessible, but given coefficients with three or four significant digits, the operation is excessively tedious, and is one in which mistakes are easy to make, and if the coefficients are irrational it is troublesome to decide the degree of accuracy advisable at each stage. And in the classical process, although the coefficients of the variables, the first derivatives of the functions, change but slightly from step to step of the approximation, the labour of solving one set of linear equations is not in the least diminished in virtue of the work done in solving the earlier sets, and, of course, increases with the degree of accuracy maintained. In the alternative process which is to be described, use is made of the fact that the coefficients vary but little, and subsequently it is pointed out that the process is applicable if the coefficients are actually constant and can be used in the solution of a set of linear algebraic equations with effect if the coefficients are complicated or irrational.

If the coefficients and constant terms in one set of linear algebraic equations differ but little from the coefficients and constant terms in another set, the values of the variables which satisfy the one set differ from the values which satisfy the other by amounts which are small compared with the values themselves, it being assumed that the determinants of the coefficients are not small. Thus, if λ_1, μ_1, ..., ν_3 are any close approximations to

$$F_x(a_1,\ b_1,\ c_1),\quad F_y(a_1,\ b_1,\ c_1),\quad \ldots,\quad H_z(a_1,\ b_1,\ c_1),$$

the values of the nine first derivatives of three functions $F(x, y, z)$, $G(x, y, z)$, $H(x, y, z)$ for values a_1, b_1, c_1 of the variables for which the functions are known to be small simultaneously, and if a''_2, b''_2, c''_2 are such as to satisfy the equations

$$\lambda_1 a''_2+\mu_1 b''_2+\nu_1 c''_2 = -F(a_1,\ b_1,\ c_1),$$
$$\lambda_2 a''_2+\mu_2 b''_2+\nu_2 c''_2 = -G(a_1,\ b_1,\ c_1),$$
$$\lambda_3 a''_2+\mu_3 b''_2+\nu_3 c''_2 = -H(a_1,\ b_1,\ c_1),$$

then a_1+a_2'', b_1+b_2'', c_1+c_2'', which we may denote by r_2'', s_2'', t_2'', differ from the second approximations r_2', s_2', t_2' of the classical solution by amounts of the order $g_1''f_2''$, where f_2'' is the greatest of the moduli of the three terms a_2'', b_2'', c_2'' and g_1'' is an approximation factor* depending on the accuracy with which $\lambda_1, \mu_1, \ldots, \nu_3$ represent

$$F_x(a_1, b_1, c_1), \quad F_y(a_1, b_1, c_1), \quad \ldots, \quad H_z(a_1, b_1, c_1).$$

Since r_2', s_2', t_2' differ from the exact solutions r, s, t by amounts of order $f_2''^2$, r_2'', s_2'', t_2'' differ from r, s, t by amounts of order $(g_1''+f_2'')f_2''$; if f_2'' is of a higher order than g_1'', the approximation furnished by r_2'', s_2'', t_2'' is not as close as that furnished by r_2', s_2', t_2', but the somewhat arbitrary nature of $\lambda_1, \mu_1, \ldots, \nu_3$ enables us to take for them rational numbers which can all be comparatively simple without the value of g_1'' becoming unduly large, and so to render the calculation of r_2'', s_2'', t_2'' a much simpler matter than the calculation of r_2', s_2', t_2'.

To obtain a closer approximation than that given by r_2'', s_2'', t_2'', we have to solve linear equations which either are equations such as

$$a_3''F_x(r_2'', s_2'', t_2'')+b_3''F_y(r_2'', s_2'', t_2'')+c_3''F_z(r_2'', s_2'', t_2'') = -F(r_2'', s_2'', t_2''),$$

or are equations whose coefficients and constant terms differ little from those of these equations. Now it is assumed throughout that none of the second derivatives of the functions F, G, H are large compared with the largest of the first derivatives, and it follows that numbers $\lambda_1, \mu_1, \ldots, \nu_3$, which are approximations to

$$F_x(a_1, b_1, c_1), \quad F_y(a_1, b_1, c_1), \quad \ldots, \quad H_z(a_1, b_1, c_1),$$

are approximations also to

$$F_x(r_2'', s_2'', t_2''), \quad F_y(r_2'', s_2'', t_2''), \quad \ldots, \quad H_z(r_2'', s_2'', t_2''),$$

and indeed to the values of the first derivatives for any values of the arguments not differing greatly from the values r, s, t, which we are endeavouring to calculate. Hence, instead of solving equations such as that last written, we may solve equations such as

$$\lambda_1 a_3''+\mu_1 b_3''+\nu_1 c_3'' = -F(r_2'', s_2'', t_2''),$$

which differ only in the constant terms from the equations of the set solved to find a_2'', b_2'', c_2''; the sums r_3'', s_3'', t_3'', that is, $r_2''+a_3''$, $s_2''+b_3''$, $t_2''+c_3''$, differ from r, s, t by amounts of order $(g_2''+f_3'')f_3''$, where f_3'' is the greatest of the moduli of a_3'', b_3'', c_3'' and g_2'' is an approximation factor differing but slightly from g_1''.

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* On the nature of this factor, see Section III below.

The process may be continued indefinitely: the solutions are found as the sums of series $\Sigma a_m''$, $\Sigma b_m''$, $\Sigma c_m''$ whose terms are calculated in successive triads from sets of equations such as

$$\lambda_1 a_{m+1}''+\mu_1 b_{m+1}''+\nu_1 c_{m+1}'' = -F(r_m'', s_m'', t_m''),$$

$$\lambda_2 a_{m+1}''+\mu_2 b_{m+1}''+\nu_2 c_{m+1}'' = -G(r_m'', s_m'', t_m''),$$

$$\lambda_3 a_{m+1}''+\mu_3 b_{m+1}''+\nu_3 c_{m+1}'' = -H(r_m'', s_m'', t_m''),$$

the arguments r_m'', s_m'', t_m'' being the m-th partial sums of the series $\Sigma a_m''$, $\Sigma b_m''$, $\Sigma c_m''$, and the coefficients λ_1, μ_1, ..., ν_3 being the same at every step; the remainder of each series after m terms is of order $(g_{m-1}''+f_m'')f_m''$, where f_m'' is the greatest of the moduli of a_m'', b_m'', c_m'', and g_{m-1}'' depends on the accuracy with which λ_1, μ_1, ..., ν_3 represent

$$F_x(r_{m-1}'', s_{m-1}'', t_{m-1}''),\ F_y(r_{m-1}'', s_{m-1}'', t_{m-1}''),\ \ldots,\ H_z(r_{m-1}'', s_{m-1}'', t_{m-1}''),$$

and so tends as m increases to a definite limit g'' dependent on the nearness of the coefficients to $F_x(r, s, t)$, $F_y(r, s, t)$, ..., $H_z(r, s, t)$. Since f_m'' tends to zero, but g_{m-1}'' as a rule does not, we may say that in general the remainder of each series is of order $g''f_m''$, where g'' is a fractional approximation factor, though in exceptional cases the remainder may be of order $f_m''^2$.

When we have to solve a number of sets of linear equations with common rational coefficients but different constants, the most tedious part of the work, and, what is equally important in practice, the part of the work in which the greatest care is needed if mistakes are to be avoided, can be performed once for all. From the set of coefficients λ_1, μ_1, ..., ν_3 a set ρ_1'', σ_1'', ..., τ_3'', the reciprocal set with the sign of every member changed, can be found such that the set of relations

$$\lambda_1 a+\mu_1 b+\nu_1 c = -u,\quad \lambda_2 a+\mu_2 b+\nu_2 c = -v,\quad \lambda_3 a+\mu_3 b+\nu_3 c = -w,$$

between three variables a, b, c and three variables u, v, w is equivalent to the set of relations

$$a = \rho_1''u+\sigma_1''v+\tau_1''w,\quad b = \rho_2''u+\sigma_2''v+\tau_2''w,\quad c = \rho_3''u+\sigma_3''v+\tau_3''w,$$

and the nine coefficients ρ_1'', σ_1'', ..., τ_3'' having once been calculated, the successive triads of which we are in search are obtained from equations such as

$$a''_{m+1} = \rho''_1 F(r''_m, s''_m, t''_m) + \sigma''_1 G(r''_m, s''_m, t''_m) + \tau''_1 H(r''_m, s''_m, t''_m).$$

A last simplification suggests itself immediately. If $\lambda_1, \mu_1, \ldots, \nu_3$ are rational, the coefficients $\rho''_1, \sigma''_1, \ldots, \tau''_3$ are rational, but they may be complicated fractions. It is pointless that the relations such as

$$a = \rho''_1 u + \sigma''_1 v + \tau''_1 w$$

should represent the relations such as

$$\lambda_1 a + \mu_1 b + \nu_1 c = -u,$$

with greater accuracy than that with which the coefficients $\lambda_1, \mu_1, \ldots, \nu_3$ represent the derivatives $F_x, F_y, \ldots, H_z$, and therefore we may facilitate calculation by substituting for the coefficients $\rho''_1, \sigma''_1, \ldots, \tau''_3$ any coefficients $\rho_1, \sigma_1, \ldots, \tau_3$, which do not differ greatly from them, the effect being to substitute for the set of approximation factors $g''_1, g''_2, \ldots$, a set of approximation factors $g_1, g_2, \ldots$ tending to a limit g which may be either larger or smaller than the limit g''. So, finally, the solutions r, s, t are the sums of series Σa_m, Σb_m, Σc_m, whose terms are calculated in successive triads from the equations

$$a_{m+1} = \rho_1 F(r_m, s_m, t_m) + \sigma_1 G(r_m, s_m, t_m) + \tau_1 H(r_m, s_m, t_m),$$

$$b_{m+1} = \rho_2 F(r_m, s_m, t_m) + \sigma_2 G(r_m, s_m, t_m) + \tau_2 H(r_m, s_m, t_m),$$

$$c_{m+1} = \rho_3 F(r_m, s_m, t_m) + \sigma_3 G(r_m, s_m, t_m) + \tau_3 H(r_m, s_m, t_m),$$

the coefficients $\rho_1, \sigma_1, \ldots, \tau_3$ being constant throughout and simple in form, and the arguments r_m, s_m, t_m being the m-th partial sums of the series Σa_m, Σb_m, Σc_m themselves. The restriction to three equations in three variables has been purely a matter of convenience, and, in general, we have the theorem :

If it is known that an approximate simultaneous solution of any number n of independent functional equations

$$F_p(x_1, x_2, \ldots, x_n) = 0 \quad (p = 1, 2, \ldots, n),$$

in the same number of variables is given by

$$x_p = a_{p1} \quad (p = 1, 2, \ldots, n),$$

then rational coefficients ρ_{st} *can readily be obtained, such that, if sets of* *numbers* a_{pm} *are calculated in succession from the formulæ*

$$a_{p,\,m+1} = \sum_{q=1}^{n} \rho_{qp} F_q(r_{1m},\, r_{2m},\, \ldots,\, r_{nm}) \quad (p = 1,\, 2,\, \ldots,\, n),$$

where

$$r_{lm} = \sum_{k=1}^{m} a_{lk} \quad (l = 1,\, 2,\, \ldots,\, n),$$

then the n *series* $\sum_{m=1}^{\infty} a_{pm}$ *are convergent, and if the sum of the series* Σa_{pm} *is* r_p, *the functional equations are all satisfied for the set of values* $r_1, r_2, \ldots, r_n$ *of the variables; the series are ultimately dominated by geometric series with a common ratio dependent on the differences between the values chosen for the coefficients and the values of the quotients*

$$(-)^{s+t+1} \frac{\partial(F_1,\, F_1,\, \ldots,\, F_{t-1},\, F_{t+1},\, \ldots,\, F_{n-1},\, F_n)}{\partial\,(r_1,\, r_2,\, \ldots,\, r_{s-1},\, r_{s+1},\, \ldots,\, r_{n-1},\, r_n)} \Big/ \frac{\partial(F_1,\, F_2,\, \ldots,\, F_n)}{\partial\,(r_1,\, r_2,\, \ldots,\, r_n)},$$

unless these differences all happen to vanish, in which case the dominating series is as in the classical method of approximation a series of the form

$$c\,(1+k+k^2+k^4+k^8+\ldots).$$

Theoretically, the classical method is more powerful than the method described, since its dominating series ultimately converges more rapidly than any geometric series, but in practice each step of the classical method is incomparably more troublesome than the whole group of steps recommended here for advancing the approximation by the same amount.

If in any branch of applied mathematics the solution of sets of numerical functional equations became a daily necessity, it might be worth while to have tabulated sets of rational coefficients ρ_{st} corresponding approximately to standard sets of coefficients λ_{ts}; the undertaking would be weighty, for even with only three equations the number of entries would be very large if the number of different values, positive, zero, and negative, which each coefficient was allowed to assume was adequate.

Before returning to the particular set of functional equations for whose solution the method of this paper was devised, let us refer to the application of the method to the two simplest cases; to find that there is still something to be said on familiar problems is always an encouragement to research.

III.

We have already remarked that if the coefficients of a set of simultaneous linear equations are irrational or are complicated, the solution, to a preassigned degree of accuracy, is troublesome: we can apply our general method to deduce the solution by successive steps from equations with simple coefficients. Thus to solve three equations

$$f_1x+g_1y+h_1z=p, \quad f_2x+g_2y+h_2z=q, \quad f_3x+g_3y+h_3z=r,$$

we take any numbers $\lambda_1, \mu_1, \ldots, \nu_3$ which do not differ greatly from $f_1, g_1, \ldots, h_3$, and find simple rational numbers $\rho_1, \sigma_1, \ldots, \tau_3$, such that the equations

$$\lambda_1x+\mu_1y+\nu_1z=-u, \quad \lambda_2x+\mu_2y+\nu_2z=-v, \quad \lambda_3x+\mu_3y+\nu_3z=-w,$$

between two sets of variables x, y, z and u, v, w are approximately equivalent to the equations

$$x=\rho_1u+\sigma_1v+\tau_1w, \quad y=\rho_2u+\sigma_2v+\tau_2w, \quad z=\rho_3u+\sigma_3v+\tau_3w;$$

then, if

$$\left.\begin{aligned} x_1&=-(\rho_1p+\sigma_1q+\tau_1r)\\ y_1&=-(\rho_2p+\sigma_2q+\tau_2r)\\ z_1&=-(\rho_3p+\sigma_3q+\tau_3r)\end{aligned}\right\} \qquad \left.\begin{aligned} p_2&=p-(f_1x_1+g_1y_1+h_1z_1)\\ q_2&=q-(f_2x_1+g_2y_1+h_2z_1)\\ r_2&=r-(f_3x_1+g_3y_1+h_3z_1)\end{aligned}\right\}$$

$$\left.\begin{aligned} x_2&=-(\rho_1p_2+\sigma_1q_2+\tau_1r_2)\\ y_2&=-(\rho_2p_2+\sigma_2q_2+\tau_2r_2)\\ z_2&=-(\rho_3p_2+\sigma_3q_2+\tau_3r_2)\end{aligned}\right\} \qquad \left.\begin{aligned} p_3&=p_2-(f_1x_2+g_1y_2+h_1z_2)\\ q_3&=q_2-(f_2x_2+g_2y_2+h_2z_2)\\ r_3&=r_2-(f_3x_2+g_3y_2+h_3z_2)\end{aligned}\right\}$$

and so on, the series $x_1+x_2+\ldots$, $y_1+y_2+\ldots$, $z_1+z_2+\ldots$ tend to the actual values of x, y, z satisfying the proposed equations. The method is particularly useful if the original equations have irrational coefficients, since it renders it unnecessary to determine in advance how far accuracy in approximating to the values of the coefficients is significant in the attainment of any required degree of accuracy in the solutions, and it renders it possible to make use of any given approximate solution in searching for an approximation still closer. So true is it that when coefficients are complicated an economy is effected by this process, that the simplest way of conducting an approximation to a solution of a set of functional equations by the classical method is to solve by this method the various sets of linear equations that arise: this mixture of methods is naturally less satisfactory than a frank desertion of the classical method, but when in any case a point is reached from which linear equations would complete the solution to the required degree of accuracy, to write down these linear equations enables the calculator to avoid further reference to tables.

In the case of linear algebraic equations it is easy to shew the validity of the process employed. With the three equations just discussed, we have

$$x_{m+1} =$$

$$\{1+(\rho_1 f_1+\sigma_1 f_2+\tau_1 f_3)\}\, x_m+(\rho_1 g_1+\sigma_1 g_2+\tau_1 g_3)\, y_m+(\rho_1 h_1+\sigma_1 h_2+\tau_1 h_3)\, z_m,$$

and similar expressions for y_{m+1}, z_{m+1}. The nine coefficients

$$1+(\rho_1 f_1+\sigma_1 f_2+\tau_1 f_3),\ (\rho_1 g_1+\sigma_1 g_2+\tau_1 g_3),\ \ldots,\ 1+(\rho_3 h_1+\sigma_3 h_2+\tau_3 h_3)$$

can be made as small as necessary by proper choice of the nine numbers $\rho_1, \sigma_1, \ldots, \tau_3$, and it is evident that if the greatest of their moduli is not greater than $\frac{1}{3}g$, and g is a proper fraction, the series

$$x_1+x_2+\ldots,\ y_1+y_2+\ldots,\ z_1+z_2+\ldots$$

all converge not less rapidly than a geometric series with ratio g. In practice the method may often be found to succeed even if the greatest modulus is not less than $\frac{1}{3}$, for the signs of the various terms do not as a rule combine in the most unfavourable manner conceivable. The method being proved valid for linear algebraic equations, its validity for equations of any form is a consequence of the validity of the classical process.

The other simple case of which we wish to speak is that of a single functional equation. What we say is that if a_1 is a first approximation to a root of a functional equation

$$F(x) = 0,$$

and $-\rho$ is any number not very different from the value of the reciprocal of dF/dx when x is equal to a_1, then the root itself is the sum of the series Σa_m whose partial sums are calculated in succession from the formulæ

$$\left.\begin{aligned} a_2 &= \rho F(a_1) \\ r_2 &= a_1+a_2 \end{aligned}\right\} \quad \left.\begin{aligned} a_3 &= \rho F(r_2) \\ r_3 &= r_2+a_2 \end{aligned}\right\} \quad \left.\begin{aligned} a_4 &= \rho F(r_3) \\ r_4 &= r_3+a_3 \end{aligned}\right\} \quad \left.\begin{aligned} a_5 &= \rho F(r_4) \\ r_5 &= r_4+a_4 \end{aligned}\right\}$$

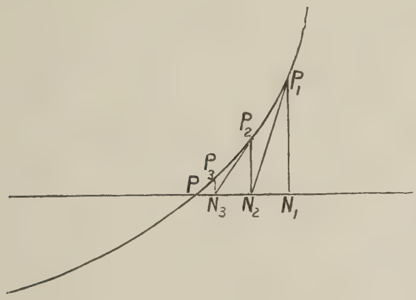
and so on.

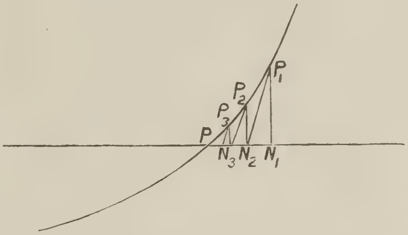
For example, taking $2-x^2$ for $F(x)$ and $1\frac{1}{2}$ as a first approximation, we naturally take ρ to be $\frac{1}{3}$, and we have

$$a_2 = -\ {\cdot}083, \quad a_3 = -\ {\cdot}0026, \quad a_4 = -\ {\cdot}00018, \quad a_5 = -\ {\cdot}000006,$$

$$r_2 = 1{\cdot}417, \quad r_3 = 1{\cdot}4144, \quad r_4 = 1{\cdot}41422, \quad r_5 = 1{\cdot}414214.$$

*The process of solving a single functional equation may be illustrated graphically, the relation of the method of this paper to the classical method being made apparent. The problem is, given a point N_1 on the axis of x known to be not far from the point P in which a curve $y = F(x)$ meets that axis, to construct a sequence of points N_1, N_2, N_3, ..., having P for limiting point. In the older method, from N_1 is drawn the ordinate N_1P_1 to the curve (the line parallel to the axis of y, not necessarily the line perpendicular to the axis of x), and the tangent P_1N_2 to the curve at P_1 cuts the axis of x in the second point N_2 of the sequence; N_3 lies on the tangent at the point P_2 in which the line N_2P_2 parallel to N_1P_1 cuts the curve, and so on. In the present method, P_1 is found from N_1, P_2 from N_2, and so on, as before, but the lines P_1N_2, P_2N_3, ... by which N_2 is found from P_1, N_3 from P_2, and so on, are not tangents to the curve, but are lines all parallel to some direction not differing greatly from that of the tangent at P_1. If actual tangents are used, tables must be consulted for the determination not only of the lengths of the ordinates N_1P_1, N_2P_2, ..., but also of the slopes of the tangents P_1N_2, P_2N_3, ...; if the lines P_1N_2, P_2N_3, ... have a common direction, the triangles $N_2N_1P_1$, $N_3N_2P_2$, ... are all similar, and when the ratio of N_2N_1 to N_1P_1 has been found it is only for the values of the ordinates N_1P_1, N_2P_2, ... that reference need be made to tables. This graphical consideration brings out clearly another point. The rapidity of the approximation depends on the closeness of the





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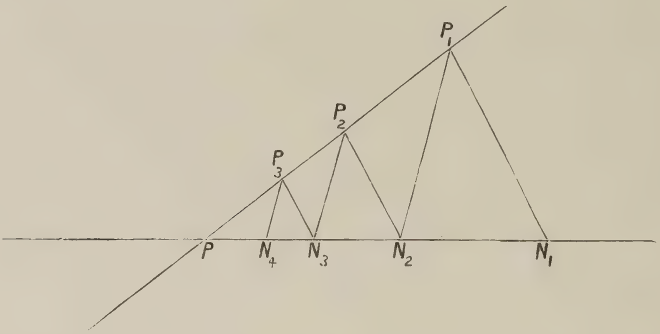
* The remainder of this section has been added since the reading of the paper; the graphical illustration was suggested by one of the referees.

common direction of P_1N_2, P_2N_3, ... to the direction of the tangent at P, the point to be found, but the approximation may be valid even if the two directions differ considerably: if PP_1 is actually a straight line, and N_2 is



any point between N_1 and the image of N_1 in P, then by drawing N_2P_2 parallel to N_1P_1, P_2N_3 parallel to P_1N_2, N_3P_3 parallel to N_1P_1, P_3N_4 parallel to P_1N_2, and so on, we can construct a sequence of points converging to P, the convergence being the faster the nearer N_2 is taken to P. Similarly in the more general case, there may be much latitude in the choice of the common direction of P_1N_2, P_2N_3, and the rest, and because of this latitude a choice may be made which gives a simple value to the ratio of N_2N_1 to N_1P_1, of N_3N_2 to N_2P_2, and so on.

Thus we see a connection between the method of this paper and a method used* in a number of problems in applied mathematics. The simplest ratio which N_2N_1 can bear to N_1P_1 is unity, and to draw P_1N_2 in such a direction as to make N_2N_1 equal to N_1P_1 is to evaluate the root of the equation $F(x) = 0$ by the steps

$$a_2 = -F(a_1), \quad a_3 = -F(a_1+a_2), \quad a_4 = -F(a_1+a_2+a_3), \quad \ldots,$$

that is,

$$r_2 = r_1 - F(r_1), \quad r_3 = r_2 - F(r_2), \quad r_4 = r_3 - F(r_3), \quad \ldots.$$

This process is specially useful when the equation to be solved has the particular form

$$x = f(x),$$

for if we write

$$F(x) = x - f(x)$$

we see that the approximation then advances by the steps

$$r_2 = f(r_1), \quad r_3 = f(r_2), \quad r_4 = f(r_3), \quad \ldots,$$

---

* I am indebted to Dr. Bromwich for drawing my attention to this method.

which are so simple as to justify considerable slowness in the convergence. It can be shewn that convergence is certain (though it may be slow), if $f'(x)$ is positive and less than unity in the neighbourhood of the root, and since if $f^{-1}(x)$ is the function inverse to $f(x)$, the equation $x=f(x)$ is identical with the equation $x=f^{-1}(x)$ and $df^{-1}(x)/dx$ is the reciprocal of $df(x)/dx$, it follows that the root of the equation $x=f(x)$ can be found either by the sequence

$$r_2=f(r_1), \quad r_3=f(r_2), \quad r_4=f(r_3), \quad \ldots,$$

or by the sequence

$$r_2=f^{-1}(r_1), \quad r_3=f^{-1}(r_2), \quad r_4=f^{-1}(r_3), \quad \ldots,$$

provided only that $f'(x)$ is positive and different from unity near the required root.

To more equations than one, even to a set given in such a form as

$$x=f(x, y, z), \quad y=g(x, y, z), \quad z=h(x, y, z),$$

this process is not necessarily adaptable, but the knowledge of the process may well be an encouragement to the use of very crude approximations in choosing the rational coefficients which we have denoted in general by

$$\rho_{11}, \rho_{21}, \ldots, \rho_{nn}.$$

IV.

In illustration of the general method we have described, we give some details of the calculation in the case of the equations connected with the covering puzzle. First, a rough calculation, of which no account need be given, shows that 36°, 36°, 72°, 66° is a better approximation than 36°, 36°, 72°, 72° to the solution of the equations. To have only acute angles to consider, we substitute $36°-\alpha$, $36°-2\beta$, $72°+\gamma$, $66°-\eta$ for θ, ϕ, ψ, χ, and the equations solved are

$$-2\sin(36°-\alpha)+\cos(36°-\alpha-\beta+\gamma)+\sin(18°+\alpha+\beta+\gamma)=0,$$

$$-2\sin(36°-2\beta)+\cos(30°-\alpha-\beta-\eta)+\sin(12°+\alpha+\beta-\eta)=0,$$

$$-2\sin(36°-\alpha)-\cos(12°-\alpha-\eta)+2\sin(2\alpha+\gamma)+\sin(30°+\alpha-\eta)$$
$$+\cos(18°-2\beta+\gamma)+\sin(36°+2\beta+\gamma)=0,$$

$$-\cos(6°+2\alpha+2\gamma+\eta)-\cos(6°-2\beta-2\gamma+\eta)+\cos(42°-2\alpha-2\gamma+\eta)$$
$$+\sin(24°-2\beta+2\gamma+\eta)+2\sin(24°+\eta)=0.$$

The equations which the classical process requires us first to solve are

$$\begin{aligned}
3{\cdot}157\alpha'+1{\cdot}539\beta'+{\cdot}363\gamma' &= {\cdot}058,\\
1{\cdot}478\alpha'+4{\cdot}714\beta' \qquad\qquad -{\cdot}478\eta' &= {\cdot}102,\\
6{\cdot}276\alpha'+2{\cdot}236\beta'+2{\cdot}500\gamma'-1{\cdot}074\eta' &= {\cdot}115,\\
1{\cdot}547\alpha'-2{\cdot}036\beta'+3{\cdot}165\gamma'+2{\cdot}280\eta' &= {\cdot}026,
\end{aligned}$$

α, β, γ, η being expressed in radians; no attempt is made to solve these equations, but the set

$$\begin{aligned}
3\tfrac{1}{6}\alpha+1\tfrac{1}{2}\beta+\tfrac{1}{3}\gamma \qquad\qquad &= -t,\\
1\tfrac{1}{2}\alpha+4\tfrac{3}{4}\beta \qquad\qquad -\tfrac{1}{2}\eta &= -u,\\
6\tfrac{1}{4}\alpha+2\tfrac{1}{4}\beta+2\tfrac{1}{2}\gamma-\eta &= -v,\\
1\tfrac{1}{2}\alpha-2\beta+3\tfrac{1}{6}\gamma+2\tfrac{1}{4}\eta &= -w,
\end{aligned}$$

that is to say

$$\begin{aligned}
19\alpha+9\beta+2\gamma \qquad\qquad &= -6t,\\
6\alpha+19\beta- \qquad\qquad 2\eta &= -4u,\\
25\alpha+9\beta+10\gamma-4\eta &= -4v,\\
18\alpha-24\beta+38\gamma+27\eta &= -12w,
\end{aligned}$$

is inverted, not accurately, but to the approximate form

$$\begin{aligned}
\alpha &= -(\tfrac{1}{2}-\tfrac{1}{12})t+\tfrac{1}{7}u+\tfrac{1}{30}w,\\
\beta &= \tfrac{1}{16}t-(\tfrac{1}{4}+\tfrac{1}{60})u+\tfrac{1}{30}v-\tfrac{1}{30}w,\\
\gamma &= (1-\tfrac{1}{4}-\tfrac{1}{40})t-\tfrac{1}{6}u-(\tfrac{1}{4}+\tfrac{1}{30})v-\tfrac{1}{6}w,\\
\eta &= -(1-\tfrac{1}{3}-\tfrac{1}{60})t-\tfrac{1}{9}u+(\tfrac{1}{2}-\tfrac{1}{16})v-(\tfrac{1}{4}+\tfrac{1}{40})w,
\end{aligned}$$

the form* of coefficient used here being the most convenient. Substituting $-{\cdot}058$, $-{\cdot}102$, $-{\cdot}115$, $-{\cdot}026$ for t, u, v, w, for a second approximation to the values of α, β, γ, η (simultaneous zeroes being the first

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* If a set of equations with integral coefficients is inverted accurately, the coefficients in the reciprocal set are fractions with a common denominator Δ, the determinant of the original coefficients, and if the set of numbers Δ, $\Delta/2$, $\Delta/3$, ..., $\Delta/(k-1)$ is written down, k being any integer greater than 9, an approximation of any desired accuracy to each coefficient in the convenient form

$$e_0 s_0+\frac{e_1}{s_1}+\frac{e_2}{10s_2}+\frac{e_3}{100s_3}+\frac{e_4}{1000s_4}+\ldots,$$

where each of the letters $e_0, e_1, e_2, \ldots$ stands for one of the three numbers 1, 0, -1, and each of the letters $s_0, s_1, s_2, \ldots$ for a positive integer less than k, can be written down at sight.

approximation), we have[*]

$$
\begin{array}{rrrrrrrr}
 & & & & & \underline{70} & & \\
 & & & & & 4 & & \\
 & & & & & 4 & & \underline{84} \\
 & & & \underline{29} & & 29 & & 1 \\
 & & & 1 & & 17 & & 7 \\
 & & & 2 & & 1 & & 7 \\
 & & & 26 & & 15 & & 11 \\
\alpha_2 = & \cdot 029 & \beta_2 = & -\cdot 004 & \gamma_2 = & -\cdot 058 & \eta_2 = & \cdot 058 \\
 & -\ 5 & & -\ \underline{4} & & & & -\ 19 \\
 & -\ 15 & & -\ 8 & & & & -\ 1 \\
 & -\ \underline{1} & & & & & & -\ \underline{58} \\
 & -\ 21 & & & & & & -\ 78
\end{array}
$$

that is,

$$\alpha_2 = \cdot 008 = 28', \quad \beta_2 = \cdot 021 = 1^\circ 12', \quad \gamma_2 = \cdot 012 = 41', \quad \eta_2 = \cdot 006 = 21'.$$

To find a third approximation, the values of the functions on the left-hand sides of the equations for the values $\alpha_2, \beta_2, \gamma_2, \eta_2$ of the variables are found, this time to five places of decimals; the calculation requires only addition and subtraction of numbers read from tables, and the values found are ·00439, ·00663, ·00516, —·00170. By substitution of these values for t, u, v, w in the formulæ already used, the third terms in the approximation are found to be

$$\alpha_3 = -\cdot 00094 = -3', \qquad \beta_3 = -\cdot 00127 = -4',$$

$$\gamma_3 = \cdot 00090 = 3', \qquad \eta_3 = -\cdot 00087 = -3',$$

and these are, in fact, true to a single minute. For continuing the approximation, the linear equations to which the functional equations are, to seven places of decimals, now equivalent, are written down, the variables being expressed in seconds and denoted by a, b, c, e, and the coefficients being found from the difference columns in the tables. For example, the term $\cos(36^\circ - \alpha - \beta + \gamma)$ in the first function is replaced by

$$\cos\{36^\circ - (\alpha_2 + \beta_2 - \gamma_2) - (\alpha_3 + \beta_3 - \gamma_3) - (a + b - c)\},$$

---

[*] This arrangement of the figures, though not elegant in appearance, is the most convenient in practice: the terms are calculated in order, and each one is placed above or below those already written according as it is positive or negative. Separate addition of the positive and negative components is desirable on account of the uncertainty of the sign which is to prevail. After this we give only the results at the various stages, but nothing is actually omitted except such columns as these.

that is, by $\cos\{35^\circ 11' - (a+b-c)\}$, and this in turn by

$$\{8173125 + (1676/60)(a+b-c)\} \times 10^{-7}.$$

The equations so found, multiplied by 60×10^7, are

$$\begin{aligned}
9137a + 4405b + 1053c \phantom{{}-3085e} &= -121500,\\
4203a + 13879b \phantom{{}+7252c} - 1459e &= -155580,\\
18311a + 6168b + 7252c - 3085e &= -182040,\\
4610a - 5597b + 9683c + 6666e &= -41460,
\end{aligned}$$

equations in which the coefficients of the variables, divided by 2909, that is, by 10^7 times the number of radians in a minute of arc, necessarily differ but little from the rational coefficients in the earlier equations connecting t, u, v, w with α, β, γ, η. Four equations such as

$$9137a + 4405b + 1053c = -k,$$

connecting four variables k, l, m, n with the four variables a, b, c, e are therefore approximately equivalent to four equations of which the first is

$$2909a = -(\tfrac{1}{2} - \tfrac{1}{12})\,k + \tfrac{1}{7}l + \tfrac{1}{30}n.$$

First approximations to a, b, c, e, or fourth approximations to α, β, γ, η, are immediately found to be $-9''$, $-10''$, $1''$, $-10''$; these cannot be trusted to a second, but to proceed one stage further it is necessary only to find four numbers k_2, l_2, m_2, n_2 by substituting these last terms in such equations as

$$k_2 = 9137a_1 + 4405b_1 + 1053c_1 + 121500,$$

and then to find the final terms in the approximation from such equations as

$$2909a_2 = -(\tfrac{1}{2} - \tfrac{1}{12})\,k_2 + \tfrac{1}{7}l_2 + \tfrac{1}{30}n_2.$$

It is found that, to two significant figures,

$$k_2 = -3700, \quad l_2 = -6400, \quad m_2 = -6300, \quad n_2 = -1000,$$

and that b_2 is slightly less than $\frac{1}{2}''$, while a_2, c_2, e_2 are about $\frac{1}{5}''$, $\frac{1}{9}''$, $\frac{1}{4}''$. The conclusion is that the required solution of the given equations, correct to the nearest second, is

$$\theta = 35^\circ 35' 9'', \quad \phi = 33^\circ 44' 19'', \quad \psi = 72^\circ 44' 1'', \quad \chi = 65^\circ 42' 10''.$$

The value of $\theta + \frac{1}{2}\phi$, the angle whose cosine is the ratio of the radii of the circles, is $52^\circ 27' 18\frac{1}{3}''$, accurate in point of fact to at least $\frac{1}{20}''$, and the cosine is ·6094183, with an error less than 2 in the last place: we may say confidently that covering is possible if the ratio exceeds ·6094185, impossible if the ratio is less than ·6094180. The ratio of $a-c$ to b is $\sin 1^\circ 33' 43\tfrac{1}{3}''/\sin 72^\circ 44' 1''$, that is, ·028545.

The smallness of $(a-c)/b$ suggests a supplementary question, that of the least value of a/b for which covering is possible while three of the small circles actually pass through the centre of the large circle. To solve this, we have only to take instead of the equation expressing that $2\theta+\phi$ is a maximum, the geometrical condition that C and K coincide, a condition expressed by the equations

$$\chi = \psi = 2\phi = 2\theta.$$

The angle θ must satisfy the equation

$$2\cos\theta\cos\tfrac{3}{2}\theta = 1,$$

that is,

$$1-\cos\tfrac{5}{2}\theta-\cos\tfrac{1}{2}\theta = 0,$$

so that $\tfrac{1}{2}\theta$ satisfies the equation

$$F(\xi) \equiv 1-\cos 5\xi-\cos\xi = 0,$$

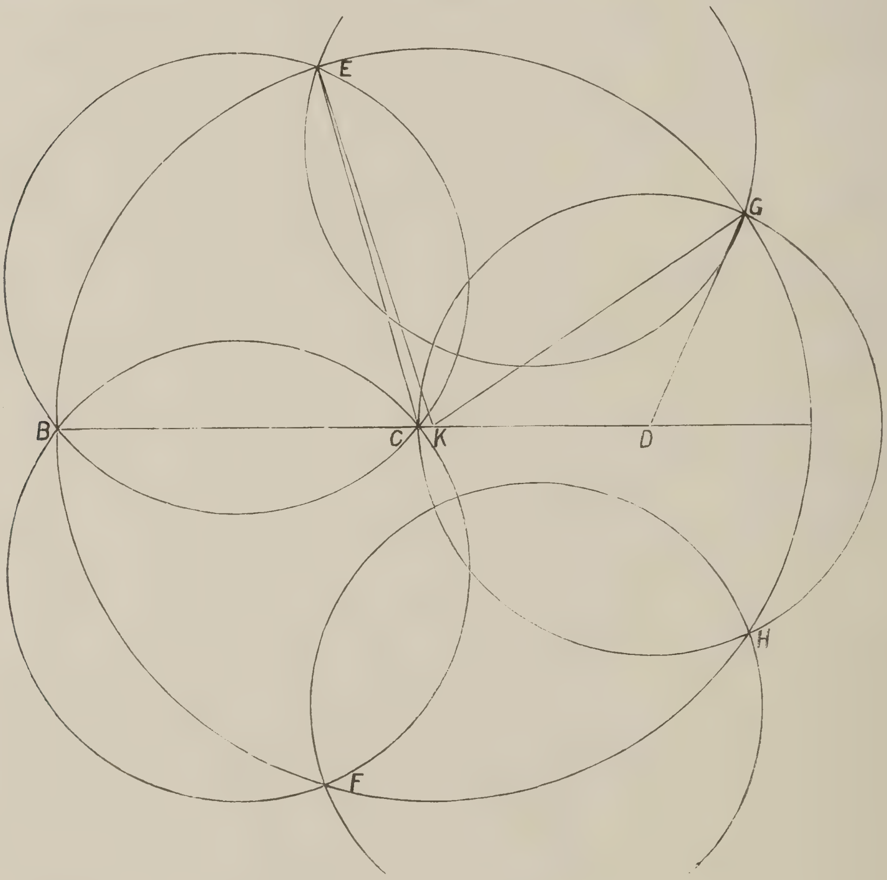
a first approximation to the required root of this equation being 18°. A good approximation to $1/(5\sin 90^\circ+\sin 18^\circ)$ is $(\tfrac{1}{5}-\tfrac{1}{90})$, and the root is the sum of a series $\xi_1+\xi_2+\ldots$, whose terms are found in succession from the formulæ

$$\xi_1 = 18^\circ, \quad \xi_2 = -(\tfrac{1}{5}-\tfrac{1}{90})F(\xi_1), \quad \xi_3 = -(\tfrac{1}{5}-\tfrac{1}{90})F(\xi_1+\xi_2), \quad \ldots.$$

It is found that in fact $F(\xi_1+\xi_2+\xi_3)$ vanishes to seven places of decimals, $\xi_1+\xi_2+\xi_3$ being equal to $17^\circ 28' 16\tfrac{1}{4}''$, so that the ratio of the radii, being the cosine of $\tfrac{3}{2}\theta$, that is, of $52^\circ 24' 48\tfrac{3}{4}''$, is ·6099579.

Perhaps the most curious feature of the whole problem is the nearness of this ratio, on the one hand to the smallest ratio permitting covering, and on the other hand to the smallest ratio which allows the five small circles all to pass through the centre of the large circle, this last ratio being $\tfrac{1}{2}\sec 36^\circ$, that is, ·6180340. The difference between the smallest ratio allowing covering and the last ratio found is just large enough to take effect in practice, but the lack of precision in the painted circle and the thickness of the discs prevent the accuracy which would be necessary if a distinction was to be made between the first two arrangements discussed, and if my readers can perceive the centre of the large circle they may proceed to pocket as many of the showman's rewards as they feel themselves to have earned by reading these pages.

[My brother, Mr. B. M. Neville, to whom I am indebted for the drawing to scale which accompanies this paper, finds that unless the large circle has a diameter of about a metre the possibility of the best construction, when the construction through the centre is inadequate, cannot be made evident. To him is due the discovery of a most convenient and accessible form of covering disc—the pieces of parchment sold as jam covers: provided with five of these, one has only to draw on a sheet of paper a large circle of appropriate radius; the transparency of the small circles is a great advantage.]



ON THE STEADY ROTATION OF A SOLID OF REVOLUTION IN A VISCOUS FLUID

By G. B. JEFFERY.

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THE only possible motion of a viscous fluid which is symmetrical about an axis, and for which the stream lines are circles having their centres on the axis and their planes perpendicular to the axis, is the motion generated by the rotation of two infinite concentric circular cylinders about their common axis.* If, however, the motion is slow so that the squares and products of the velocity components may be neglected, there are other possible solutions. Let ϖ, ϕ, z be cylindrical coordinates, and u, v, w the corresponding components of velocity, then neglecting the squares and products of u, v, w, the equations of steady motion in the symmetrical case are

$$\frac{1}{\nu}\frac{\partial\chi}{\partial\varpi}+\nabla^2 u-\frac{u}{\varpi^2}=0,$$

$$\nabla^2 v-\frac{v}{\varpi^2}=0,$$

$$\frac{1}{\nu}\frac{\partial\chi}{\partial z}+\nabla^2 w=0,$$

where ν is the kinematic viscosity, and $\chi=-V-p/\rho$, where V is the potential of the external forces, p the mean pressure, and ρ the density. These equations are satisfied by $\chi=\text{const.}$, $u=w=0$, while v is a function of ϖ, z only and

$$\nabla^2 v-\frac{v}{\varpi^2}=0. \tag{1}$$

Writing $v=\psi/\sin\phi$,

this equation becomes $\nabla^2\psi=0$.

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* "The Equations of Motion of a Viscous Fluid," *Phil. Mag.*, April 1915, p. 445.

Hence if we have any solution of Laplace's equation of the form $f(\varpi, z)\sin\phi$, a solution of the present problem is given by

$$v = f(\varpi, z).$$

If the motion is generated by the rotation of a solid of revolution about the axis of z with angular velocity ω, we have to make v vanish for large values of ϖ while, on the hypothesis of no slipping at the solid-fluid surface, we have $v = \varpi\omega$ over the surface of the solid.

If α, β, γ are a system of orthogonal curvilinear coordinates, and u', v', w' the corresponding components of velocity, and if elements of arc measured along the normals to the surfaces $\alpha, \beta, \gamma = \text{const.}$, are $\delta\alpha/h_1$, $\delta\beta/h_2$, $\delta\gamma/h_3$ respectively, we have with the usual notation for stress components*

$$\widehat{\alpha\gamma} = \mu\left\{\frac{h_3}{h_1}\frac{\partial}{\partial\gamma}(h_1 u') + \frac{h_1}{h_3}\frac{\partial}{\partial\alpha}(h_3 w')\right\},$$

where μ is the coefficient of viscosity.

Let α, β be conjugate functions of ϖ, z, while $\gamma = \phi$, then

$$h_1 = h_2 = \left\{\left(\frac{\partial\alpha}{\partial\varpi}\right)^2 + \left(\frac{\partial\beta}{\partial\varpi}\right)^2\right\}^{\frac{1}{2}}, \qquad h_3 = \frac{1}{\varpi}.$$

If the solid is defined by one of the surfaces $\alpha = \text{const.}$, we have for the tangential stress on its surface in the direction perpendicular to the axis

$$\mu h_1 \varpi \frac{\partial}{\partial\alpha}\left(\frac{v}{\varpi}\right),$$

or the total couple exerted by the fluid on the solid is

$$G = 2\pi\mu\int \varpi^3 \frac{\partial}{\partial\alpha}\left(\frac{v}{\varpi}\right) d\beta, \tag{2}$$

the value of the integrand being taken on the surface of the solid and the integration extending round the contour of the solid.

If n, s are measured along the normal and the arc of the contour respectively

$$G = 2\pi\mu\int \varpi^3 \frac{\partial}{\partial n}\left(\frac{v}{\varpi}\right) ds. \tag{3}$$

We have a solution of Laplace's equation in spherical polar coordinates

$$\psi = \sin\phi \sum_{n=1}^{n=\infty}\left\{A_n r^n + B_n r^{-n-1}\right\} P_n^1(\cos\theta),$$

where $P_n^1(\cos\theta)$ is the associated Legendre function of the first kind, of degree n and of the first order. Hence we have a solution of the present problem

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* Lamé, *Coordonnées Curvilignes*, p. 284.

$$v = \sum_{n=1}^{\infty} \{A_n r^n + B_n r^{-n-1}\} P_n^1(\cos\theta).$$

As a particular case take $n = 1$. We have

$$v = \{Ar + B/r^2\} \sin\theta,$$

which is the well known solution for the motion of a fluid contained between two concentric spheres which are constrained to rotate about the same diameter. In the case of a single sphere of radius a rotating with angular velocity ω in an infinite fluid, we have

$$v = \frac{a^3\omega}{r^2}\sin\theta, \qquad G = 8\pi\mu\omega a^3. \tag{4}$$

The general case of the rotation of an ellipsoid of any shape about a principal axis has been solved in terms of ellipsoidal harmonics by Mr. Edwardes (*Quarterly Journal of Mathematics*, Vol. XXVI, p. 70). The simpler case of an ellipsoid of revolution can, however, be very readily solved by the method of this paper.

Rotation of an Ovary Ellipsoid.

Take coordinates defined by

$$z + i\varpi = c\cosh(\xi + i\eta),$$

so that $$z = c\cosh\xi\cos\eta, \qquad \varpi = c\sinh\xi\sin\eta.$$

The surfaces $\xi =$ const. are a series of confocal ovary ellipsoids, while the surfaces $\eta =$ const. are a set of confocal hyperboloids of revolution, the foci in each case being $z = \pm c$, $\varpi = 0$. Well known solutions of Laplace's equation are

$$P_n^m(\cos\eta)\, P_n^m(\cosh\xi)\sin m\phi,$$

$$P_n^m(\cos\eta)\, Q_n^m(\cosh\xi)\sin m\phi,$$

and hence we have solutions of the problem in hand

$$v = P_n^1(\cos\eta)\, P_n^1(\cosh\xi),$$

and $$v = P_n^1(\cos\eta)\, Q_n^1(\cosh\xi),$$

the former is finite when $\cosh\xi = 1$, *i.e.*, on the line joining the foci, but it becomes infinite with ξ. The second solution, on the other hand, becomes infinite on the line joining the foci, but vanishes at infinity. It is therefore applicable to the case of an ellipsoid in an infinite extent of fluid. We will, however, take the more general case of the motion of a fluid between two confocal ovary ellipsoids rotating with different angular velocities. For this purpose both solutions are required. Put $n = 1$ in the above solutions and assume

$$v = \sin\eta\,\{AP_1^1(\cosh\xi)+BQ_1^1(\cosh\xi)\}.$$

If the two surfaces are defined by $\xi = \xi_0, \xi_1$, and if their angular velocities are ω_0, ω_1 respectively, the boundary conditions give

$$AP_1^1(\cosh\xi_0)+BQ_1^1(\cosh\xi_0) = c\omega_0\sinh\xi_0,$$

$$AP_1^1(\cosh\xi_1)+BQ_1^1(\cosh\xi_1) = c\omega_1\sinh\xi_1.$$

We may conveniently write

$$f(\xi) = \log\coth\frac{\xi}{2} - \frac{\cosh\xi}{\sinh^2\xi}.$$

Then $\quad P_1^1(\cosh\xi) = \sinh\xi \quad$ and $\quad Q_1^1(\cosh\xi) = \sinh\xi\, f(\xi).$

Solving for A, B, we have

$$v = c\sinh\xi\sin\eta\left\{\omega_0\frac{f(\xi)-f(\xi_1)}{f(\xi_0)-f(\xi_1)}+\omega_1\frac{f(\xi_0)-f(\xi)}{f(\xi_0)-f(\xi_1)}\right\}.$$

To calculate the couple on either ellipsoid necessary to maintain the rotations we may use the formula (2), and we have at once

$$G = \tfrac{16}{3}\pi\mu c^3\frac{\omega_0-\omega_1}{f(\xi_0)-f(\xi_1)}.$$

In the case of a single ellipsoid ($\xi = \xi_0$) in an infinite extent of fluid, we have, putting $\xi_1 = \infty$, $\omega_1 = 0$,

$$v = \omega_0 c\sinh\xi\sin\eta\frac{f(\xi)}{f(\xi_0)},$$

and

$$G = \tfrac{16}{3}\pi\mu\omega_0 c^3/f(\xi_0).$$

If a, b are the polar and equatorial radii respectively,

$$a = c\cosh\xi_0, \qquad b = c\sinh\xi_0,$$

and we have

$$G = \tfrac{16}{3}\pi\mu\omega_0 c^3\Big/\left\{\tfrac{1}{2}\log\left(\frac{a+c}{a-c}\right)-\frac{ac}{b^2}\right\}, \tag{5}$$

where

$$c^2 = a^2-b^2.$$

When $b \to a$, $c \to 0$, and

$$G = 8\pi\mu\omega_0 a^3,$$

which is the value already given for a sphere.

Rotation of a Planetary Ellipsoid.

In the case of an oblate or planetary ellipsoid we take coordinates defined by

$$z + i\varpi = c\sinh(\xi + i\eta),$$

so that

$$z = c\sinh\xi\cos\eta, \qquad \varpi = c\cosh\xi\sin\eta.$$

The surfaces $\xi = \text{const.}$ and $\eta = \text{const.}$ are ellipsoids and hyperboloids of revolution having a common focal circle, $z = 0$, $\varpi = c$. We have as solutions of Laplace's equation*

$$P_n^m(\cos\eta)\, p_n^m(\sinh\xi)\sin m\phi, \qquad P_n^m(\cos\eta)\, q_n^m(\sinh\xi)\sin m\phi,$$

where

$$p_n^m(\zeta) = (\zeta^2+1)^{\frac{1}{2}m}\frac{d^m}{d\zeta^m}p_n(\zeta), \qquad q_n^m(\zeta) = (\zeta^2+1)^{\frac{1}{2}m}\frac{d^m}{d\zeta^m}q_n(\zeta),$$

the functions p_n, q_n being connected with the ordinary Legendre functions by the relations

$$p_n(\zeta) = i^{-n}P_n(i\zeta), \qquad q_n(\zeta) = i^{n+1}Q_n(i\zeta).$$

Hence we have solutions

$$v = P_n^1(\cos\eta)\, p_n^1(\sinh\xi),$$

$$v = P_n^1(\cos\eta)\, q_n^1(\sinh\xi).$$

As in the previous case the former becomes infinite, while the latter vanishes at infinity.

Let the two ellipsoids ξ_0, ξ_1 have angular velocities ω_0, ω_1. Assume

$$v = \{A p_1^1(\sinh\xi) + B q_1^1(\sinh\xi)\}\sin\eta.$$

The boundary conditions give

$$A p_1^1(\sinh\xi_0) + B q_1^1(\sinh\xi_0) = c\omega_0\cosh\xi_0,$$

$$A p_1^1(\sinh\xi_1) + B q_1^1(\sinh\xi_1) = c\omega_1\cosh\xi_1.$$

Now

$$p_1^1(\sinh\xi) = \cosh\xi,$$

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* Lamb, *Hydrodynamics*, p. 136.

and writing
$$F(\xi) = \frac{\sinh\xi}{\cosh^2\xi} - \cot^{-1}\sinh\xi,$$
we have
$$q_1^1(\sinh\xi) = \cosh\xi\, F(\xi).$$
Solving for A, B,
$$v = c\cosh\xi\sin\eta\left\{\omega_0\frac{F(\xi_1)-F(\xi)}{F(\xi_1)-F(\xi_0)} + \omega_1\frac{F(\xi)-F(\xi_0)}{F(\xi_1)-F(\xi_0)}\right\}.$$

Using equation (2) to evaluate the couple which must be applied to either ellipsoid to maintain the rotation, we have without difficulty
$$G = \tfrac{16}{3}\pi\mu c^3\frac{\omega_1-\omega_0}{F(\xi_1)-F(\xi_0)}.$$

If the fluid is not bounded externally, so that we have a single ellipsoid in an infinite extent of fluid, we may put $\omega_1 = 0$ and $\xi_1 = \infty$. In this case
$$v = c\cosh\xi\sin\eta\,\omega_0\,F(\xi)/F(\xi_0),$$
and
$$G = \tfrac{16}{3}\pi\mu c^3\omega_0\,F(\xi_0).$$
If a, b are the polar and equatorial radii respectively
$$a = c\sinh\xi_0, \quad b = c\cosh\xi_0,$$
and we have
$$G = \tfrac{16}{3}\pi\mu\omega_0 c^3\Big/\left\{\frac{ac}{b^2} - \cot^{-1}\frac{a}{c}\right\},$$
where
$$b^2-a^2 = c^2.$$

We may note that this is the form taken by the result (5) for an ovary ellipsoid when b is greater than a for
$$\tfrac{1}{2}\log\frac{a+i\sqrt{(b^2-a^2)}}{a-i\sqrt{(b^2-a^2)}} = i\cot^{-1}\frac{a}{\sqrt{(b^2-a^2)}}.$$

The Rotation of a Circular Disc in an Infinite Fluid.

The solution of this problem may be obtained by putting $\xi_0 = 0$ in the solution for a planetary ellipsoid. Hence
$$v = -\frac{2\omega_0}{\pi}c\cosh\xi\sin\eta\,F(\xi), \tag{6}$$
and the couple necessary to maintain the rotation is
$$G = \tfrac{32}{3}\mu\omega_0 c^3,$$
where c is the radius of the disc.

From (6) the normal gradient of the velocity over the surface of the disc can be calculated,

$$\left.\frac{\partial v}{\partial z}\right)_{\xi=0} = -\frac{4}{\pi}\,\omega_0\varpi\,\left.\frac{\partial \xi}{\partial z}\right)_{\xi=0}$$

$$= -\frac{4}{\pi}\,\omega_0\,\frac{\varpi}{\sqrt{(c^2-\varpi^2)}}. \qquad (7)$$

It appears that, while the velocity is everywhere finite, the velocity gradient, and therefore the shearing stress, become infinite at the edge of the disc. It is probable that in the neighbourhood of the sharp edge the condition of no relative motion between the solid and the fluid breaks down. It would be interesting to see whether this infinity disappeared in a more accurate solution which did not neglect the terms involving the squares of the velocity components.

Another solution of this problem may be obtained in terms of Bessel functions, for we have a solution of Laplace's equation

$$e^{-kz}J_1(k\varpi)\sin\phi,$$

and hence a solution of equation (1)

$$v = \int_0^\infty f(k)e^{-kz}J_1(k\varpi)\,dk.$$

In the case of the disc we have

$$v = \omega_0\varpi \quad for \quad z=0,\ \varpi<c, \qquad (8)$$

$$\frac{\partial v}{\partial z} = 0 \quad for \quad z=0,\ \varpi>c, \qquad (9)$$

and hence $f(k)$ is determined by

$$\int_0^\infty f(k)\,J_1(k\varpi)\,dk = \omega_0\varpi \quad (\varpi<c),$$

$$\int_0^\infty kf(k)J_1(k\varpi)\,dk = 0 \quad (\varpi>c).$$

It has not been found possible to solve these integral equations directly, but if we make use of (7), together with (9), and the theorem* that if

$$-\frac{\partial v}{\partial z} = F(\varpi) \quad for \quad z=0,$$

then $$v = \int_0^\infty e^{-kz}J_1(k\varpi)\int_0^\infty F(\lambda)\,J_1(k\lambda)\,\lambda\,d\lambda\,dk,$$

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* Gray and Mathews, *Bessel Functions*, p. 80. The infinity of F at $\varpi=c$ does not invalidate this theorem.

we obtain without difficulty

$$v = \frac{4}{\pi}\,\omega_0 \int_0^\infty e^{-kz}(\sin ck - ck\cos ck)\, J_1(k\varpi)\,\frac{dk}{k^2},$$

and this will be found to satisfy both conditions (8) and (9).

The Rotation of Two Non-Concentric Spheres.

This method may be applied to the problem of the motion of a viscous fluid generated by the rotation of any two spheres with different angular velocities about their common diameter. If we take coordinates defined by

$$\xi + i\eta = \log\frac{\varpi + i(z+a)}{\varpi + i(z-a)}, \qquad (10)$$

in any meridian plane the curves $\xi = \text{const.}$, $\eta = \text{const.}$ are the systems of coaxial circles *about* the points $z = \pm a$ and *through* these points respectively. The surfaces $\xi = \text{const.}$ will be a family of coaxial spheres having the common radical plane z, $\xi = 0$, and we can choose the axes of reference and the constant a so that any two given spheres are members of this family. We have a solution of Laplace's equation in these coordinates of which the following is a particular case*

$$\sqrt{(\cosh\xi - \cos\eta)}\,\Sigma\left\{A_n\cosh(n+\tfrac{1}{2})\xi + B_n\sinh(n+\tfrac{1}{2})\xi\right\} P_n^1(\cos\eta)\sin\phi.$$

It follows that a solution of the present problem is given by

$$v = \sqrt{(\cosh\xi - \cos\eta)}\,\Sigma\left\{A_n\cosh(n+\tfrac{1}{2})\,\xi + B_n\sinh(n+\tfrac{1}{2})\,\xi\right\} P_n^1(\cos\eta).$$

Let the two spheres be defined by $\xi = \xi_1$, ξ_2, and let them have angular velocities ω_1, ω_2. Without loss of generality we may take ξ_1 positive, and ξ_2 positive or negative according as the two spheres do or do not lie one within the other. We have to determine A_n, B_n so that v is equal to $\varpi\omega_1$, when $\xi = \xi_1$, and equal to $\varpi\omega_2$, when $\xi = \xi_2$. From (10),

$$\varpi = \frac{a\sin\eta}{\cosh\xi - \cos\eta},$$

and we obtain

$$A_n\sinh(n+\tfrac{1}{2})(\xi_1 - \xi_2)$$
$$= -2\sqrt{2}\,a\left[\omega_1 e^{-(n+\frac{1}{2})\xi_1}\sinh(n+\tfrac{1}{2})\,\xi_2 - \omega_2 e^{\pm(n+\frac{1}{2})\xi_2}\sinh(n+\tfrac{1}{2})\,\xi_1\right],$$
$$B_n\sinh(n+\tfrac{1}{2})(\xi_1 - \xi_2)$$
$$= 2\sqrt{2}\,a\left[\omega_1 e^{-(n+\frac{1}{2})\xi_1}\cosh(n+\tfrac{1}{2})\,\xi_2 - \omega_2 e^{\pm(n+\frac{1}{2})\xi_2}\cosh(n+\tfrac{1}{2})\,\xi_1\right],$$

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* Heine, *Kugelfunctionen*, II, p. 268; see also a paper by the author "On a Form of the Solution of Laplace's Equation," *Proc. Roy. Soc.*, A, 87, 1912, p. 109.

where the upper or lower sign is taken according as ξ_2 is negative or positive.

We will investigate the resultant couple on either sphere

$$\frac{\partial}{\partial\xi}\left(\frac{v}{\varpi}\right) = \frac{3\sinh\xi}{2a\sin\eta}\,v$$

$$+\frac{1}{a}(\cosh\xi-\zeta)\sum_{n=1}^{\infty}(n+\tfrac{1}{2})\{A_n\sinh(n+\tfrac{1}{2})\xi+B_n\cosh(n+\tfrac{1}{2})\xi\}\frac{d}{d\zeta}P_n(\zeta),$$

where $$\zeta = \cos\eta.$$

The couple on the sphere ξ_1 is given by

$$G_1 = 2\pi\mu\int_0^{\pi}\varpi^3\frac{\partial}{\partial\xi}\left(\frac{v}{\varpi}\right)d\eta \quad (\xi=\xi_1)$$

$$= 3\pi\mu a^3\omega_1\sinh\xi_1\int_{-1}^{+1}\frac{(1-\zeta^2)\,d\zeta}{(\cosh\xi_1-\zeta)^4}$$

$$+2\pi\mu a^2\sum_{n=1}^{\infty}(n+\tfrac{1}{2})\{A_n\sinh(n+\tfrac{1}{2})\xi_1+B_n\cosh(n+\tfrac{1}{2})\xi_1\}I_n,$$

where $$I_n = \int_{-1}^{+1}\frac{1-\zeta^2}{(\cosh\xi_1-\zeta)^{\frac{3}{2}}}\frac{dP_n(\zeta)}{d\zeta}d\zeta.$$

Now $$\int_{-1}^{+1}\frac{1-\zeta^2}{(\cosh\xi_1-\zeta)^4}d\zeta = \tfrac{4}{3}\operatorname{cosech}^4\xi_1,$$

and it remains to evaluate I_n. Now

$$\frac{1}{(\cosh\xi_1-\zeta)^{\frac{1}{2}}} = \sqrt{2}\sum_{m=0}^{m=\infty}e^{-(m+\frac{1}{2})\xi_1}P_m(\zeta).$$

Differentiating with respect to ζ,

$$\frac{1}{(\cosh\xi_1-\zeta)^{\frac{3}{2}}} = 2\sqrt{2}\sum_{m=1}^{m=\infty}e^{-(m+\frac{1}{2})\xi_1}\frac{dP_m(\zeta)}{d\zeta},$$

and so $$I_n = 2\sqrt{2}\sum_{m=1}^{\infty}e^{-(m+\frac{1}{2})\xi_1}\int_{-1}^{+1}P_m^1(\zeta)\,P_n^1(\zeta)\,d\zeta$$

$$= 2\sqrt{2}\,e^{-(n+\frac{1}{2})\xi_1}\frac{2}{2n+1}\frac{(n+1)!}{(n-1)!}$$

$$= 4\sqrt{2}\,\frac{n(n+1)}{2n+1}e^{-(n+\frac{1}{2})\xi_1}.$$

Hence

$$G_1 = 4\pi\mu a^3\omega_1 \operatorname{cosech}^3 \xi_1$$
$$+4\sqrt{2}\,\pi\mu a^2 \sum_{n=1}^{\infty} n(n+1)\left\{A_n \sinh\left(n+\tfrac{1}{2}\right)\xi_1 + B_n \cosh\left(n+\tfrac{1}{2}\right)\xi_1\right\} e^{-(n+\frac{1}{2})\xi_1},$$

or, inserting the values of the constants,

$$G_1 = 4\pi\mu a^3\omega_1 \operatorname{cosech}^3 \xi_1$$
$$+16\pi\mu a^3 \sum_{n=1}^{\infty} n(n+1)\left\{\omega_1 e^{-(2n+1)\xi_1} \coth\left(n+\tfrac{1}{2}\right)(\xi_1-\xi_2)\right.$$
$$\left. -\omega_2 e^{-(n+\frac{1}{2})(\xi_1\mp\xi_2)} \operatorname{cosech}\left(n+\tfrac{1}{2}\right)(\xi_1-\xi_2)\right\}.$$

The two series multiplying ω_1 and ω_2 are convergent, whatever ξ_2 may be, provided that ξ_1 is positive; they may therefore be summed separately.

Now, when $\xi > 0$, we have

$$\operatorname{cosech}^3 \xi = 8e^{-3\xi}/(1-e^{-2\xi})^3 = 4 \sum_{n=1}^{\infty} n(n+1)\, e^{-(2n+1)\xi},$$

and so the terms containing ω_1 in G_1 can be written

$$8\pi\mu a^3\omega_1 \operatorname{cosech}^3 \xi_1 + 16\pi\mu a^3\omega_1 \sum_{n=1}^{\infty} n(n+1)\, e^{-(2n+1)\xi_1}\left\{\coth\left(n+\tfrac{1}{2}\right)(\xi_1-\xi_2)-1\right\}.$$

Also $$\coth\left(n+\tfrac{1}{2}\right)(\xi_1-\xi_2)-1 = 2 \sum_{n=1}^{\infty} e^{-(2n+1)m(\xi_1-\xi_2)},$$

and so $$\sum_{n=1}^{\infty} n(n+1)\, e^{-(2n+1)\xi_1}\left\{\coth\left(n+\tfrac{1}{2}\right)(\xi_1-\xi_2)-1\right\}$$
$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 2n(n+1)\, e^{-(2n+1)\{(m+1)\xi_1-m\xi_2\}}.$$

The terms in this double series are all positive, and the series is convergent when summed in the present way; we may therefore interchange the order of summation, and we have

$$\tfrac{1}{2} \sum_{m=1}^{\infty} \operatorname{cosech}^3 \{(m+1)\xi_1 - m\xi_2\}.$$

We now see that the terms containing ω_1 in G_1 can be brought into the single sum

$$8\pi\mu a^3\omega_1 \sum_{m=0}^{\infty} \operatorname{cosech}^3 \{(m+1)\xi_1 - m\xi_2\}.$$

The terms containing ω_2 can be transformed similarly, using the identities

$$\sum_{n=1}^{\infty} n(n+1)\, e^{-(n+\frac{1}{2})(\xi_1\mp\xi_2)} \operatorname{cosech}\left(n+\tfrac{1}{2}\right)(\xi_1-\xi_2)$$
$$= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} 2n(n+1)\, e^{-(2n+1)[(m+1)\xi_1-(m\pm\frac{1}{2}+\frac{1}{2})\xi_2]}$$
$$= \tfrac{1}{2} \sum_{m=0}^{\infty} \operatorname{cosech}^3 \left[(m+1)\xi_1 - \left(m+\tfrac{1}{2}\pm\tfrac{1}{2}\right)\xi_2\right].$$

On combining the results we readily obtain

$$G_1 = 8\pi\mu a^3\Big[\omega_1 \sum_{m=0}^{\infty} \operatorname{cosech}^3[(m+1)\xi_1 - m\xi_2] \\ - \omega_2 \sum_{m=0}^{\infty} \operatorname{cosech}^3[(m+1)\xi_1 - (m+\tfrac{1}{2}\pm\tfrac{1}{2})\xi_2]\Big].$$

If one sphere encloses the other, ξ_2 is positive, and the lower sign must be taken, and

$$G_1 = 8\pi\mu a^3(\omega_1-\omega_2) \sum_{m=0}^{\infty} \operatorname{cosech}^3[(m+1)\xi_1 - m\xi_2]. \tag{11}$$

If, on the other hand, ξ_2 is negative, the spheres are separate and the upper sign must be taken. In this case

$$G_1 = 8\pi\mu a^3\Big[\omega_1 \sum_{m=0}^{\infty} \operatorname{cosech}^3[(m+1)\xi_1 - m\xi_2] \\ - \omega_2 \sum_{m=0}^{\infty} \operatorname{cosech}^3 (m+1)(\xi_1-\xi_2)\Big]. \tag{12}$$

Suppose that the sphere ξ_2 is constrained to rotate with a given spin ω_2. There will be a certain value of ω_1 for which G_1 vanishes. This is the steady angular velocity with which the sphere ξ_1 would rotate if allowed to move freely. From (11) we see that in the case when one sphere encloses the other this gives $\omega_1 = \omega_2$. In this case, then, if one sphere be allowed to move freely it will acquire the same angular velocity as the other sphere, and the fluid will move as a rigid body. In the case of two non-enclosing spheres, if ξ_2 is constrained to rotate with angular velocity ω_2, while ξ_1 is left free to move under the fluid stresses, it will rotate with angular velocity

$$\omega_1 = \omega_2 \frac{\sum_{m=0}^{\infty} \operatorname{cosech}^3(m+1)(\xi_1-\xi_2)}{\sum_{m=0}^{\infty} \operatorname{cosech}^3[(m+1)\xi_1 - m\xi_2]}.$$

In the case of two equal spheres $\xi_2 = -\xi_1$, and from (10) we obtain

$$\frac{\text{diameter of either sphere}}{\text{distance apart of centres}} = \operatorname{sech}\xi_1.$$

The character of this influence of one sphere upon the other is exhibited in the following table:—

Two Equal Spheres—One free.

ξ_1	$\frac{\text{Diameter}}{\text{Centre Distance}}$	$\frac{\omega_1}{\omega_2}$
·2	·9803	·1278
·4	·9250	·1023
·6	·8435	·0759
·8	·7477	·0524
1·0	·6481	·0340
1·2	·5523	·0211
1·4	·4649	·0126
1·6	·3880	·0073
1·8	·3218	·0042
2·0	·2658	·0023
2·5	·1631	·0005
3·0	·0993	·0001

It appears that even when the distance between the spheres is as small as one fiftieth part of the distance between their centres, one sphere communicates only one eighth of its spin to the other sphere.

If we put $\xi_2 = 0$ and $\omega_2 = 0$, we obtain the solution for the rotation of a sphere in a viscous fluid in the presence of a fixed infinite plane perpendicular to the axis of rotation. The plane will cause the sphere to experience an increased resistance to its rotation, and we may compare the resisting couple with its value in the absence of the plane.

The Increase to the Resistance to the Rotation of a Sphere owing to the Presence of an Infinite Plane.

ξ_1	$\frac{\text{Radius of Sphere}}{\text{Distance of Centre from Plane}}$	Ratio of Increase of Couple due to Plane.
·2	·9803	1·171
·4	·9250	1·126
·6	·8435	1·087
·8	·7477	1·057
1·0	·6481	1·036
1·2	·5523	1·022
1·4	·4649	1·013
1·6	·3880	1·007
1·8	·3218	1·004
2·0	·2658	1·002
2·5	·1631	1·0005
3·0	·0993	1·0001

Here again the effect is surprisingly small. If the fixed plane is brought so close to the rotating sphere that their distance apart is but one-fiftieth of the radius of the latter, the couple required to maintain the rotation is only increased by 17 per cent. These results, it will be noted, are independent of the degree of viscosity of the fluid.

A PSEUDO-SPHERE WHOSE EQUATION IS EXPRESSIBLE IN TERMS OF ELLIPTIC FUNCTIONS

By J. R. Wilton.

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As the general form of the pseudo-sphere has not yet been obtained there is some interest in the determination of simple particular cases, such as that of Serret's surface, in which the two radii of curvature are equal, the common value being $\iota\mu$. A pseudo-sphere whose equation in its most general form contains eight arbitrary constants, of which six are due merely to change of position and orientation in space, may be obtained very simply, as follows.

If we attempt to find the developable surfaces which satisfy the equation

$$s = f(z,\, p,\, q), \tag{1}$$

we shall have to determine the functions ϕ and ψ in

$$z = \lambda x + \phi(\lambda) y + \psi(\lambda),$$

$$0 = x + \phi' y + \psi',$$

so that (1) is satisfied. On eliminating x this is easily seen to lead to the relation

$$\phi' = (\phi'' y + \psi'') f\,\{(\phi - \lambda\phi')y + \psi - \lambda\psi',\, \phi,\, \psi\},$$

which, for general forms of f, can only be satisfied by taking $\phi = \lambda\phi'$, and therefore $\phi'' = 0$.

Thus, putting $\phi = b\lambda/a$, we obtain a particular solution of (1) in the form

$$z = \psi\,(ax + by),$$

where

$$ab\psi'' = f(\psi,\, a\psi',\, b\psi').$$

In the particular case of the equation $s = \mu^{-2} \sin z$, which, in conformity with the notation of Forsyth's *Differential Geometry*, § 54, we shall take in the form

$$\frac{\partial^2 \omega}{\partial p\, \partial q} = \frac{1}{\mu^2} \sin \omega,$$

the solution of this type may be expressed (i) in the form

$$\left.\begin{aligned} \mu k(-ab)^{\frac{1}{2}}\,\phi &= ap+bq+c, \\ \sin\tfrac{1}{2}\omega &= \operatorname{sn}\phi \quad (\operatorname{mod} k), \end{aligned}\right\}$$

the arbitrary constants being a, b, c and k; (ii) in the rather more convenient Weierstrassian form

$$\left.\begin{aligned} u &= ap+bq+c, \\ \wp(u) &= e_2\cos^2\tfrac{1}{2}\omega+e_3\sin^2\tfrac{1}{2}\omega, \end{aligned}\right\} \tag{2}$$

with the relation $$\mu^2 ab\,(e_2-e_3) = 1. \tag{3}$$

For the sake of definiteness we may assume that the roots of $\wp'$ are real and that $e_1 > e_2 > e_3$, so that

$$\wp' = (e_2-e_3)(e_1-\wp)^{\frac{1}{2}}\sin z.$$

We proceed to the determination of the Cartesian form of the equation of the pseudo-sphere defined by equations (2) and (3).

With the notation of Forsyth, *loc. cit.*, § 37, the equations for X and x are

$$\left.\begin{aligned} 2\mu\,(e_2-\wp)^{\frac{1}{2}}(\wp-e_3)^{\frac{1}{2}}X_1-(e_1+2\wp)\,x_1+(e_2-e_3)\,x_2 &= 0, \\ x_{11}+2a\mu\,(e_1-\wp)^{\frac{1}{2}}X_1 &= 0, \\ \mu\,(e_2-e_3)\,x_{12} &= 2(e_2-\wp)^{\frac{1}{2}}(\wp-e_3)^{\frac{1}{2}}X; \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} 2\mu\,(e_2-\wp)^{\frac{1}{2}}(\wp-e_3)^{\frac{1}{2}}X_2+(e_2-e_3)\,x_1-(e_1+2\wp)\,x_2 &= 0, \\ \mu\,(e_2-e_3)\,x_{12} &= 2(e_2-\wp)^{\frac{1}{2}}(\wp-e_3)^{\frac{1}{2}}X, \\ x_{22}+2b\mu\,(e_1-\wp)^{\frac{1}{2}}X_2 &= 0. \end{aligned}\right\} \tag{5}$$

In equations (4) we regard q as constant, the dependent variables being X, x_1 and x_2. Putting

$$X = (e_1-\wp)^{\frac{1}{2}}f'(\wp),$$

so that
$$x_1 = \mu a\,\{A-2(e_1-\wp)\,f'-f\},$$
$$x_2 = \mu bf,$$

where A is an arbitrary function of q, we find

$$\wp'^2 f''+2(e_1^2+e_2e_3+2e_1\wp-\wp^2)f'+\{e_1+2\wp-(e_2-e_3)\,b/a\}\,f = A\,(e_1+2\wp),$$

in which accents denote differentiation with regard to $\wp$.

The particular integral of this equation is easily seen to lead to a solution of (4) and (5) in the form

$$\left.\begin{aligned} \mu X &= 2C\,(e_1-\wp)^{\frac{1}{2}}, \\ x_1 &= aC\,\{e_1+2\wp+(e_2-e_3)\,b/a\}, \\ x_2 &= bC\,\{e_1+2\wp+(e_2-e_3)\,a/b\}, \end{aligned}\right\} \tag{6}$$

where C is an arbitrary constant.

To complete the determination of the pseudo-sphere we require the complementary function of the equation for f, which may be taken in the form

$$\frac{d^2f}{du^2}+\frac{4}{\wp'}\,(e_1-\wp)(e_1+2\wp)\,\frac{df}{du}+\{e_1+2\wp+(e_2-e_3)\,b/a\}\,f=0. \tag{7}$$

It is easy to verify that both integrals of (7) are uniform functions of u over the whole plane, and therefore that we may assume a solution in the form

$$f(u)=\frac{\sigma(u+\alpha)\,\sigma(u+\beta)}{\sigma^2(u)}\,e^{[\lambda-\zeta(\alpha)-\zeta(\beta)]\,u}.$$

Substituting this value we readily find, as, for instance, on pp. 469–70 of Vol. IV of Forsyth's *Theory of Differential Equations*, the relation

$$\begin{aligned} 8\wp+\frac{4\lambda}{\wp'}\,(e_1-\wp)(e_1+2\wp)+\left[\lambda-\nu+\frac{2}{\wp'}\,(e_1-\wp)\,(e_1+2\wp)\right]\frac{\wp_1'-\wp'}{\wp_1-\wp} \\ +\left[\lambda+\nu+\frac{2}{\wp'}\,(e_1-\wp)(e_1+2\wp)\right]\frac{\wp_2'-\wp'}{\wp_2-\wp}+3\,(\wp_1+\wp_2) \\ +\lambda^2+e_1+(e_2-e_3)\,b/a\equiv 0, \end{aligned} \tag{8}$$

where the subscripts 1 and 2 denote the arguments α and β respectively, and

$$\nu=\tfrac{1}{2}\,(\wp_1'+\wp_2')/(\wp_1-\wp_2).$$

The apparent singularities of (8) are $u=0$, ω, ω'; these lead, after reduction, to the equations

$$\begin{gathered} \beta=\alpha+\omega, \\ -\frac{1}{\mu^2b^2}=-(e_2-e_3)\,\frac{a}{b}=\wp_1+\wp_2+e_1=\tfrac{1}{4}\,\frac{\wp_1'^2}{(\wp_1-e_1)^2}, \\ \lambda=-2\,\frac{(\wp_1-\wp_2)^2}{\wp_1'+\wp_2'}=\frac{\wp_1'}{\wp_1-e_1}-\frac{\wp_1''}{\wp_1'}. \end{gathered} \tag{9}$$

Thus, putting $\kappa=\zeta_1+\zeta_2-\lambda$,

we have $f(u)=H\phi+K\psi$,

where

$$\left.\begin{aligned}\phi &= \frac{\sigma(u+a)\,\sigma(u+a+\omega)}{\sigma(a)\,\sigma(a+\omega)\,\sigma^2(u)}\,e^{-\kappa u},\\ \psi &= \frac{\sigma(u-a)\,\sigma(u-a-\omega)}{\sigma(a)\,\sigma(a+\omega)\,\sigma^2(u)}\,e^{\kappa u},\end{aligned}\right\} \tag{10}$$

and H and K are functions of q, which by direct substitution in equations (5) are readily found to be

$$H = A\,e^{-b\lambda q}, \qquad K = B e^{b\lambda q},$$

where A and B are arbitrary constants.

Hence the general solution of (4) and (5) is given by

$$f_1 = A_1 e^{-b\lambda q}\phi + B_1 e^{b\lambda q}\psi + C_1(2\wp - \wp_1 - \wp_2),$$

$$x_1 = -a\left\{\frac{2}{\wp'}(e_1 - \wp)\,\frac{\partial f_1}{\partial u} + f_1 + C_1\lambda^2\right\},$$

$$x_2 = bf_1,$$

where use has been made of equations (6); while y and z are given by precisely similar equations with different constants. Let subscripts 2 give the value of y and subscripts 3 the value of z.

The equations connecting the nine constants $A_1, \ldots, C_3$ are most easily determined from the relation

$$x_2^2 + y_2^2 + z_2^2 = 1,$$

which leads, after rather a long reduction, to

$$A_1,\ A_2,\ A_3 = \frac{\mu k}{\lambda}\operatorname{cosec}\frac{\beta-\gamma}{2}\,(\cos\beta,\ \sin\beta,\ \iota),$$

$$B_1,\ B_2,\ B_3 = \frac{\mu}{\lambda k}\operatorname{cosec}\frac{\beta-\gamma}{2}\,(\cos\gamma,\ \sin\gamma,\ \iota),$$

$$C_1,\ C_2,\ C_3 = \frac{\iota\mu}{\lambda}\operatorname{cosec}\frac{\beta-\gamma}{2}\left(\cos\frac{\beta+\gamma}{2},\ \sin\frac{\beta+\gamma}{2},\ \iota\cos\frac{\beta-\gamma}{2}\right),$$

where k, β and γ are arbitrary.

This is, by Bonnet's theorem, the complete tale of relations satisfied by $A_1, \ldots, C_3$.

From the value of ϕ in (10) we easily deduce

$$\frac{2(e_1-\wp)}{\wp'}\,\frac{d\phi}{du} + \phi = \frac{e_1-\wp_1}{\wp_1'}\;\frac{\wp'(u+a)}{\wp(u+a)-e_1}\,\phi,$$

and the integral of the right-hand side is readily found to be

$$-\frac{\sigma(u+2a+\omega)}{\sigma(2a+\omega)\,\sigma(u)}\,e^{-\kappa u}.$$

Hence, making use of (9), and putting

$$v = bq,$$

$$\left.\begin{aligned} P &= \frac{\sigma(u+2a+\omega)}{\sigma(2a+\omega)\,\sigma(u)}\,e^{-\kappa u-\lambda v}, \\ Q &= \frac{\sigma(u-2a-\omega)}{\sigma(2a+\omega)\,\sigma(u)}\,e^{\kappa u+\lambda v}, \\ -R &= 2\zeta(u)+\left\{\frac{(e_2-e_3)^2}{\wp_1+\wp_2+e_1}-e_1\right\}(u-v)+(\wp_1+\wp_2)\,v, \end{aligned}\right\} \qquad (11)$$

we find the most general form of the pseudo-sphere given by (2), namely,

$$\begin{aligned} \frac{\lambda}{\mu}(x-l)\sin\frac{\beta-\gamma}{2} &= kP\cos\beta+\frac{1}{k}\,Q\cos\gamma+\iota R\cos\frac{\beta+\gamma}{2}, \\ \frac{\lambda}{\mu}(y-m)\sin\frac{\beta-\gamma}{2} &= kP\sin\beta+\frac{1}{k}\,Q\sin\gamma+\iota R\sin\frac{\beta+\gamma}{2}, \\ \frac{\lambda}{\mu}(z-n)\sin\frac{\beta-\gamma}{2} &= \iota kP \qquad +\frac{\iota}{k}\,Q \qquad -R\cos\frac{\beta-\gamma}{2}, \end{aligned}$$

where a, β, γ, k, l, m, n and the periods ω and ω' are arbitrary constants.

The radii of curvature are

$$\mu\left\{\frac{e_2-\wp(u)}{\wp(u)-e_3}\right\}^{\frac{1}{2}}, \qquad -\mu\left\{\frac{\wp(u)-e_3}{e_2-\wp(u)}\right\}^{\frac{1}{2}}.$$

By change of origin and rotation of the axes the pseudo-sphere just obtained can be reduced to the form

$$\begin{aligned} \frac{\lambda}{2\mu}(x-\iota y) &= -P, \\ \frac{\lambda}{2\mu}(x+\iota y) &= \quad Q, \\ \frac{\lambda}{\mu}\,z \quad &= \quad \iota R. \end{aligned}$$

The explicit form of the equation can be obtained by actual elimination of the parameters u and v from equations (11).

Using cylindrical coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

we have

$$\lambda^2 r^2/4\mu^2 = -PQ = \wp(u) - \wp(2a+\omega)$$

$$= \wp(u) - \wp(\beta), \text{ say,} \tag{12}$$

giving u as a function of r; and

$$e^{2\iota\theta} = -\frac{Q}{P} = \frac{\sigma(\beta-u)}{\sigma(\beta+u)}\, e^{2\kappa u + 2\lambda v},$$

giving v as a function of r and θ.

After some straightforward reductions the explicit equation of the pseudo-sphere is found to be

$$\frac{\iota z}{c} = 2\,\frac{\wp(\beta)-e_1}{\wp'(\beta)}\,\{\zeta(u) - [\wp(\beta)+e_1]\,u\} - \zeta(\beta)\,u + \tfrac{1}{2}\log\frac{\sigma(\beta+u)}{\sigma(\beta-u)} + \iota\theta, \tag{13}$$

where

$$c = \mu\left\{\frac{\wp(\beta)-e_2}{e_1-e_2}\,\frac{\wp(\beta)-e_3}{e_1-e_3}\right\}^{\frac{1}{2}},$$

β is an arbitrary constant, and u is given by (12), *i.e.*,

$$r^2 = \mu^2\,\frac{\wp(\beta)-e_1}{(e_1-e_2)(e_1-e_3)}\,\{\wp(u) - \wp(\beta)\}. \tag{14}$$

Dr. Bromwich points out that, as equation (13) is of the form

$$z = f(r) + c\theta,$$

the pseudo-sphere might have been obtained with considerably less analysis—once the form is known—by means of the equation

$$1 + \frac{c^2}{r^2} + \left(\frac{df}{dr}\right)^2 = \left(\frac{r^2}{\mu^2} + A\right)^{-1}, \tag{15}$$

in which c and A are arbitrary constants.

Substituting the actual value of z given by (13), we readily verify that equation (15) is satisfied if

$$A = \frac{\{\wp(\beta)-e_1\}^2}{(e_1-e_2)(e_1-e_3)}.$$

Thus equation (13) must be regarded as containing only two independent arbitrary constants.

Owing to the occurrence of imaginary quantities in equation (13) it is perhaps of interest to note that the surface may be real. A particular case, found by putting $e_1 = e_2$, is that of the pseudo-sphere

$$z = \sqrt{(a^2-r^2)} - a\log[\{a+\sqrt{(a^2-r^2)}\}/r] + c\theta,$$

in which $$\mu^2 = a^2+c^2,$$

the constants a and c being otherwise arbitrary.

Added May 19*th*, 1915.

It is easy to verify that a surface of constant *mean* curvature $2/a$, of the same form as the surface in (13), is given by the equation

$$\frac{z}{a} = \zeta(u)+\left\{\frac{1-(e_2-e_3)^2}{4}+e_1\right\}u-\{\wp(a)-e_1\}^{\frac{1}{2}}\left\{\tfrac{1}{2}\log\frac{\sigma(u+a)}{\sigma(u-a)}-\zeta(a)u+\iota\theta\right\},$$

where $$\wp(a) = \tfrac{1}{4}(1-e_1)^2-e_2e_3,$$

$$r^2 = a^2\{\wp(a)-\wp(u)\}.$$

Note by Dr. T. J. I'A. Bromwich.

When the equation to a surface is written in terms of cylindrical coordinates z, r, θ, in the form

$$z = F(r,\ \theta),$$

it is known that the measure of curvature (Gauss's) is equal to

$$\{r^2F_{11}(F_{22}+rF_1)-(rF_{12}-F_2)^2\}/(r^2+r^2F_1^2+F_2^2)^2,$$

where the suffixes 1 and 2 indicate partial differentiation with respect to r and θ, respectively.

If we take the surfaces of the type found by Dr. Wilton

$$z = f(r)+c\theta,$$

where c is a constant, the condition for a pseudo-sphere becomes

$$-\frac{1}{\mu^2} = \frac{r^3f_1f_{11}-c^2}{(r^2+r^2f_1^2+c^2)^2}.$$

Then, writing $q = 1+f_1^2+c^2/r^2$, we have

$$\frac{dq}{dr} = 2\left(f_1f_{11}-\frac{c^2}{r^3}\right),$$

so that the condition for the pseudo-sphere becomes

$$-\frac{1}{\mu^2} = \frac{1}{2rq^2}\,\frac{dq}{dr},$$

leading to $$\frac{r^2}{\mu^2}+A = \frac{1}{q},$$

where A is a constant of integration. This is the equation quoted in (15) of Dr. Wilton's paper; it leads at once to the formula

$$f(r)=\int\frac{dr}{r}\left\{\frac{\mu^2r^2-(r^2+c^2)(r^2+A\mu^2)}{r^2+A\mu^2}\right\}^{\frac{1}{2}};$$

and thus $f(r)$ can be evaluated, by using Dr. Wilton's substitution (14), in the form given in equation (13).

The special case $A=0$, gives

$$f(r)=\int\frac{dr}{r}(\mu^2-c^2-r^2)^{\frac{1}{2}},$$

reducing to the elementary integral

$$f(r)=\sqrt{(a^2-r^2)}-a\log\left[\left\{a+\sqrt{(a^2-r^2)}\right\}/r\right],$$

where $$a^2=\mu^2-c^2.$$

HIGHLY COMPOSITE NUMBERS

By S. RAMANUJAN.

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I.

Introduction and Summary of Results.

1. The number $d(N)$ of divisors of N varies with extreme irregularity as N tends to infinity, tending itself to infinity or remaining small according to the form of N. In this paper I prove a large number of results which add a good deal to our knowledge of the behaviour of $d(N)$.

It was proved by Dirichlet* that

$$\frac{d(1)+d(2)+d(3)+\ldots+d(N)}{N} = \log N+2\gamma-1+O\left(\frac{1}{\sqrt{N}}\right),^{\dagger}$$

where γ is the Eulerian constant. Voronöi‡ and Landau§ have shown that the error term may be replaced by $O(N^{-\frac{2}{3}+\epsilon})$, or indeed $O(N^{-\frac{2}{3}}\log N)$. It seems not unlikely that the real value of the error is of the form $O(N^{-\frac{3}{4}+\epsilon})$, but this is as yet unproved. Mr. Hardy has, however, shown recently‖ that the equation

$$\frac{d(1)+d(2)+d(3)+\ldots+d(N)}{N} = \log N+2\gamma-1+o(N^{-\frac{3}{4}})$$

is certainly false. He has also proved that

$$d(1)+d(2)+\ldots+d(N-1)+\tfrac{1}{2}d(N)-N\log N-(2\gamma-1)N-\tfrac{1}{4}$$
$$= \sqrt{N}\sum_{1}^{\infty}\frac{d(n)}{\sqrt{n}}\left[H_1\{4\pi\sqrt{(nN)}\}-Y_1\{4\pi\sqrt{(nN)}\}\right],$$

---

* *Werke*, Vol. 2, p. 49.

† $f=O(\phi)$ means that a constant exists such that $|f|<K\phi$: $f=o(\phi)$ means that $f/\phi\to 0$.

‡ *Crelle's Journal*, Vol. 126, p. 241.

§ *Göttinger Nachrichten*, 1912.

‖ *Comptes Rendus*, May 10, 1915.

where Y_0 is the ordinary second solution of Bessel's equation and

$$H_1(x) = \frac{2}{\pi} \int_1^\infty \frac{we^{-xw}\,dw}{\sqrt{(w^2-1)}}\,;$$

and that the series on the right-hand side is the sum of the series

$$\frac{N^{\frac{1}{4}}}{\pi\sqrt{2}} \sum_1^\infty \frac{d(n)}{n^{\frac{3}{4}}} \cos\left\{4\pi\sqrt{(nN)} - \tfrac{1}{4}\pi\right\},$$

and an absolutely and uniformly convergent series.

The "average" order of $d(N)$ is thus known with considerable accuracy. In this paper I consider, not the average order of $d(N)$, but its maximum order. This problem has been much less studied. It is obvious that

$$d(N) < 2\sqrt{N}.$$

It was shown by Wigert* that

$$\text{(i)} \qquad d(N) < 2^{\frac{\log N}{\log\log N}(1+\epsilon)}$$

for all positive values of ϵ and all sufficiently large values of N, and that

$$\text{(ii)} \qquad d(N) > 2^{\frac{\log N}{\log\log N}(1-\epsilon)}$$

for an infinity of values of N. From (i) it follows in particular that

$$d(N) < N^{\delta}$$

for all positive values of δ and all sufficiently large values of N.

Wigert proves (i) by purely elementary reasoning, but uses the "Prime Number Theorem"† to prove (ii). This is, however, unnecessary, the inequality (ii) being also capable of elementary proof. In § 5 I show, by elementary reasoning, that

$$d(N) < 2^{\frac{\log N}{\log\log N} + O\frac{\log N}{(\log\log N)^2}}$$

for all values of N, and

$$d(N) > 2^{\frac{\log N}{\log\log N} + O\frac{\log N}{(\log\log N)^2}}$$

for an infinity of values of N. I also show later on that, if we assume all known results concerning the distribution of primes, then

$$d(N) < 2^{Li(\log N) + O\left[\log N e^{-a\sqrt{(\log\log N)}}\right]}$$

---

* *Arkiv för Matematik*, Vol. 3, No. 18.

† The theorem that $$\pi(x) \sim \frac{x}{\log x},$$ $\pi(x)$ being the number of primes not exceeding x.

for all values of N, and

$$d(N) > 2^{Li(\log N) + O[\log N e^{-a\sqrt{(\log\log N)}}]}$$

for an infinity of values of N, where a is a positive constant.

I then adopt a different point of view. I define a highly composite number as a number whose number of divisors exceeds that of all its predecessors. Writing such a number in the form

$$N = 2^{a_2}.3^{a_3}.5^{a_5}\dots p^{a_p},$$

I prove that

$$a_2 \geqslant a_3 \geqslant a_5 \geqslant \dots \geqslant a_p,$$

and that

$$a_p = 1,$$

for all highly composite values of N except 4 and 36.

I then go on to prove the indices near the beginning form a decreasing sequence in the stricter sense, *i.e.*, that

$$a_2 > a_3 > a_5 > \dots > a_\lambda,$$

where λ is a certain function of p.

Near the end groups of equal indices may occur, and I prove that there are actually groups of indices equal to

$$1,\ 2,\ 3,\ 4,\ \dots,\ \mu,$$

where μ again is a certain function of p. I also prove that if λ is fairly small in comparison with p, then

$$a_\lambda \log\lambda \sim \frac{\log p}{\log 2};$$

and that the later indices can be assigned with an error of at most unity.

I prove also that two successive highly composite numbers are asymptotically equivalent, *i.e.*, that the ratio of two consecutive such numbers tends to unity. These are the most striking results. More precise ones will be found in the body of the paper. These results give us a fairly accurate idea of the structure of a highly composite number.

I then select from the general aggregate of highly composite numbers a special set which I call "superior highly composite numbers." I determine completely the general form of all such numbers, and I show how a combination of the idea of a superior highly composite number with the assumption of the truth of the Riemann hypothesis concerning the roots of the ζ-function leads to even more precise results concerning the maximum order of $d(N)$. These results naturally differ from all which precede in that they depend on the truth of a hitherto unproved hypothesis.

II.

Elementary Results concerning the Order of $d(N)$.

2. Let $d(N)$ denote the number of divisors of N, and let

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_n^{a_n}, \tag{1}$$

where $p_1, p_2, p_3, \dots, p_n$ are a given set of n primes. Then

$$d(N) = (1+a_1)(1+a_2)(1+a_3) \dots (1+a_n). \tag{2}$$

From (1) we see that

$$\begin{aligned} &(1/n) \log (p_1 p_2 p_3 \dots p_n N) \\ &= (1/n) \{(1+a_1) \log p_1 + (1+a_2) \log p_2 + \dots + (1+a_n) \log p_n\} \\ &> \{(1+a_1)(1+a_2)(1+a_3) \dots (1+a_n) \log p_1 \log p_2 \dots \log p_n\}^{1/n}. \end{aligned}$$

Hence we have

$$d(N) < \frac{\{(1/n) \log (p_1 p_2 p_3 \dots p_n N)\}^n}{\log p_1 \log p_2 \log p_3 \dots \log p_n}, \tag{3}$$

for all values of N.

We shall now consider how near to this limit it is possible to make $d(N)$ by choice of the indices $a_1, a_2, a_3, \dots, a_n$. Let us suppose that

$$1+a_m = v \frac{\log p_n}{\log p_m} + \epsilon_m \quad (m = 1, 2, 3, \dots, n), \tag{4}$$

where v is a large integer and $-\frac{1}{2} < \epsilon_m < \frac{1}{2}$. Then, from (4), it is evident that

$$\epsilon_n = 0. \tag{5}$$

Hence, by a well known theorem due to Dirichlet*, it is possible to choose values of v as large as we please and such that

$$|\epsilon_1| < \epsilon, \ |\epsilon_2| < \epsilon, \ |\epsilon_3| < \epsilon, \ \dots, \ |\epsilon_{n-1}| < \epsilon, \tag{6}$$

where $\epsilon \leqslant v^{-1/(n-1)}$. Now let

$$t = v \log p_n, \qquad \delta_m = \epsilon_m \log p_m. \tag{7}$$

Then from (1), (4) and (7) we have

$$\log (p_1 p_2 p_3 \dots p_n N) = nt + \sum_1^n \delta_m. \tag{8}$$

---

* *Werke*, Vol. 1, p. 635.

Similarly, from (2), (4) and (7), we see that

$$(9)\quad d(N) = \frac{(t+\delta_1)(t+\delta_2)\ldots(t+\delta_n)}{\log p_1 \log p_2 \log p_3 \ldots \log p_n}$$

$$= \frac{t^n \exp\left\{\dfrac{\Sigma\delta_n}{t} - \dfrac{\Sigma\delta_n^2}{2t^2} + \dfrac{\Sigma\delta_n^3}{3t^3} - \ldots\right\}}{\log p_1 \log p_2 \log p_3 \ldots \log p_n}$$

$$= \left(t + \frac{\Sigma\delta_n}{n}\right)^n \frac{\exp\left\{-\dfrac{n\Sigma\delta_n^2 - (\Sigma\delta_n)^2}{2nt^2} + \dfrac{n^2\Sigma\delta_n^3 - (\Sigma\delta_n)^3}{3n^2t^3} - \ldots\right\}}{\log p_1 \log p_2 \log p_3 \ldots \log p_n}$$

$$= \frac{\{(1/n)\log(p_1p_2p_3\ldots p_nN)\}^n}{\log p_1 \log p_2 \ldots \log p_n}\left[1 - \tfrac{1}{2}(\log N)^{-2}\{n^2\Sigma\delta_n^2 - n(\Sigma\delta_n)^2\} + \ldots\right],$$

in virtue of (8). From (6), (7), and (9) it follows that it is possible to choose the indices $a_1, a_2, \ldots, a_n$, so that

$$(10)\qquad d(N) = \frac{\{(1/n)\log(p_1p_2p_3\ldots p_nN)\}^n}{\log p_1 \log p_2 \ldots \log p_n}\{1 - O(\log N)^{-2n/(n-1)}\},$$

where the symbol O has its ordinary meaning.

The following examples show how close an approximation to $d(N)$ may be given by the right-hand side of (3). If

$$N = 2^{72}\,.\,7^{25},$$

then, according to (3), we have

$$(11)\qquad d(N) < 1898{\cdot}00000685\ldots;$$

and as a matter of fact $d(N) = 1898$. Similarly, taking

$$N = 2^{568}\,.\,3^{358},$$

we have, by (3),

$$(12)\qquad d(N) < 204271{\cdot}000000372\ldots;$$

while the actual value of $d(N)$ is 204271. In a similar manner, when

$$N = 2^{64}\,.\,3^{40}\,.\,5^{27},$$

we have, by (3),

$$(13)\qquad d(N) < 74620{\cdot}00412\ldots;$$

while actually $d(N) = 74620.$

3. Now let us suppose that, while the number n of different prime factors of N remains fixed, the primes p_ν, as well as the indices a_ν, are allowed to vary. It is evident that $d(N)$, considered as a function of N, is greatest when the primes p_ν are the first n primes, say 2, 3, 5, ..., p, where p is the n-th prime. It therefore follows from (3) that

$$d(N) < \frac{\{(1/n)\log(2.3.5\ldots p.N)\}^n}{\log 2 \log 3 \log 5 \ldots \log p}, \tag{14}$$

and from (10) that it is possible to choose the indices so that

$$d(N) = \frac{\{(1/n)\log(2.3.5\ldots p.N)\}^n}{\log 2 \log 3 \log 5 \ldots \log p}\{1-O(\log N)^{-2n/(n-1)}\}. \tag{15}$$

4. Before we proceed to consider the most general case, in which nothing is known about N, we must prove certain preliminary results. Let $\pi(x)$ denote the number of primes not exceeding x, and let

$$\vartheta(x) = \log 2+\log 3+\log 5+\ldots+\log p,$$

and

$$\varpi(x) = \log 2 . \log 3 . \log 5 \;\ldots\ldots\; \log p,$$

where p is the largest prime not greater than x; also let $\phi(t)$ be a function of t such that $\phi'(t)$ is continuous between 2 and x. Then

$$\begin{aligned} \int_2^x \pi(t)\,\phi'(t)\,dt &= \int_2^3 \phi'(t)\,dt+2\int_3^5 \phi'(t)\,dt+3\int_5^7 \phi'(t)\,dt \\ &\quad +4\int_7^{11} \phi'(t)\,dt+\ldots+\pi(x)\int_p^x \phi'(t)\,dt \\ &= \{\phi(3)-\phi(2)\}+2\{\phi(5)-\phi(3)\}+3\{\phi(7)-\phi(5)\} \\ &\quad +4\{\phi(11)-\phi(7)\}+\ldots+\pi(x)\{\phi(x)-\phi(p)\} \\ &= \pi(x)\,\phi(x)-\{\phi(2)+\phi(3)+\phi(5)+\ldots+\phi(p)\}. \end{aligned} \tag{16}$$

As an example let us suppose that $\phi(t) = \log t$. Then we have

$$\pi(x)\log x-\vartheta(x) = \int_2^x \frac{\pi(t)}{t}\,dt. \tag{17}$$

Again let us suppose that $\phi(t) = \log\log t$. Then we see that

$$\pi(x)\log\log x-\log\varpi(x) = \int_2^x \frac{\pi(t)}{t\log t}\,dt. \tag{18}$$

But $$\int_2^x \frac{\pi(t)}{t\log t}\,dt = \frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}\,dt+\int_2^x \left(\frac{1}{u(\log u)^2}\int_2^u \frac{\pi(t)}{t}\,dt\right)du.$$

Hence we have

$$(19)\quad \pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)}\right\}-\log\varpi(x)$$

$$=\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\}+\frac{1}{\log x}\int_2^x\frac{\pi(t)}{t}\,dt+\int_2^x\left(\frac{1}{u(\log u)^2}\int_2^u\frac{\pi(t)}{t}\,dt\right)du.$$

But $$\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\}=\pi(x)\log\left\{1-\frac{\pi(x)\log x-\vartheta(x)}{\pi(x)\log x}\right\}$$

$$=\pi(x)\log\left\{\frac{1}{\pi(x)\log x}\int_2^x\frac{\pi(t)}{t}\,dt\right\}<-\frac{1}{\log x}\int_2^x\frac{\pi(t)}{t}\,dt;$$

and so

$$(20)\qquad \pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\}+\frac{1}{\log x}\int_2^x\frac{\pi(t)}{t}\,dt<0.$$

Again,

$$\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\}=-\pi(x)\log\left\{1+\frac{\pi(x)\log x-\vartheta(x)}{\vartheta(x)}\right\}$$

$$=-\pi(x)\log\left\{1+\frac{1}{\vartheta(x)}\int_2^x\frac{\pi(t)}{t}\,dt\right\}>-\frac{\pi(x)}{\vartheta(x)}\int_2^x\frac{\pi(t)}{t}\,dt;$$

and so

$$(21)\quad \pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\}+\frac{1}{\log x}\int_2^x\frac{\pi(t)}{t}\,dt>-\frac{\pi(x)\log x-\vartheta(x)}{\vartheta(x)\log x}\int_2^x\frac{\pi(t)}{t}\,dt$$

$$=-\frac{1}{\vartheta(x)\log x}\left\{\int_2^x\frac{\pi(t)}{t}\,dt\right\}^2.$$

It follows from (19), (20) and (21) that

$$\int_2^x\left(\frac{1}{u(\log u)^2}\int_2^u\frac{\pi(t)}{t}\,dt\right)du>\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)}\right\}-\log\varpi(x)$$

$$>\int_2^x\left(\frac{1}{u(\log u)^2}\int_2^u\frac{\pi(t)}{t}\,dt\right)du-\frac{1}{\vartheta(x)\log x}\left\{\int_2^x\frac{\pi(t)}{t}\,dt\right\}^2.$$

Now it is easily proved by elementary methods* that

$$\pi(x)=O\left(\frac{x}{\log x}\right),\quad \frac{1}{\vartheta(x)}=O\left(\frac{1}{x}\right);$$

and so $$\int_2^x\frac{\pi(t)}{t}\,dt=O\left(\frac{x}{\log x}\right).$$

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* See Landau, *Handbuch*, pp. 71 *et seq.*

Hence

$$\int_2^x \left(\frac{1}{u(\log u)^2}\int_2^u \frac{\pi(t)}{t}\,dt\right)du = \int_2^x O\left\{\frac{1}{(\log u)^3}\right\}du = O\left\{\frac{x}{(\log x)^3}\right\};$$

and

$$\frac{1}{\vartheta(x)\log x}\left\{\int_2^x \frac{\pi(t)}{t}\,dt\right\}^2 = \frac{1}{\vartheta(x)\log x}\,O\left\{\frac{x^2}{(\log x)^2}\right\} = O\left\{\frac{x}{(\log x)^3}\right\}.$$

Hence we see that

$$\frac{\{\vartheta(x)/\pi(x)\}^{\pi(x)}}{\varpi(x)} = e^{O\,[x/(\log x)^3]}. \tag{22}$$

5. We proceed to consider the case in which nothing is known about N. Let

$$N' = 2^{a_1}.3^{a_2}.5^{a_3}\ldots p^{a_n}.$$

Then it is evident that $d(N) = d(N')$, and that

$$\vartheta(p) \leqslant \log N' \leqslant \log N. \tag{23}$$

It follows from (3) that

(24)

$$d(N) = d(N') < \frac{1}{\varpi(p)}\left\{\frac{\vartheta(p)+\log N'}{\pi(p)}\right\}^{\pi(p)}$$

$$\leqslant \left\{1+\frac{\log N}{\vartheta(p)}\right\}^{\pi(p)} \frac{\{\vartheta(p)/\pi(p)\}^{\pi(p)}}{\varpi(p)}$$

$$= \left\{1+\frac{\log N}{\vartheta(p)}\right\}^{\pi(p)} e^{O\,[p/(\log p)^3]} = \left\{1+\frac{\log N}{\vartheta(p)}\right\}^{\pi(p)+O\,[p/(\log p)^3]},$$

in virtue of (22) and (23). But from (17) we know that

$$\pi(p)\log p - \vartheta(p) = O\left(\frac{p}{\log p}\right);$$

and so $\quad \vartheta(p) = \pi(p)\{\log p + O(1)\} = \pi(p)\{\log\vartheta(p)+O(1)\}.$

Hence

$$\pi(p) = \vartheta(p)\left\{\frac{1}{\log\vartheta(p)} + O\frac{1}{\{\log\vartheta(p)\}^2}\right\}. \tag{25}$$

It follows from (24) and (25) that

$$d(N) \leqslant \left\{1+\frac{\log N}{\vartheta(p)}\right\}^{\frac{\vartheta(p)}{\log\vartheta(p)} + O\frac{\vartheta(p)}{[\log\vartheta(p)]^2}}.$$

Writing t instead of $\vartheta(p)$, we have

$$d(N) \leqslant \left(1+\frac{\log N}{t}\right)^{\frac{t}{\log t}+O\frac{t}{(\log t)^2}}; \tag{26}$$

and from (23) we have

$$t \leqslant \log N. \tag{27}$$

Now, if N is a function of t, the order of the right hand side of (26), considered as a function of N, is increased when N is decreased in comparison with t, and decreased when N is increased in comparison with t. Thus the most unfavourable hypothesis is that N, considered as a function of t, is as small as is compatible with the relation (27). We may therefore write $\log N$ for t in (26). Hence

$$d(N) < 2^{\frac{\log N}{\log\log N}+O\frac{\log N}{(\log\log N)^2}}, \tag{28}$$

for all values of N.*

The inequality (28) has been proved by purely elementary reasoning. We have not assumed, for example, the prime number theorem, expressed by the relation

$$\pi(x) \sim \frac{x}{\log x}.^{\dagger}$$

We can also, without assuming this theorem, show that the right-hand side of (28) is actually the order of $d(N)$ for an infinity of values of N. Let us suppose that

$$N = 2 \,.\, 3 \,.\, 5 \,.\, 7 \ldots p.$$

Then $$d(N) = 2^{\pi(p)} = 2^{\frac{t}{\log t}+O\frac{t}{(\log t)^2}},$$

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[*] If we assume *nothing* about $\pi(x)$, we can show that

$$d(N) < 2^{\frac{\log N}{\log\log N}+O\frac{\log N \log\log\log N}{(\log\log N)^2}}.$$

If we assume the prime number theorem, and nothing more, we can show that

$$d(N) < 2^{\frac{\log N}{\log\log N}+[1+o(1)]\frac{\log N}{(\log\log N)^2}}.$$

If we assume that $$\pi(x) = \frac{x}{\log x} + O\frac{x}{(\log x)^2},$$

we can show that $$d(N) < 2^{\frac{\log N}{\log\log N}+\frac{\log N}{(\log\log N)^2}+O\frac{\log N}{(\log\log N)^3}}.$$

[†] $\phi(x) \sim \psi(x)$ means that $\phi(x)/\psi(x) \to 1$ as $x \to \infty$.

in virtue of (25). Since $\log N = \vartheta(p) = t$, we see that

$$d(N) = 2^{\frac{\log N}{\log\log N} + O\frac{\log N}{(\log\log N)^2}},$$

for an infinity of values of N. Hence the maximum order of $d(N)$ is

$$2^{\frac{\log N}{\log\log N} + O\frac{\log N}{(\log\log N)^2}}.$$

III.

The Structure of Highly Composite Numbers.

6. A number N may be said to be a highly composite number, if $d(N') < d(N)$ for all values of N' less than N. It is easy to see from the definition that, if N is highly composite and $d(N') > d(N)$, then there is at least one highly composite number M, such that

$$N < M \leqslant N'. \tag{29}$$

If N and N' are consecutive highly composite numbers, then $d(M) \leqslant d(N)$ for all values of M between N and N'. It is obvious that

$$d(N) < d(2N) \tag{30}$$

for all values of N. It follows from (29) and (30) that, if N is highly composite, then there is at least one highly composite number M such that $N < M \leqslant 2N$. That is to say, there is at least one highly composite number N, such that

$$x < N \leqslant 2x, \tag{31}$$

if

$$x \geqslant 1.$$

7. I do not know of any method for determining consecutive highly composite numbers except by trial. The following table gives the consecutive highly composite values of N, and the corresponding values of $d(N)$ and $dd(N)$, up to $d(N) = 10080$.

The numbers marked with the asterisk in the table are called superior highly composite numbers. Their definition and properties will be found in §§ 32, 33.

$dd(N)$	$d(N)$	N
2	$2 = 2$	*$2 = 2$
2	$3 = 3$	$4 = 2^2$
3	$4 = 2^2$	*$6 = 2.3$
4	$6 = 2.3$	*$12 = 2^2.3$
4	$8 = 2^3$	$24 = 2^3.3$
3	$9 = 3^2$	$36 = 2^2.3^2$
4	$10 = 2.5$	$48 = 2^4.3$
6	$12 = 2^2.3$	*$60 = 2^2.3.5$
5	$16 = 2^4$	*$120 = 2^3.3.5$
6	$18 = 2.3^2$	$180 = 2^2.3^2.5$
6	$20 = 2^2.5$	$240 = 2^4.3.5$
8	$24 = 2^3.3$	*$360 = 2^3.3^2.5$
8	$30 = 2.3.5$	$720 = 2^4.3^2.5$
6	$32 = 2^5$	$840 = 2^3.3.5.7$
9	$36 = 2^2.3^2$	$1260 = 2^2.3^2.5.7$
8	$40 = 2^3.5$	$1680 = 2^4.3.5.7$
10	$48 = 2^4.3$	*$2520 = 2^3.3^2.5.7$
12	$60 = 2^2.3.5$	*$5040 = 2^4.3^2.5.7$
7	$64 = 2^6$	$7560 = 2^3.3^3.5.7$
12	$72 = 2^3.3^2$	$10080 = 2^5.3^2.5.7$
10	$80 = 2^4.5$	$15120 = 2^4.3^3.5.7$
12	$84 = 2^2.3.7$	$20160 = 2^6.3^2.5.7$
12	$90 = 2.3^2.5$	$25200 = 2^4.3^2.5^2.7$
12	$96 = 2^5.3$	$27720 = 2^3.3^2.5.7.11$
9	$100 = 2^2.5^2$	$45360 = 2^4.3^4.5.7$
12	$108 = 2^2.3^3$	$50400 = 2^5.3^2.5^2.7$
16	$120 = 2^3.3.5$	*$55440 = 2^4.3^2.5.7.11$
8	$128 = 2^7$	$83160 = 2^3.3^3.5.7.11$
15	$144 = 2^4.3^2$	$110880 = 2^5.3^2.5.7.11$
12	$160 = 2^5.5$	$166320 = 2^4.3^3.5.7.11$
16	$168 = 2^3.3.7$	$221760 = 2^6.3^2.5.7.11$
18	$180 = 2^2.3^2.5$	$277200 = 2^4.3^2.5^2.7.11$
14	$192 = 2^6.3$	$332640 = 2^5.3^3.5.7.11$
12	$200 = 2^3.5^2$	$498960 = 2^4.3^4.5.7.11$
16	$216 = 2^3.3^3$	$554400 = 2^5.3^2.5^2.7.11$
12	$224 = 2^5.7$	$665280 = 2^6.3^3.5.7.11$
20	$240 = 2^4.3.5$	*$720720 = 2^4.3^2.5.7.11.13$
9	$256 = 2^8$	$1081080 = 2^3.3^3.5.7.11.13$
18	$288 = 2^5.3^2$	*$1441440 = 2^5.3^2.5.7.11.13$
14	$320 = 2^6.5$	$2162160 = 2^4.3^3.5.7.11.13$
20	$336 = 2^4.3.7$	$2882880 = 2^6.3^2.5.7.11.13$
24	$360 = 2^3.3^2.5$	$3603600 = 2^4.3^2.5^2.7.11.13$
16	$384 = 2^7.3$	*$4324320 = 2^5.3^3.5.7.11.13$
15	$400 = 2^4.5^2$	$6486480 = 2^4.3^4.5.7.11.13$
20	$432 = 2^4.3^3$	$7207200 = 2^5.3^2.5^2.7.11.13$

$dd(N)$	$d(N)$	N
14	$448 = 2^6.7$	$8648640 = 2^6.3^3.5.7.11.13$
24	$480 = 2^5.3.5$	$10810800 = 2^4.3^3.5^2.7.11.13$
24	$504 = 2^3.3^2.7$	$14414400 = 2^6.3^2.5^2.7.11.13$
10	$512 = 2^9$	$17297280 = 2^7.3^3.5.7.11.13$
21	$576 = 2^6.3^2$	*$21621600 = 2^5.3^3.5^2.7.11.13$
24	$600 = 2^3.3.5^2$	$32432400 = 2^4.3^4.5^2.7.11.13$
16	$640 = 2^7.5$	$36756720 = 2^4.3^3.5.7.11.13.17$
24	$672 = 2^5.3.7$	$43243200 = 2^6.3^3.5^2.7.11.13$
30	$720 = 2^4.3^2.5$	$61261200 = 2^4.3^2.5^2.7.11.13.17$
18	$768 = 2^8.3$	$73513440 = 2^5.3^3.5.7.11.13.17$
18	$800 = 2^5.5^2$	$110270160 = 2^4.3^4.5.7.11.13.17$
24	$864 = 2^5.3^3$	$122522400 = 2^5.3^2.5^2.7.11.13.17$
16	$896 = 2^7.7$	$147026880 = 2^6.3^3.5.7.11.13.17$
28	$960 = 2^6.3.5$	$183783600 = 2^4.3^3.5^2.7.11.13.17$
30	$1008 = 2^4.3^2.7$	$245044800 = 2^6.3^2.5^2.7.11.13.17$
11	$1024 = 2^{10}$	$294053760 = 2^7.3^3.5.7.11.13.17$
24	$1152 = 2^7.3^2$	*$367567200 = 2^5.3^3.5^2.7.11.13.17$
30	$1200 = 2^4.3.5^2$	$551350800 = 2^4.3^4.5^2.7.11.13.17$
18	$1280 = 2^8.5$	$698377680 = 2^4.3^3.5.7.11.13.17.19$
28	$1344 = 2^6.3.7$	$735134400 = 2^6.3^3.5^2.7.11.13.17$
36	$1440 = 2^5.3^2.5$	$1102701600 = 2^5.3^4.5^2.7.11.13.17$
20	$1536 = 2^9.3$	$1396755360 = 2^5.3^3.5.7.11.13.17.19$
21	$1600 = 2^6.5^2$	$2095133040 = 2^4.3^4.5.7.11.13.17.19$
40	$1680 = 2^4.3.5.7$	$2205403200 = 2^6.3^4.5^2.7.11.13.17$
28	$1728 = 2^6.3^3$	$2327925600 = 2^5.3^2.5^2.7.11.13.17.19$
18	$1792 = 2^8.7$	$2793510720 = 2^6.3^3.5.7.11.13.17.19$
32	$1920 = 2^7.3.5$	$3491888400 = 2^4.3^3.5^2.7.11.13.17.19$
36	$2016 = 2^5.3^2.7$	$4655851200 = 2^6.3^2.5^2.7.11.13.17.19$
12	$2048 = 2^{11}$	$5587021440 = 2^7.3^3.5.7.11.13.17.19$
27	$2304 = 2^8.3^2$	*$6983776800 = 2^5.3^3.5^2.7.11.13.17.19$
36	$2400 = 2^5.3.5^2$	$10475665200 = 2^4.3^4.5^2.7.11.13.17.19$
32	$2688 = 2^7.3.7$	*$13967553600 = 2^6.3^3.5^2.7.11.13.17.19$
42	$2880 = 2^6.3^2.5$	$20951330400 = 2^5.3^4.5^2.7.11.13.17.19$
22	$3072 = 2^{10}.3$	$27935107200 = 2^7.3^3.5^2.7.11.13.17.19$
48	$3360 = 2^5.3.5.7$	$41902660800 = 2^6.3^4.5^2.7.11.13.17.19$
32	$3456 = 2^7.3^3$	$48886437600 = 2^5.3^3.5^2.7^2.11.13.17.19$
20	$3584 = 2^9.7$	$64250746560 = 2^6.3^3.5.7.11.13.17.19.23$
45	$3600 = 2^4.3^2.5^2$	$73329656400 = 2^4.3^4.5^2.7^2.11.13.17.19$
36	$3840 = 2^8.3.5$	$80313433200 = 2^4.3^3.5^2.7.11.13.17.19.23$
42	$4032 = 2^6.3^2.7$	$97772875200 = 2^6.3^3.5^2.7^2.11.13.17.19$
13	$4096 = 2^{12}$	$128501493120 = 2^7.3^3.5.7.11.13.17.19.23$
48	$4320 = 2^5.3^3.5$	$146659312800 = 2^5.3^4.5^2.7^2.11.13.17.19$
30	$4608 = 2^9.3^2$	$160626866400 = 2^5.3^3.5^2.7.11.13.17.19.23$
42	$4800 = 2^6.3.5^2$	$240940299600 = 2^4.3^4.5^2.7.11.13.17.19.23$
36	$5376 = 2^8.3.7$	*$321253732800 = 2^6.3^3.5^2.7.11.13.17.19.23$

$dd(N)$	$d(N)$	N
48	$5760 = 2^7.3^2.5$	$481880599200 = 2^5.3^4.5^2.7.11.13.17.19.23$
24	$6144 = 2^{11}.3$	$642507465600 = 2^7.3^3.5^2.7.11.13.17.19.23$
56	$6720 = 2^6.3.5.7$	$963761198400 = 2^6.3^4.5^2.7.11.13.17.19.23$
36	$6912 = 2^8.3^3$	$1124388064800 = 2^5.3^3.5^2.7^2.11.13.17.19.23$
22	$7168 = 2^{10}.7$	$1606268664000 = 2^6.3^3.5^3.7.11.13.17.19.23$
54	$7200 = 2^5.3^2.5^2$	$1686582097200 = 2^4.3^4.5^2.7^2.11.13.17.19.23$
40	$7680 = 2^9.3.5$	$1927522396800 = 2^7.3^4.5^2.7.11.13.17.19.23$
48	$8064 = 2^7.3^2.7$	$^*2248776129600 = 2^6.3^3.5^2.7^2.11.13.17.19.23$
14	$8192 = 2^{13}$	$3212537328000 = 2^7.3^3.5^3.7.11.13.17.19.23$
56	$8640 = 2^6.3^3.5$	$3373164194400 = 2^5.3^4.5^2.7^2.11.13.17.19.23$
33	$9216 = 2^{10}.3^2$	$4497552259200 = 2^7.3^3.5^2.7^2.11.13.17.19.23$
72	$10080 = 2^5.3^2.5.7$	$6746328388800 = 2^6.3^4.5^2.7^2.11.13.17.19.23$

8. Now let us consider what must be the nature of N in order that N should be a highly composite number. In the first place it must be of the form

$$2^{a_2}.3^{a_3}.5^{a_5}.7^{a_7}\dots p_1^{a_{p_1}},$$

where

$$a_2 \geqslant a_3 \geqslant a_5 \geqslant \dots \geqslant a_{p_1} \geqslant 1. \tag{32}$$

This follows at once from the fact that

$$d(\varpi_2^{a_2}\varpi_3^{a_3}\varpi_5^{a_5}\dots\varpi_{p_1}^{a_{p_1}}) = d(2^{a_2}.3^{a_3}.5^{a_5}\dots p_1^{a_{p_1}}),$$

for all prime values of $\varpi_2, \varpi_3, \varpi_5, \dots, \varpi_{p_1}$.

It follows from the definition that, if N is highly composite and $N' < N$, then $d(N')$ must be less than $d(N)$. For example, $\frac{5}{6}N < N$, and so $d(\frac{5}{6}N) < d(N)$. Hence

$$\left(1+\frac{1}{a_2}\right)\left(1+\frac{1}{a_3}\right) > \left(1+\frac{1}{1+a_5}\right),$$

provided that N is a multiple of 3.

It is convenient to write

$$a_\lambda = 0 \quad (\lambda > p_1). \tag{33}$$

Thus if N is not a multiple of 5 then a_5 should be considered as 0.

Again, a_{p_1} must be less than or equal to 2 for all values of p_1. For let P_1 be the prime next above p_1. Then it can be shown that $P_1 < p_1^2$ for all values of p_1.* Now, if a_{p_1} is greater than 2, let

$$N' = \frac{NP_1}{p_1^2}.$$

Then N' is an integer less than N, and so $d(N') < d(N)$. Hence

$$(1+a_{p_1}) > 2(a_{p_1}-1),$$

or

$$3 > a_{p_1},$$

which contradicts our hypothesis. Hence

$$a_{p_1} \leqslant 2, \tag{34}$$

for all values of p_1.

Now let p_1'', p_1', p_1, P_1, P_1' be consecutive primes in ascending order. Then, if $p_1 \geqslant 5$, $a_{p_1''}$ must be less than or equal to 4. For, if this were not so, we could suppose that

$$N' = \frac{NP_1}{(p_1'')^3}.$$

But it can easily be shown that, if $p_1 \geqslant 5$, then

$$(p_1'')^3 > P_1;$$

and so $N' < N$ and $d(N') < d(N)$. Hence

$$(1+a_{p_1''}) > 2\,(a_{p_1''}-2). \tag{35}$$

But since $a_{p_1''} \geqslant 5$, it is evident that

$$(1+a_{p_1''}) \leqslant 2\,(a_{p_1''}-2),$$

which contradicts (35); therefore, if $p_1 \geqslant 5$, then

$$a_{p_1''} \leqslant 4. \tag{36}$$

Now let

$$N' = \frac{Np_1''P_1}{p_1'\,p_1}.$$

---

* It can be proved by elementary methods that, if $x \geqslant 1$, there is at least one prime p such that $x < p \leqslant 2x$. This result is known as Bertrand's Postulate: for a proof, see Landau, *Handbuch*, p. 89. It follows at once that $P_1 < p_1^2$, if $p_1 > 2$; and the inequality is obviously true when $p_1 = 2$. Some similar results used later in this and the next section may be proved in the same kind of way. It is for some purposes sufficient to know that there is always a prime p such that $x < p < 3x$, and the proof of this is easier than that of Bertrand's Postulate. These inequalities are enough, for example, to show that

$$\log P_1 = \log p_1 + O(1).$$

It is easy to verify that, if $5 \leqslant p_1 \leqslant 19$, then

$$p_1 p_1 > p_1'' P_1 ;$$

and so $N' < N$ and $d(N') < d(N)$. Hence

$$(1+a_{p_1})(1+a_{p_1'})(1+a_{p_1''}) > 2a_{p_1} a_{p_1'} (2+a_{p_1''}),$$

or

$$\left(1+\frac{1}{a_{p_1}}\right)\left(1+\frac{1}{a_{p_1'}}\right) > 2\left(1+\frac{1}{1+a_{p_1''}}\right).$$

But from (36) we know that $1+a_{p_1'} \leqslant 5$. Hence

$$\left(1+\frac{1}{a_{p_1}}\right)\left(1+\frac{1}{a_{p_1'}}\right) > 2\tfrac{2}{5}. \tag{37}$$

From this it follows that $a_{p_1} = 1$. For, if $a_{p_1} \geqslant 2$, then

$$\left(1+\frac{1}{a_{p_1}}\right)\left(1+\frac{1}{a_{p_1'}}\right) \leqslant 2\tfrac{1}{4},$$

in virtue of (32). This contradicts (37). Hence, if $5 \leqslant p_1 \leqslant 19$, then

$$a_{p_1} = 1. \tag{38}$$

Next let $$N' = NP_1 P_1' / (p_1 p_1' p_1'').$$

It can easily be shown that, if $p_1 \geqslant 11$, then

$$P_1 P_1' < p_1 p_1' p_1'' ;$$

and so $N' < N$ and $d(N') < d(N)$. Hence

$$(1+a_{p_1})(1+a_{p_1'})(1+a_{p_1''}) > 4a_{p_1} a_{p_1'} a_{p_1''},$$

or

$$\left(1+\frac{1}{a_{p_1}}\right)\left(1+\frac{1}{a_{p_1}}\right)\left(1+\frac{1}{a_{p_1''}}\right) > 4. \tag{39}$$

From this we infer that a_{p_1} must be 1. For, if $a_{p_1} \geqslant 2$, it follows from (32) that

$$\left(1+\frac{1}{a_{p_1}}\right)\left(1+\frac{1}{a_{p_1'}}\right)\left(1+\frac{1}{a_{p_1''}}\right) \leqslant 3\tfrac{3}{8},$$

which contradicts (39). Hence we see that, if $p_1 \geqslant 11$, then

$$a_{p_1} = 1. \tag{40}$$

It follows from (38) and (40) that, if $p_1 \geqslant 5$, then

$$a_{p_1} = 1. \tag{41}$$

But if $p_1 = 2$ or 3, then from (34) it is clear that

$$a_{p_1} = 1 \textit{ or } 2. \tag{42}$$

It follows that $a_{p_1} = 1$ for all highly composite numbers, except for 2^2, and perhaps for certain numbers of the form $2^a . 3^2$. In the latter case $a \geqslant 2$. It is easy to show that, if $a \geqslant 3$, $2^a . 3^2$ cannot be highly composite. For if we suppose that

$$N' = 2^{a-1} . 3 . 5,$$

then it is evident that $N' < N$ and $d(N') < d(N)$, and so

$$3(1+a) > 4a,$$

or

$$a < 3.$$

Hence it is clear that a cannot have any other value except 2. Moreover we can see by actual trial that 2^2 and $2^2 . 3^2$ are highly composite. *Hence*

$$a_{p_1} = 1 \tag{43}$$

for all highly composite values of N save 4 *and* 36, *when*

$$a_{p_1} = 2.$$

Hereafter when we use this result it is to be understood that 4 and 36 are exceptions.

9. It follows from (32) and (43) that N must be of the form

$$\begin{aligned} & 2 . 3 . 5 . 7 \ldots\ldots p_1 \\ \times & 2 . 3 . 5 . 7 \ldots p_2 \\ \times & 2 . 3 . 5 \ldots p_3 \\ \times & \ldots\ldots\ldots\ldots \\ & \ldots\ldots \end{aligned} \tag{44}$$

where $p_1 > p_2 \geqslant p_3 \geqslant p_4 \geqslant \ldots$ and the number of rows is a_2.

Let P_r be the prime next above p_r, so that

$$\log P_r = \log p_r + O(1), \tag{45}$$

in virtue of Bertrand's Postulate. Then it is evident that

$$a_{p_r} \geqslant r, \quad a_{P_r} \leqslant r-1; \tag{46}$$

and so

$$a_P \leqslant a_{p_r} - 1. \tag{47}$$

It is to be understood that

(48) $$a_{p_1} = 0,$$

in virtue of (33).

It is clear from the form of (44) that r can never exceed a_2, and that

(49) $$p_{a_\lambda} = \lambda.$$

10. Now let $$N' = \frac{N}{\nu}\lambda^{[\log\nu/\log\lambda]},*$$

where $\nu \leqslant p_1$, so that N' is an integer. Then it is evident that $N' < N$ and $d(N') < d(N)$, and so

$$(1+a_\nu)(1+a_\lambda) > a_\nu\left(1+a_\lambda+\left[\frac{\log\nu}{\log\lambda}\right]\right),$$

or

(50) $$(1+a_\lambda) > a_\nu\left[\frac{\log\nu}{\log\lambda}\right].$$

Since the right-hand side vanishes when $\nu > p_1$, we see that (50) is true for all values of λ and ν.

Again let $$N' = N\mu\lambda^{-1-[\log\mu/\log\lambda]},$$

where $[\log\mu/\log\lambda] < a_\lambda$, so that N' is an integer. Then it is evident that $N' < N$ and $d(N') < d(N)$, and so

(51) $$(1+a_\mu)(1+a_\lambda) > (2+a_\mu)\left(a_\lambda-\left[\frac{\log\mu}{\log\lambda}\right]\right).$$

Since the right-hand side is less than or equal to 0 when

$$a_\lambda \leqslant [\log\mu/\log\lambda],$$

we see that (51) is true for all values of λ and μ. From (51) it evidently follows that

(52) $$(1+a_\lambda) < (2+a_\mu)\left[\frac{\log(\lambda\mu)}{\log\lambda}\right].$$

From (50) and (52) it is clear that

(53) $$a_\nu\left[\frac{\log\nu}{\log\lambda}\right] \leqslant a_\lambda \leqslant a_\mu+(2+a_\mu)\left[\frac{\log\mu}{\log\lambda}\right],$$

for all values of λ, μ and ν.

---

* $[x]$ denotes as usual the integral part of x.

Now let us suppose that $\nu = p_1$ and $\mu = P_1$, so that $a_\nu = 1$ and $a_\mu = 0$. Then we see that

$$\left[\frac{\log p_1}{\log \lambda}\right] \leqslant a_\lambda \leqslant 2\left[\frac{\log P_1}{\log \lambda}\right], \tag{54}$$

for all values of λ. Thus, for example, we have

$$\begin{aligned} p_1 &= 3, & 1 &\leqslant a_2 \leqslant 4; \\ p_1 &= 5, & 2 &\leqslant a_2 \leqslant 4; \\ p_1 &= 7, & 2 &\leqslant a_2 \leqslant 6; \\ p_1 &= 11, & 3 &\leqslant a_2 \leqslant 6; \end{aligned}$$

and so on. It follows from (54) that, if $\lambda \leqslant p_1$, then

$$a_\lambda \log \lambda = O(\log p_1), \qquad a_\lambda \log \lambda \neq o(\log p_1). \tag{55}$$

11. Again let

$$N' = N\lambda^{[\sqrt{\{(1+a_\lambda+a_\mu)\log\mu/\log\lambda\}}]}\,\mu^{-1-[\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}}]},$$

and let us assume for the moment that

$$a_\mu > \sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}},$$

in order that N' may be an integer. Then $N' < N$ and $d(N') < d(N)$, and so

$$\begin{aligned} (1+a_\lambda)(1+a_\mu) > \; & \left\{1+a_\lambda+[\sqrt{\{(1+a_\lambda+a_\mu)\log\mu/\log\lambda\}}]\right\} \\ & \times\left\{a_\mu-[\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}}]\right\} \\ > \; & \left\{a_\lambda+\sqrt{\{(1+a_\lambda+a_\mu)\log\mu/\log\lambda\}}\right\} \\ & \times\left\{a_\mu-\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}}\right\}. \end{aligned} \tag{56}$$

It is evident that the right-hand side of (56) becomes negative when

$$a_\mu < \sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}},$$

while the left-hand side remains positive, and so the result is still true. Hence

$$a_\mu \log\mu - a_\lambda \log\lambda < 2\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda\,\log\mu\}}, \tag{57}$$

for all values of λ and μ. Interchanging λ and μ in (57), we obtain

$$a_\lambda \log\lambda - a_\mu \log\mu < 2\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda\,\log\mu\}}. \tag{58}$$

From (57) and (58) it evidently follows that

$$(59)\qquad |a_\lambda \log\lambda - a_\mu \log\mu| < 2\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda\,\log\mu\}},$$

for all values of λ and μ. It follows from this and (55) that, if λ and μ are neither greater than p_1, then

$$(60)\qquad a_\lambda\log\lambda - a_\mu\log\mu = O\sqrt{\{\log p_1 \log(\lambda\mu)\}},$$

and so that *if* $\log\lambda = o(\log p_1)$, *then*

$$(61)\qquad a_2\log 2 \sim a_3\log 3 \sim a_5\log 5 \sim \ldots \sim a_\lambda\log\lambda.$$

12. It can easily be shown by elementary algebra that, if x, y, m, and n are not negative, and if

$$|x-y| < 2\sqrt{(mx+ny+mn)},$$

then

$$(62)\qquad \begin{cases} |\sqrt{(x+n)}-\sqrt{(y+m)}| < \sqrt{(m+n)}; \\ |\sqrt{(x+n)}-\sqrt{(m+n)}| < \sqrt{(y+m)}. \end{cases}$$

From (62) and (59) it follows that

$$(63)\qquad |\sqrt{\{(1+a_\lambda)\log\lambda\}} - \sqrt{\{(1+a_\mu)\log\mu\}}| < \sqrt{\{\log(\lambda\mu)\}},$$

and

$$(64)\qquad |\sqrt{\{(1+a_\lambda)\log\lambda\}} - \sqrt{\{\log(\lambda\mu)\}}| < \sqrt{\{(1+a_\mu)\log\mu\}},$$

for all values of λ and μ. If, in particular, we put $\mu=2$ in (63), we obtain

$$(65)\quad \sqrt{\{(1+a_2)\log 2\}} - \sqrt{\{\log(2\lambda)\}} < \sqrt{\{(1+a_\lambda)\log\lambda\}} < \sqrt{\{(1+a_2)\log 2\}} + \sqrt{\{\log(2\lambda)\}},$$

for all values of λ. Again, from (63), we have

$$(1+a_\lambda)\log\lambda < (\sqrt{\{(1+a_\nu)\log\nu\}} + \sqrt{\{\log(\lambda\nu)\}})^2,$$

or

$$(66)\qquad a_\lambda\log\lambda < (1+a_\nu)\log\nu + \log\nu + 2\sqrt{\{(1+a_\nu)\log\nu\,\log(\lambda\nu)\}}.$$

Now let us suppose that $\lambda \leqslant \mu$. Then, from (66), it follows that

$$(67)\quad \begin{aligned} a_\lambda\log\lambda + \log\mu &< (1+a_\nu)\log\nu + \log(\mu\nu) + 2\sqrt{\{(1+a_\nu)\log\nu\,\log(\lambda\nu)\}} \\ &\leqslant (1+a_\nu)\log\nu + \log(\mu\nu) + 2\sqrt{\{(1+a_\nu)\log\nu\,\log(\mu\nu)\}} \\ &= \{\sqrt{\{(1+a_\nu)\log\nu\}} + \sqrt{\log(\mu\nu)}\}^2, \end{aligned}$$

with the condition that $\lambda \leqslant \mu$. Similarly we can show that

$$(67') \qquad a_\lambda \log \lambda + \log \mu > \left\{ \sqrt{\{(1+a_\nu) \log \nu\}} - \sqrt{\log (\mu\nu)} \right\}^2,$$

with the condition that $\lambda \leqslant \mu$.

13. Now let

$$N' = \frac{N}{\lambda} 2^{[\log \lambda / \{\pi(\mu) \log 2\}]} 3^{[\log \lambda / \{\pi(\mu) \log 3\}]} \ldots \mu^{[\log \lambda / \{\pi(\mu) \log \mu\}]},$$

where $\pi(\mu) \log \mu < \log \lambda \leqslant \log p_1$. Then it is evident that N' is an integer less than N, and so $d(N') < d(N)$. Hence

$$\left(1 + \frac{1}{a_\lambda}\right) (1+a_2)(1+a_3)(1+a_5) \ldots (1+a_\mu)$$

$$> \left\{ a_2 + \frac{\log \lambda}{\pi(\mu) \log 2} \right\} \left\{ a_3 + \frac{\log \lambda}{\pi(\mu) \log 3} \right\} \ldots \left\{ a_\mu + \frac{\log \lambda}{\mu(\mu) \log \mu} \right\};$$

that is

$$\left\{ a_2 \log 2 + \frac{\log \lambda}{\pi(\mu)} \right\} \left\{ a_3 \log 3 + \frac{\log \lambda}{\pi(\mu)} \right\} \ldots \left\{ a_\mu \log \mu + \frac{\log \lambda}{\pi(\mu)} \right\}$$

$$< \left(1 + \frac{1}{a_\lambda}\right) (a_2 \log 2 + \log 2)(a_3 \log 3 + \log 3) \ldots (a_\mu \log \mu + \log \mu)$$

$$\leqslant \left(1 + \frac{1}{a_\lambda}\right) (a_2 \log 2 + \log \mu)(a_3 \log 3 + \log \mu) \ldots (a_\mu \log \mu + \log \mu).$$

In other words,

$$(68) \quad \left(1 + \frac{1}{a_\lambda}\right)$$

$$> \left\{ 1 + \frac{\dfrac{\log \lambda}{\pi(\mu)} - \log \mu}{a_2 \log 2 + \log \mu} \right\} \left\{ 1 + \frac{\dfrac{\log \lambda}{\pi(\mu)} - \log \mu}{a_3 \log 3 + \log \mu} \right\} \ldots \left\{ 1 + \frac{\dfrac{\log \lambda}{\pi(\mu)} - \log \mu}{a_\mu \log \mu + \log \mu} \right\}$$

$$> \left\{ 1 + \frac{\dfrac{\log \lambda}{\pi(\mu)} - \log \mu}{\left(\sqrt{\{(1+a_\nu) \log \nu\}} + \sqrt{\log (\mu\nu)}\right)^2} \right\}^{\pi(\mu)},$$

where ν is any prime, in virtue of (67). From (68) it follows that

$$(69) \qquad \sqrt{\{(1+a_\nu) \log \nu\}} + \sqrt{\log (\mu\nu)} > \sqrt{\left\{ \frac{\dfrac{\log \lambda}{\pi(\mu)} - \log \mu}{\left(1 + \dfrac{1}{a_\lambda}\right)^{1/\pi(\mu)} - 1} \right\}},$$

provided that $\pi(\mu) \log \mu < \log \lambda \leqslant \log p_1$.

14. Again let

$$N' = N\lambda\, 2^{-1-[\log\lambda/\{\pi(\mu)\log 2\}]}\, 3^{-1-[\log\lambda/\{\pi(\mu)\log 3\}]} \dots \mu^{-1-[\log\lambda/\{\pi(\mu)\log\mu\}]},$$

where $\mu \leqslant p_1$ and $\lambda > \mu$. Let us assume for the moment that

$$a_\kappa \log\kappa > \frac{\log\lambda}{\pi(\mu)},$$

for all values of κ less than or equal to μ, so that N' may be an integer. Then by arguments similar to those of the previous section, we can show that

$$\frac{1+a_\lambda}{2+a_\lambda} > \left\{1 - \frac{\dfrac{\log\lambda}{\pi(\mu)} + \log\mu}{(\sqrt{\{(1+a_\nu)\log\nu\}} - \sqrt{\log(\mu\nu)})^2}\right\}^{\pi(\mu)}. \tag{70}$$

From this it follows that

$$|\sqrt{\{(1+a_\nu)\log\nu\}} - \sqrt{\log(\mu\nu)} < \sqrt{\left\{\frac{\dfrac{\log\lambda}{\pi(\mu)} + \log\mu}{1 - \left(\dfrac{1+a_\lambda}{2+a_\lambda}\right)^{1/\pi(\mu)}}\right\}}, \tag{71}$$

provided that $\mu \leqslant p_1$ and $\mu < \lambda$. The condition that

$$a_\kappa \log\kappa > \{\log\lambda/\pi(\mu)\}$$

is unnecessary because we know from (67′) that

$$|\sqrt{\{(1+a_\nu)\log\nu\}} - \sqrt{\log(\mu\nu)}| < \sqrt{(a_\kappa\log\kappa + \log\mu)} \leqslant \sqrt{\left\{\frac{\log\lambda}{\pi(\mu)} + \log\mu\right\}}, \tag{72}$$

when

$$a_\kappa \log\kappa \leqslant \{\log\lambda/\pi(\mu)\},$$

and the last term in (72) is evidently less than the right-hand side of (71).

15. We shall consider in this and the following sections some important deductions from the preceding formulæ. Putting $\nu = 2$ in (69) and (71), we obtain

$$\sqrt{\{(1+a_2)\log 2\}} > \sqrt{\left\{\frac{\dfrac{\log\lambda}{\pi(\mu)} - \log\mu}{\left(1+\dfrac{1}{a_\lambda}\right)^{1/\pi(\mu)} - 1}\right\}} - \sqrt{\log(2\mu)}, \tag{73}$$

provided that $\pi(\mu)\log\mu < \log\lambda \leqslant \log p_1$, and

$$\sqrt{\{(1+a_2)\log 2\}} < \sqrt{\left\{\frac{\dfrac{\log\lambda}{\pi(\mu)} + \log\mu}{1 - \left(\dfrac{1+a_\lambda}{2+a_\lambda}\right)^{1/\pi(\mu)}}\right\}} + \sqrt{\log(2\mu)}, \tag{74}$$

provided that $\mu \leqslant p_1$, and $\mu < \lambda$. Now supposing that $\lambda = p_1$ in (73), and $\lambda = P_1$ in (74), we obtain

$$\sqrt{\{(1+a_2)\log 2\}} > \sqrt{\left\{\frac{\dfrac{\log p_1}{\pi(\mu)} - \log \mu}{2^{1/\pi(\mu)} - 1}\right\}} - \sqrt{\log (2\mu)}, \tag{75}$$

provided that $\pi(\mu) \log \mu < \log p_1$, and

$$\sqrt{\{(1+a_2)\log 2\}} < \sqrt{\left\{\frac{\dfrac{\log P_1}{\pi(\mu)} + \log \mu}{1 - 2^{-1/\pi(\mu)}}\right\}} + \sqrt{\log (2\mu)}, \tag{76}$$

provided that $\mu \leqslant p_1$. In (75) and (76) μ can be so chosen as to obtain the best possible inequality for a_2. If p_1 is too small, we may abandon this result in favour of

$$\left[\frac{\log p_1}{\log 2}\right] \leqslant a_2 \leqslant 2\left[\frac{\log P_1}{\log 2}\right], \tag{77}$$

which is obtained from (54) by putting $\lambda = 2$.

After having obtained in this way what information we can about a_2, we may use (73) and (74) to obtain information about a_λ. Here also we have to choose μ so as to obtain the best possible inequality for a_λ. But if λ is too small we may, instead of this, use

$$\begin{aligned}\sqrt{\{(1+a_2)\log 2\}} - \sqrt{\log(2\lambda)} &< \sqrt{\{(1+a_\lambda)\log \lambda\}} \\ &< \sqrt{\{(1+a_2)\log 2\}} + \sqrt{\log(2\lambda)},\end{aligned} \tag{78}$$

which is obtained by putting $\mu = 2$ in (63).

16. Now let us consider the order of a_2. From (73) it is evident that, if $\pi(\mu)\log\mu < \log\lambda \leqslant \log p_1$, then

$$(1+a_2)\log 2 + \log(2\mu) + 2\sqrt{\{(1+a_2)\log 2 \log(2\mu)\}} > \frac{\dfrac{\log \lambda}{\pi(\mu)} - \log \mu}{\left(1 + \dfrac{1}{a_\lambda}\right)^{1/\pi(\mu)} - 1}. \tag{79}$$

But we know that for positive values of x,

$$\frac{1}{e^x - 1} = \frac{1}{x} + O(1), \qquad \frac{1}{e^x - 1} = O\left(\frac{1}{x}\right).$$

Hence
$$\frac{\log\lambda}{\pi(\mu)}\,\frac{1}{\left(1+\frac{1}{a_\lambda}\right)^{1/\pi(\mu)}-1}=\frac{\log\lambda}{\pi(\mu)}\left\{\frac{\pi(\mu)}{\log\left(1+\frac{1}{a_\lambda}\right)}+O(1)\right\}$$
$$=\frac{\log\lambda}{\log\left(1+\frac{1}{a_\lambda}\right)}+O\left\{\frac{\log\lambda}{\pi(\mu)}\right\};$$

and
$$\frac{\log\mu}{\left(1+\frac{1}{a_\lambda}\right)^{1/\pi(\mu)}-1}=O\left\{\frac{\pi(\mu)\log\mu}{\log\left(1+\frac{1}{a_\lambda}\right)}\right\}=O(\mu a_\lambda).$$

Again from (55) we know that $a_2=O(\log p_1)$. Hence (79) may be written as

(80) $$a_2\log 2+O\sqrt{(\log p_1\log\mu)}+O(\log\mu)$$
$$\geqslant\frac{\log\lambda}{\log\left(1+\frac{1}{a_\lambda}\right)}+O\left\{\frac{\log\lambda}{\pi(\mu)}\right\}+O(\mu a_\lambda).$$

But
$$\log\mu=O(\mu a_\lambda),$$
$$\mu a_\lambda=\frac{\mu}{\log\lambda}(a_\lambda\log\lambda)=O\left(\frac{\mu\log p_1}{\log\lambda}\right),$$
$$\frac{\log\lambda}{\pi(\mu)}=O\left\{\frac{\log\lambda\,\log\mu}{\mu}\right\}.$$

Again
$$\frac{\log\lambda\,\log\mu}{\mu}+\frac{\mu\log p_1}{\log\lambda}>2\sqrt{(\log p_1\,\log\mu)};$$

and so
$$\sqrt{(\log p_1\,\log\mu)}=O\left(\frac{\log\lambda\,\log\mu}{\mu}+\frac{\mu\log p_1}{\log\lambda}\right).$$

Hence (80) may be replaced by

(81) $$a_2\log 2\geqslant\frac{\log\lambda}{\log\left(1+\frac{1}{a_\lambda}\right)}+O\left(\frac{\log\lambda\,\log\mu}{\mu}+\frac{\mu\log p_1}{\log\lambda}\right),$$

provided that $\pi(\mu)\log\mu<\log\lambda\leqslant\log p_1$. Similarly, from (74), we can show that

(82) $$a_2\log 2\leqslant\frac{\log\lambda}{\log\left(1+\frac{1}{1+a_\lambda}\right)}+O\left(\frac{\log\lambda\,\log\mu}{\mu}+\frac{\mu\log p_1}{\log\lambda}\right),$$

provided that $\mu \leqslant p_1$ and $\mu < \lambda$. Now supposing that $\lambda = p_1$ in (81), and $\lambda = P_1$ in (82), and also that

$$\mu = O\sqrt{(\log p_1 \log\log p_1)}, \qquad \mu \neq o\sqrt{(\log p_1 \log\log p_1)},^*$$

we obtain

$$(83) \qquad \begin{cases} a_2 \log 2 \geqslant \dfrac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}, \\[2ex] a_2 \log 2 \leqslant \dfrac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}. \end{cases}$$

From (83) it evidently follows that

$$(84) \qquad a_2 \log 2 = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}.$$

And it follows from this and (60) that if $\lambda \leqslant p_1$ then

$$(85) \qquad a_\lambda \log \lambda = \frac{\log p_1}{\log 2} + O\left\{\sqrt{(\log p_1 \log \lambda)} + \sqrt{(\log p_1 \log\log p_1)}\right\}.$$

Hence, if $\log \lambda = o(\log p_1)$, we have

$$(86) \qquad a_2 \log 2 \sim a_3 \log 3 \sim a_5 \log 5 \sim \ldots \sim a_\lambda \log \lambda \sim \frac{\log p_1}{\log 2}.$$

17. The relations (86) give us information about the order of a_λ when λ is sufficiently small compared to p_1, in fact, when λ is of the form p_1^ϵ, where $\epsilon \to 0$. Such values of λ constitute but a small part of its total range of variation, and it is clear that further formulæ must be proved before we can gain an adequate idea of the general behaviour of a_λ. From (81), (82) and (84) it follows that

$$(87) \qquad \begin{cases} \dfrac{\log \lambda}{\log\left(1 + \dfrac{1}{a_\lambda}\right)} \leqslant \dfrac{\log p_1}{\log 2} + O\left\{\dfrac{\log \lambda \log \mu}{\mu} + \dfrac{\mu \log p_1}{\log \lambda} + \sqrt{(\log p_1 \log\log p_1)}\right\}, \\[3ex] \dfrac{\log \lambda}{\log\left(1 + \dfrac{1}{1 + a_\lambda}\right)} \geqslant \dfrac{\log p_1}{\log 2} + O\left\{\dfrac{\log \lambda \log \mu}{\mu} + \dfrac{\mu \log p_1}{\log \lambda} + \sqrt{(\log p_1 \log\log p_1)}\right\}, \end{cases}$$

provided that $\pi(\mu) \log \mu < \log \lambda \leqslant \log p_1$. From this we can easily show that, if $\pi(\mu)\log\mu < \log\lambda \leqslant \log p_1$, then

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* $f \neq o(\phi)$ is to be understood as meaning that $|f| > K\phi$, where K is a constant, and $f \neq O(\phi)$ as meaning that $|f|/\phi \to \infty$. They are *not* the mere negations of $f = o(\phi)$ and $f = O(\phi)$.

$$(88)\quad \begin{cases} a_\lambda \leqslant (2^{\log\lambda/\log p_1}-1)^{-1}+O\left\{\dfrac{\log\mu}{\mu}+\dfrac{\mu\log p_1}{(\log\lambda)^2}+\dfrac{\sqrt{(\log p_1\log\log p_1)}}{\log\lambda}\right\}, \\ a_\lambda \geqslant (2^{\log\lambda/\log p_1}-1)^{-1}-1+O\left\{\dfrac{\log\mu}{\mu}+\dfrac{\mu\log p_1}{(\log\lambda)^2}+\dfrac{\sqrt{(\log p_1\log\log p_1)}}{\log\lambda}\right\}. \end{cases}$$

Now let us suppose that

$$\log\lambda \neq o\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)}.$$

Then we can choose μ so that

$$\mu = O\left\{\log\lambda\sqrt{\left(\frac{\log\log p_1}{\log p_1}\right)}\right\},$$

$$\mu \neq o\left\{\log\lambda\sqrt{\left(\frac{\log\log p_1}{\log p_1}\right)}\right\}.$$

Now it is clear that $\log\mu = O(\log\log p_1)$, and so

$$\frac{\log\mu}{\mu} = O\left(\frac{\log\log p_1}{\mu}\right) = O\left\{\frac{\sqrt{(\log p_1\log\log p_1)}}{\log\lambda}\right\};$$

and that
$$\frac{\mu\log p_1}{(\log\lambda)^2} = O\left\{\frac{\sqrt{(\log p_1\log\log p_1)}}{\log\lambda}\right\}.$$

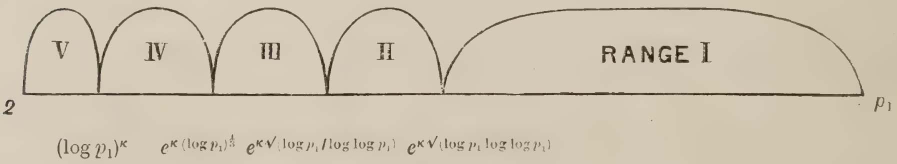
From this and (88) it follows that, if

$$\log\lambda \neq o\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)},$$

then

$$(89)\quad \begin{cases} a_\lambda \leqslant (2^{\log\lambda/\log p_1}-1)^{-1}+O\left\{\dfrac{\sqrt{(\log p_1\log\log p_1)}}{\log\lambda}\right\}, \\ a_\lambda \geqslant (2^{\log\lambda/\log p_1}-1)^{-1}-1+O\left\{\dfrac{\sqrt{(\log p_1\log\log p_1)}}{\log\lambda}\right\}. \end{cases}$$

Now we shall divide the primes from 2 to p_1 into 5 ranges thus


V
IV
III
II
RANGE I
2
p_1
$(\log p_1)^\kappa$
$e^{\kappa(\log p_1)^{\frac{1}{3}}}$
$e^{\kappa\sqrt{(\log p_1/\log\log p_1)}}$
$e^{\kappa\sqrt{(\log p_1\log\log p_1)}}$

We shall use the inequalities (89) to specify the behaviour of a_λ in ranges I and II, and the formula (85) in ranges IV and V. Range III we shall deal with differently, by a different choice of μ in the inequalities (88). We can easily see that each result in the following sections gives the most information in its particular range.

18. Range I: $\log\lambda \neq O\sqrt{(\log p_1 \log\log p_1)}$.*

Let
$$\Lambda = \left[(2^{\log\lambda/\log p_1} - 1)^{-1}\right],$$
and let
$$(2^{\log\lambda/\log p_1} - 1)^{-1} + \epsilon_\lambda,$$
where $-\frac{1}{2} < \epsilon_\lambda < \frac{1}{2}$, be an integer, so that

$$(2^{\log\lambda/\log p_1} - 1)^{-1} = \Lambda + 1 - \epsilon_\lambda \tag{90}$$

when $\epsilon_\lambda > 0$, and

$$(2^{\log\lambda/\log p_1} - 1)^{-1} = \Lambda - \epsilon_\lambda \tag{91}$$

when $\epsilon_\lambda < 0$. By our supposition we have

$$\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda} = o(1). \tag{92}$$

First let us consider the case in which

$$\epsilon_\lambda \neq O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\},$$

so that

$$\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda} = o(\epsilon_\lambda). \tag{93}$$

It follows from (89), (90), and (93) that if $\epsilon_\lambda > 0$ then

$$\begin{cases} a_\lambda \leqslant \Lambda + 1 - \epsilon_\lambda + o(\epsilon_\lambda), \\ a_\lambda \geqslant \Lambda - \epsilon_\lambda + o(\epsilon_\lambda). \end{cases} \tag{94}$$

Since $0 < \epsilon_\lambda < \frac{1}{2}$, and a_λ and Λ are integers, it follows from (94) that

$$a_\lambda \leqslant \Lambda, \quad a_\lambda > \Lambda - 1. \tag{95}$$

Hence

$$a_\lambda = \Lambda. \tag{96}$$

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* We can with a little trouble replace all equations of the type $f = O(\phi)$ which occur by inequalities of the type $|f| < K\phi$, with definite numerical constants. This would enable us to extend all the different ranges a little. For example, an equation true for

$$\log\lambda \neq O\sqrt{(\log p_1)}$$

would be replaced by an inequality true for $\log\lambda > K\sqrt{(\log p_1)}$, where K is a definite constant, and similarly $\log\lambda = o\sqrt{(\log p_1)}$ would be replaced by $\log\lambda < k\sqrt{(\log p_1)}$.

Similarly from (89), (91), and (93) we see that if $\epsilon_\lambda < 0$ then

$$\left\{\begin{array}{l} a_\lambda \leqslant \Lambda - \epsilon_\lambda + o(\epsilon_\lambda), \\ a_\lambda \geqslant \Lambda - 1 - \epsilon_\lambda + o(\epsilon_\lambda). \end{array}\right. \tag{97}$$

Since $-\frac{1}{2} < \epsilon_\lambda < 0$, it follows from (97) that the inequalities (95), and therefore the equation (96), still hold. Hence (96) holds whenever

$$\epsilon_\lambda \neq O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\}. \tag{98}$$

In particular it holds whenever

$$\epsilon_\lambda \neq o(1), \tag{99}$$

Now let us consider the case in which

$$\epsilon_\lambda = O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\}, \tag{100}$$

so that $\epsilon_\lambda = o(1)$, in virtue of (92). It follows from this and (89) and (90) that, if $\epsilon_\lambda > 0$, then

$$\left\{\begin{array}{l} a_\lambda \leqslant \Lambda + 1 + o(1), \\ a_\lambda \geqslant \Lambda + o(1). \end{array}\right. \tag{101}$$

Hence $$a_\lambda \leqslant \Lambda + 1, \quad a_\lambda \geqslant \Lambda;$$

and so

$$a_\lambda = \Lambda \quad or \quad \Lambda + 1. \tag{102}$$

Similarly from (89), (91), and (100), we see that, if $\epsilon_\lambda < 0$, then

$$\left\{\begin{array}{l} a_\lambda \leqslant \Lambda + o(1), \\ a_\lambda \geqslant \Lambda - 1 + o(1). \end{array}\right. \tag{103}$$

Hence $$a_\lambda \leqslant \Lambda, \quad a_\lambda \geqslant \Lambda - 1;$$

and so

$$a_\lambda = \Lambda \quad or \quad \Lambda - 1. \tag{104}$$

For example, let us suppose that it is required to find a_λ when $\lambda \sim p_1^{1/8}$. We have

$$(2^{\log\lambda/\log p_1} - 1)^{-1} = (2^{1/8} - 1)^{-1} + o(1) = 11.048\ldots + o(1).$$

It is evident that $\Lambda = 11$ and $\epsilon_\lambda \neq o(1)$. Hence $a_\lambda = 11$.

19. The results in the previous section may be rewritten with slight modifications, in order that the transition of a_λ from one value to another may be more clearly expressed. Let

(105) $$\lambda = p_1^{\frac{\log(1+1/x)}{\log 2}},$$

and let $x+\epsilon_x$, where $-\frac{1}{2} < \epsilon_x < \frac{1}{2}$, be an integer. Then the range of x which we are now considering is

(106) $$x = o\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)},$$

and the results of the previous section may be stated as follows. If

(107) $$\epsilon_x \neq O\left\{x\sqrt{\left(\frac{\log\log p_1}{\log p_1}\right)}\right\},$$

then

(108) $$a_\lambda = [x].$$

As a particular case of this we have

$$a_\lambda = [x],$$

when $\epsilon_x \neq o(1)$. But if

(109) $$\epsilon_x = O\left\{x\sqrt{\left(\frac{\log\log p_1}{\log p_1}\right)}\right\},$$

then when $\epsilon_x > 0$

(110) $$a_\lambda = [x] \text{ or } [x+1];$$

and when $\epsilon_x < 0$

(110′) $$a_\lambda = [x] \text{ or } [x-1].$$

20. Range II: $$\begin{cases} \log\lambda = O\sqrt{(\log p_1 \log\log p_1)}, \\ \log\lambda \neq o\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)}. \end{cases}$$

From (89) it follows that

(111) $$a_\lambda = (2^{\log\lambda/\log p_1}-1)^{-1} + O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\}.$$

But $$(2^{\log\lambda/\log p_1}-1)^{-1} = \frac{\log p_1}{\log 2 \, \log\lambda} + O(1).$$

Hence

(112) $$a_\lambda \log\lambda = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}.$$

As an example we may suppose that

$$\lambda \sim e^{\sqrt{(\log p_1)}}.$$

Then from (112) it follows that

$$a_\lambda = \frac{\sqrt{(\log p_1)}}{\log 2} + O\sqrt{(\log\log p_1)}.$$

21. Range III: $\begin{cases} \log\lambda = O\sqrt{\left(\dfrac{\log p_1}{\log\log p_1}\right)}, \\ \log\lambda \neq o(\log p_1)^{\frac{1}{3}}. \end{cases}$

Let us suppose that $\mu = O(1)$ in (88). Then we see that

$$(113)\quad a_\lambda = \frac{\log p_1}{\log 2\log\lambda} + O(1) + O\left\{\frac{\log\mu}{\mu} + \frac{\mu\log p_1}{(\log\lambda)^2} + \frac{\sqrt{(\log p_1\log\log p_1)}}{\log\lambda}\right\},$$

or

$$(114)\quad a_\lambda\log\lambda = \frac{\log p_1}{\log 2} + O\left\{\frac{\log\mu\log\lambda}{\mu} + \frac{\mu\log p_1}{\log\lambda} + \sqrt{(\log p_1\log\log p_1)}\right\}.$$

Now $$\frac{\log\mu\log\lambda}{\mu} = O(\log\lambda) = o\left(\frac{\log p_1}{\log\lambda}\right),$$

$$\frac{\mu\log p_1}{\log\lambda} = O\left(\frac{\log p_1}{\log\lambda}\right),$$

$$\sqrt{(\log p_1\log\log p_1)} = O\left(\frac{\log p_1}{\log\lambda}\right).$$

Hence

$$(115)\quad a_\lambda\log\lambda = \frac{\log p_1}{\log 2} + O\left(\frac{\log p_1}{\log\lambda}\right).$$

For example, when

$$\lambda \sim e^{(\log p_1)^{3/8}},$$

we have $$a_\lambda = \frac{(\log p_1)^{5/8}}{\log 2} + O(\log p_1)^{\frac{1}{4}}.$$

22. Range IV: $\begin{cases} \log\lambda = O(\log p_1)^{\frac{1}{3}}, \\ \log\lambda \neq o(\log\log p_1). \end{cases}$

In this case it follows from (85) that

$$(116)\quad a_\lambda\log\lambda = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1\log\lambda)}.$$

As an example in this range, when we suppose that

$$\lambda \sim e^{(\log p_1)^{\frac{1}{4}}},$$

we obtain from (116)

$$a_\lambda = \frac{(\log p_1)^{3/4}}{\log 2} + O(\log p_1)^{3/8}.$$

23. Range V: $\log \lambda = O(\log \log p_1)$.

From (85) it follows that

$$a_\lambda \log \lambda = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log \log p_1)}. \tag{117}$$

For example we may suppose that

$$\lambda \sim e^{\sqrt{(\log \log p_1)}}.$$

Then

$$a_\lambda = \frac{\log p_1}{\log 2 \sqrt{(\log \log p_1)}} + O\sqrt{(\log p_1)}.$$

24. Let λ' be the prime next below λ, so that $\lambda' \leqslant \lambda - 1$. Then it follows from (63) that

$$\sqrt{\{(1+a_{\lambda'}) \log \lambda'\}} - \sqrt{\{(1+a_\lambda) \log \lambda\}} > -\sqrt{\log(\lambda\lambda')}. \tag{118}$$

Hence

$$\sqrt{\{(1+a_{\lambda'}) \log (\lambda - 1)\}} - \sqrt{\{(1+a_\lambda) \log \lambda\}} > -\sqrt{\{2 \log \lambda\}}. \tag{119}$$

But

$$\log(\lambda - 1) < \log \lambda - \frac{1}{\lambda} < \log \lambda \left(1 - \frac{1}{2\lambda \log \lambda}\right)^2;$$

and so (119) may be replaced by

$$\sqrt{(1+a_{\lambda'})} - \sqrt{(1+a_\lambda)} > \frac{\sqrt{(1+a_{\lambda'})}}{2\lambda \log \lambda} - \sqrt{2}. \tag{120}$$

But from (54) we know that

$$1 + a_{\lambda'} \geqslant 1 + \left[\frac{\log p_1}{\log \lambda'}\right] > \frac{\log p_1}{\log \lambda'} > \frac{\log p_1}{\log \lambda}.$$

From this and (120) it follows that

$$\sqrt{(1+a_{\lambda'})} - \sqrt{(1+a_\lambda)} > \frac{\sqrt{(\log p_1)}}{2\lambda (\log \lambda)^{\frac{3}{2}}} - \sqrt{2}. \tag{121}$$

Now let us suppose that $\lambda^2(\log\lambda)^3 < \frac{1}{8}\log p_1$. Then, from (121), we have

$$\sqrt{(1+a_{\lambda'})}-\sqrt{(1+a_\lambda)} > 0,$$

or

$$a_{\lambda'} > a_\lambda. \tag{122}$$

From (122) it follows that, if $\lambda^2(\log\lambda)^3 < \frac{1}{8}\log p_1$, then

$$a_2 > a_3 > a_5 > a_7 > \ldots > a_\lambda. \tag{123}$$

In other words, *in a large highly composite number*

$$2^{a_2}.3^{a_3}.5^{a_5}.7^{a_7}\ldots p_1,$$

the indices comparatively near the beginning form a decreasing sequence in the strict sense which forbids equality. Later on groups of equal indices will in general occur.

To sum up, we have obtained fairly accurate information about a_λ for all possible values of λ. The range I is by far the most extensive, and throughout this range a_λ is known with an error never exceeding 1. The formulæ (86) hold throughout a range which includes all the remaining ranges II–V, and a considerable part of I as well, while we have obtained more precise formulæ for each individual range II–V.

25. Now let us consider the nature of p_r. It is evident that r cannot exceed a_2; *i.e.*, r cannot exceed

$$\frac{\log p_1}{(\log 2)^2}+O\sqrt{(\log p_1 \log\log p_1)}. \tag{124}$$

From (55) it evidently follows that

$$\begin{cases} a_{p_r}\log p_r = O(\log p_1), \\ a_{p_r}\log p_r \neq o(\log p_1); \end{cases} \tag{125}$$

$$\begin{cases} (1+a_{p_r})\log p_r = O(\log p_1), \\ (1+a_{p_r})\log p_r \neq o(\log p_1). \end{cases} \tag{126}$$

But from (46) we know that

$$\begin{cases} a_{p_r}\log p_r \geqslant r\log p_r, \\ (1+a_{p_r})\log p_r \leqslant r\log p_r. \end{cases} \tag{127}$$

From (125)–(127) it follows that

$$\left\{\begin{array}{l} r\log p_r = O(\log p_1), \\ r\log p_r \neq o(\log p_1); \end{array}\right. \tag{128}$$

and

$$\left\{\begin{array}{l} a_{p_r} = O(r), \\ a_{p_r} \neq o(r). \end{array}\right. \tag{129}$$

26. Supposing that $\lambda = p_r$ in (81) and $\lambda = P_r$ in (82), and remembering (128), we see that, if $r\mu = o(\log p_1)$, then

$$\log\left(1+\frac{1}{a_{p_r}}\right) \geqslant \frac{\log p_r}{a_2\log 2}\left\{1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)\right\}, \tag{130}$$

and

$$\log\left(1+\frac{1}{1+a_{P_r}}\right) \leqslant \frac{\log P_r}{a_2\log 2}\left\{1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)\right\}. \tag{131}$$

But, from (47), we have

$$\log\left(1+\frac{1}{a_{p_r}}\right) \leqslant \log\left(1+\frac{1}{1+a_{P_r}}\right).$$

Also we know that

$$\log P_r = \log p_r + O(1) = \log p_r\left\{1+O\left(\frac{1}{\log p_r}\right)\right\} = \log p_r\left\{1+O\left(\frac{r}{\log p_1}\right)\right\}.$$

Hence (131) may be replaced by

$$\log\left(1+\frac{1}{a_{p_r}}\right) \leqslant \frac{\log p_r}{a_2\log 2}\left\{1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)\right\}. \tag{132}$$

From (130) and (132) it is evident that

$$\log\left(1+\frac{1}{a_{p_r}}\right) = \frac{\log p_r}{a_2\log 2}\left\{1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)\right\}. \tag{133}$$

In a similar manner

$$\log\left(1+\frac{1}{1+a_{P_r}}\right) = \frac{\log p_r}{a_2\log 2}\left\{1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)\right\}. \tag{134}$$

Now supposing that

$$\left\{\begin{array}{l} r\mu = o(\log p_1), \\ r\mu \neq O(\log\mu), \end{array}\right. \tag{135}$$

and dividing (134) by (133), we have

$$\frac{\log\left(1+\dfrac{1}{1+a_{P_r}}\right)}{\log\left(1+\dfrac{1}{a_{p_r}}\right)} = 1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right),$$

or
$$1+\frac{1}{1+a_{P_r}} = 1+\frac{1}{a_{p_r}}+O\left\{\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)\Big/a_{p_r}\right\},$$

that is,
$$\frac{1}{1+a_{P_r}} = \frac{1}{a_{p_r}}\left\{1+O\left(\frac{\log\mu}{r\mu}+\frac{r\mu}{\log p_1}\right)\right\}.$$

Hence

(136) $$a_{p_r} = a_{P_r}+1+O\left(\frac{\log\mu}{\mu}+\frac{r^2\mu}{\log p_1}\right),$$

in virtue of (129). But $a_{P_r} \leqslant r-1$, and so

(137) $$a_{p_r} \leqslant r+O\left(\frac{\log\mu}{\mu}+\frac{r^2\mu}{\log p_1}\right).$$

But we know that $a_{p_r} \geqslant r$. Hence it is clear that

(138) $$a_{p_r} = r+O\left(\frac{\log\mu}{\mu}+\frac{r^2\mu}{\log p_1}\right).$$

From this and (136) it follows that

(139) $$a_{P_r} = r-1+O\left(\frac{\log\mu}{\mu}+\frac{r^2\mu}{\log p_1}\right),$$

provided that the conditions (135) are satisfied.

Now let us suppose that $r = o\sqrt{(\log p_1)}$. Then we can choose μ such that $r^2\mu = o(\log p_1)$ and $\mu \neq O(1)$. Consequently we have

$$\frac{\log\mu}{\mu} = o(1), \qquad \frac{r^2\mu}{\log p_1} = o(1);$$

and so it follows from (138) and (139) that

(140) $$a_{p_r} = 1+a_{P_r} = r,$$

provided that $r = o\sqrt{(\log p_1)}$. From this it is clear that, if $r = o\sqrt{(\log p_1)}$, then

(141) $$p_1 > p_2 > p_3 > p_4 > \ldots > p_r.$$

In other words, *in a large highly composite number*

$$2^{a_2}.3^{a_3}.5^{a_5}\ldots p_1,$$

the indices comparatively near the end form a sequence of the type

$$\ldots 5 \ldots 4 \ldots 3 \ldots 2 \ldots 1.$$

Near the beginning gaps in the indices will in general occur.

Again, let us suppose that $r = o(\log p_1)$, $r \neq o\sqrt{(\log p_1)}$, and $\mu = O(1)$ in (138) and (139). Then we see that

$$(142) \qquad \begin{cases} a_{p_r} = r + O\left(\dfrac{r^2}{\log p_1}\right), \\ a_{P_r} = r + O\left(\dfrac{r^2}{\log p_1}\right); \end{cases}$$

provided that $r = o(\log p_1)$ and $r \neq o\sqrt{(\log p_1)}$. But when $r \neq o(\log p_1)$, we shall use the general result, viz.,

$$(143) \qquad \begin{cases} a_{p_r} = O(r), & a_{p_r} \neq o(r), \\ a_{P_r} = O(r), & a_{P_r} \neq o(r), \end{cases}$$

which is true for all values of r except 1.

27. It follows from (87) and (128) that

$$(144) \qquad \begin{cases} \dfrac{\log p_r}{\log\left(1+\dfrac{1}{a_{p_r}}\right)} \leqslant \dfrac{\log p_1}{\log 2} + O\left\{\dfrac{\log p_1 \log \mu}{r\mu} + r\mu + \sqrt{(\log p_1 \log\log p_1)}\right\}, \\ \dfrac{\log P_r}{\log\left(1+\dfrac{1}{1+a_{P_r}}\right)} \geqslant \dfrac{\log p_1}{\log 2} + O\left\{\dfrac{\log p_1 \log \mu}{r\mu} + r\mu + \sqrt{(\log p_1 \log\log p_1)}\right\}, \end{cases}$$

with the condition that $r\mu = o(\log p_1)$. From this it can easily be shown, by arguments similar to those used in the beginning of the previous section, that

$$(145) \qquad \frac{\log p_r}{\log(1+1/r)} = \frac{\log p_1}{\log 2} + O\left\{\frac{\log p_1 \log \mu}{r\mu} + r\mu + \sqrt{(\log p_1 \log\log p_1)}\right\},$$

provided that $r\mu = o(\log p_1)$.

Now let us suppose that $r = o(\log p_1)$; then we can choose μ such that

$$\mu = o\left(\frac{\log p_1}{r}\right), \qquad \mu \neq O(1).$$

Consequently $r\mu = o(\log p_1)$ and $\log\mu = o(\mu)$, and so

$$\frac{\log p_1 \log\mu}{r\mu} = o(\log p_1).$$

From these relations and (145) it follows that *if* $r = o(\log p_1)$ *then*

$$\frac{\log p_r}{\log(1+1/r)} \sim \frac{\log p_1}{\log 2}; \tag{146}$$

that is to say that if $r = o(\log p_1)$ *then*

$$\frac{\log p_1}{\log 2} \sim \frac{\log p_2}{\log(1+\frac{1}{2})} \sim \frac{\log p_3}{\log(1+\frac{1}{3})} \sim \dots \sim \frac{\log p_r}{\log(1+1/r)}. \tag{147}$$

Again let us suppose that $r = O\sqrt{(\log p_1 \log\log p_1)}$ in (145). Then it is possible to choose μ such that

$$\begin{cases} r\mu = O\sqrt{(\log p_1 \log\log p_1)}, \\ r\mu \neq o\sqrt{(\log p_1 \log\log p_1)}. \end{cases} \tag{148}$$

It is evident that $\log\mu = O(\log\log p_1)$, and so

$$\frac{\log p_1 \log\mu}{r\mu} = O\left(\frac{\log p_1 \log\log p_1}{r\mu}\right) = O\sqrt{(\log p_1 \log\log p_1)},$$

in virtue of (148). Hence

$$\frac{\log p_r}{\log(1+1/r)} = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}, \tag{149}$$

provided that $$r = O\sqrt{(\log p_1 \log\log p_1)}.$$

Now let us suppose that $r = o(\log p_1)$, $r \neq o\sqrt{(\log p_1 \log\log p_1)}$, and $\mu = O(1)$, in (145). Then it is evident that

$$\log p_1 = O(r^2), \qquad \sqrt{(\log p_1 \log\log p_1)} = O(r),$$

and $$\frac{\log p_1 \log\mu}{r\mu} = O\left(\frac{\log p_1}{r}\right) = O(r).$$

Hence we see that

$$\frac{\log p_r}{\log(1+1/r)} = \frac{\log p_1}{\log 2} + O(r), \tag{150}$$

if $$r = o(\log p_1), \qquad r \neq o\sqrt{(\log p_1 \log\log p_1)}.$$

But, if $r \neq o(\log p_1)$, we see from (128) that

$$\left\{\begin{aligned} &\frac{\log p_r}{\log(1+1/r)} = O(\log p_1), \\ &\frac{\log p_r}{\log(1+1/r)} \neq o(\log p_1). \end{aligned}\right. \tag{151}$$

From (150) and (151) it follows that, if $r \neq o\sqrt{(\log p_1 \log\log p_1)}$, then

$$\frac{\log p_r}{\log(1+1/r)} = \frac{\log p_1}{\log 2} + O(r); \tag{152}$$

and from (149) and (152) that, if $r = o(\log p_1)$, then

$$\frac{\log p_r}{\log(1+1/r)} \sim \frac{\log p_1}{\log 2},$$

in agreement with (147). This result will, in general, fail for the largest possible values of r, which are of order $\log p_1$.

It must be remembered that all the results involving p_1 may be written in terms of N, since $p_1 = O(\log N)$ and $p_1 \neq o(\log N)$, and consequently

$$\log p_1 = \log\log N + O(1). \tag{153}$$

28. We shall now prove that successive highly composite numbers are asymptotically equivalent. Let m and n be any two positive integers which are prime to each other, such that

$$\log mn = o(\log p_1) = o(\log\log N); \tag{154}$$

and let

$$\frac{m}{n} = 2^{\delta_2}.3^{\delta_3}.5^{\delta_5}\ldots\wp^{\delta_\wp}, \tag{155}$$

Then it is evident that

$$mn = 2^{|\delta_2|}.3^{|\delta_3|}.5^{|\delta_5|}\ldots\wp^{|\delta_\wp|}. \tag{156}$$

Hence

$$\delta_\lambda \log\lambda = O(\log mn) = o(\log p_1) = o(a_\lambda \log\lambda); \tag{157}$$

so that

$$\delta_\lambda = o(a_\lambda).$$

Now

$$d\left(\frac{m}{n}N\right) = d(N)\left(1+\frac{\delta_2}{1+a_2}\right)\left(1+\frac{\delta_3}{1+a_3}\right)\cdots\left(1+\frac{\delta_\wp}{1+a_\wp}\right). \tag{158}$$

But, from (60), we know that

$$a_\lambda \log \lambda = a_2 \log 2 + O\sqrt{(\log p_1 \log \lambda)}.$$

Hence

$$(159) \quad 1+\frac{\delta_\lambda}{1+a_\lambda} = 1+\frac{\delta_\lambda \log \lambda}{a_2 \log 2} + O\left\{ |\,\delta_\lambda\,| \left(\frac{\log \lambda}{\log p_1}\right)^{\frac{3}{2}} \right\}$$

$$= 1+\frac{\delta_\lambda \log \lambda}{a_2 \log 2} + O\left\{ |\,\delta_\lambda\,| \frac{\log \lambda}{\log p_1} \sqrt{\left(\frac{\log \wp}{\log p_1}\right)} \right\}$$

$$= \exp\left\{ \frac{\delta_\lambda \log \lambda}{a_2 \log 2} + O\left(\frac{|\,\delta_\lambda\,| \log \lambda}{\log p_1} \sqrt{\frac{\log \wp}{\log p_1}}\right) + O\left(\frac{\delta_\lambda \log \lambda}{\log p_1}\right)^2 \right\}$$

$$= \exp\left\{ \frac{\delta_\lambda \log \lambda}{a_2 \log 2} + O\,\frac{|\,\delta_\lambda\,| \log \lambda}{\log p_1} \sqrt{\left(\frac{\log mn}{\log p_1}\right)} \right\}.$$

It follows from (155), (156), (158), and (159) that

$$(160) \quad d\left(\frac{m}{n}N\right) = d(N) \exp\left\{ \frac{\delta_2 \log 2 + \delta_3 \log 3 + \ldots + \delta_\wp \log \wp}{a_2 \log 2} \right.$$

$$\left. + O\,\frac{|\,\delta_2\,| \log 2 + |\,\delta_3\,| \log 3 + \ldots + |\,\delta_\wp\,| \log \wp}{\log p_1} \sqrt{\left(\frac{\log mn}{\log p_1}\right)} \right\}$$

$$= d(N)\, e^{\frac{\log (m/n)}{a_2 \log 2} + O\left(\frac{\log mn}{\log p_1}\right)^{\frac{3}{2}}}$$

$$= d(N)\, e^{\frac{1}{a_2 \log 2}\left\{ \log \frac{m}{n} + O \log mn \sqrt{\left(\frac{\log mn}{\log p_1}\right)} \right\}}.$$

Putting $m = n+1$, we see that, if

$$\log n = o(\log p_1) = o(\log \log N),$$

then

$$(161) \quad d\left\{ N\left(1+\frac{1}{n}\right) \right\} = d(N)\, e^{\frac{1}{a_2 \log 2}\left\{ \log\left(1+\frac{1}{n}\right) + O\left(\log n \sqrt{\frac{\log n}{\log p_1}}\right) \right\}}$$

$$= d(N) \left(1+\frac{1}{n}\right)^{\frac{1+O\left\{ n \log n \sqrt{\left(\frac{\log n}{\log \log N}\right)} \right\}}{a_2 \log 2}}$$

Now it is possible to choose n such that

$$n(\log n)^{\frac{3}{2}} \neq o\sqrt{(\log \log N)},$$

and

$$1+O\left\{ n \log n \sqrt{\left(\frac{\log n}{\log \log N}\right)} \right\} > O;$$

that is to say

$$(162) \qquad d\left\{ N\left(1+\frac{1}{n}\right) \right\} > d(N)$$

From this and (29) it follows that *if N is a highly composite number then the next highly composite number is of the form*

$$N+O\left\{\frac{N(\log\log\log N)^{\frac{3}{2}}}{\sqrt{(\log\log N)}}\right\}. \tag{163}$$

Hence the ratio of two consecutive highly composite numbers tends to unity.

It follows from (163) that the number of highly composite numbers not exceeding x is not of the form

$$o\left\{\frac{\log x\sqrt{(\log\log x)}}{(\log\log\log x)^{\frac{3}{2}}}\right\}.$$

29. Now let us consider the nature of $d(N)$ for highly composite values of N. From (44) we see that

$$d(N) = 2^{\pi(p_1)-\pi(p_2)}.3^{\pi(p_2)-\pi(p_3)}.4^{\pi(p_3)-\pi(p_4)}\dots(1+a_2). \tag{164}$$

From this it follows that

$$d(N) = 2^{\alpha_2}.3^{\alpha_3}.5^{\alpha_5}\dots\varpi^{\alpha_\varpi}, \tag{165}$$

where ϖ is the largest prime not exceeding $1+a_2$; and

$$a_\lambda = \pi(p_{\lambda-1})+O(p_\lambda). \tag{166}$$

It also follows that, if $\wp_1$, $\wp_2$, $\wp_3$, ..., $\wp_\lambda$ are a given set of primes, then a number $\bar{\mu}$ can be found such that the equation

$$d(N) = \wp_1^{\beta_1}.\wp_2^{\beta_2}.\wp_3^{\beta_3}\dots\wp_\mu^{\beta_\mu}\dots\wp_\lambda^{\beta_\lambda}$$

is impossible if N is a highly composite number and $\beta_\mu > \bar{\mu}$. We may state this roughly by saying that as N (a highly composite number) tends to infinity, then, not merely in N itself, but also in $d(N)$, the number of prime factors, as well as the indices, must tend to infinity. In particular such an equation as

$$d(N) = k.2^m, \tag{167}$$

where k is fixed, becomes impossible when m exceeds a certain limit depending on k.

It is easily seen from (153), (164), and (165) that

$$(168)\qquad \begin{cases} \varpi = O(a_2) = O(\log p_1) = O(\log\log N) = O\{\log\log d(N)\}, \\ \varpi \neq o(a_2) = o(\log p_1) = o(\log\log N) = o\{\log\log d(N)\}.^* \end{cases}$$

It follows from (147) that, if $\lambda = o(\log p_1)$, then

$$(169)\qquad \frac{\log a_2}{\log(1-\frac{1}{2})} \sim \frac{\log a_3}{\log(1-\frac{1}{3})} \sim \frac{\log a_5}{\log(1-\frac{1}{5})} \sim \dots \sim \frac{\log a_\lambda}{\log(1-1/\lambda)}.$$

Similarly, from (149), it follows that if $\lambda = O\sqrt{(\log p_1 \log\log p_1)}$ then

$$(170)\qquad \frac{\log(1+a_\lambda)}{\log(1-1/\lambda)} = -\frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}.$$

Again, from (152), we see that if $\lambda \neq o\sqrt{(\log p_1 \log\log p_1)}$ then

$$(171)\qquad \frac{\log(1+a_\lambda)}{\log(1-1/\lambda)} = -\frac{\log p_1}{\log 2} + O(\lambda).$$

In the left-hand side we cannot write a_λ instead of $1+a_\lambda$, as a_λ may be zero for a few values of λ.

From (165) and (170) we can show that

$$\log d(N) = a_2 \log 2 + O(a_3), \qquad \log d(N) \neq a_2 \log 2 + o(a_3);$$

and so

$$(172)\qquad \log d(N) = a_2 \log 2 + e^{\frac{\log(3/2)}{\log 2}\log p_1 + O\sqrt{(\log p_1 \log\log p_1)}}.$$

But from (163) we see that

$$\log\log d(N) = \log p_1 + O(\log\log p_1).$$

From this and (172) it follows that

$$(173)\qquad a_2 \log 2 = \log d(N) - \{\log d(N)\}^{\frac{\log(3/2)}{\log 2} + O\sqrt{\left\{\frac{\log\log\log d(N)}{\log\log d(N)}\right\}}}.$$

30. Now we shall consider the order of $dd(N)$ for highly composite values of N. It follows from (165) that

$$(174)\qquad \log dd(N) = \log(1+a_2) + \log(1+a_3) + \dots + \log(1+a_\varpi).$$

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[*] More precisely $\varpi \sim a_2$. But this involves the assumption that two consecutive primes are asymptotically equivalent. This follows at once from the prime number theorem. It appears probable that such a result cannot really be as deep as the prime number theorem, but nobody has succeeded up to now in proving it by elementary reasoning.

Now let $\lambda, \lambda', \lambda'', \ldots$ be consecutive primes in ascending order, and let

$$\lambda = O\sqrt{(\log p_1 \log\log p_1)},$$

$$\lambda \neq o\sqrt{(\log p_1 \log\log p_1)}.$$

Then, from (174), we have

$$(175)\qquad \log dd(N) = \log(1+a_2)+\log(1+a_3)+\ldots+\log(1+a_\lambda)$$
$$+\log(1+a_{\lambda'})+\log(1+a_{\lambda''})+\ldots+\log(1+a_{\varpi}).$$

But, from (170), we have

$$(176)\ \log(1+a_2)+\log(1+a_3)+\ldots+\log(1+a_\lambda)$$
$$= -\frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})(1-\tfrac{1}{5})\ldots\left(1-\frac{1}{\lambda}\right)\right\}$$
$$+O\sqrt{(\log p_1 \log\log p_1)}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\ldots\left(1-\frac{1}{\lambda}\right)\right\}.$$

It can be shown, without assuming the prime number theorem,* that

$$(177)\ -\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})(1-\tfrac{1}{5})\ldots\left(1-\frac{1}{p}\right)\right\} = \log\log p+\gamma+O\left(\frac{1}{\log p}\right),$$

where γ is the Eulerian constant. Hence

$$\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})(1-\tfrac{1}{5})\ldots\left(1-\frac{1}{p}\right)\right\} = O(\log\log p).$$

From this and (176) it follows that

$$(178)\ \log(1+a_2)+\log(1+a_3)+\ldots+\log(1+a_\lambda)$$
$$= -\frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\ldots\left(1-\frac{1}{\lambda}\right)\right\}$$
$$+O\left\{\sqrt{(\log p_1 \log\log p_1)}\log\log\lambda\right\}$$
$$= -\frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\ldots\left(1-\frac{1}{\lambda}\right)\right\}$$
$$+O\left\{\sqrt{(\log p_1 \log\log p_1)}\log\log\log p_1\right\}.$$

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* See Landau, *Handbuch*, p. 139.

Again, from (152), we see that

$$(179)\quad \log(1+a_{\lambda})+\log(1+a_{\lambda''})+\ldots+\log(1+a_{\varpi})$$

$$=-\frac{\log p_1}{\log 2}\log\left\{\left(1-\frac{1}{\lambda'}\right)\left(1-\frac{1}{\lambda''}\right)\ldots\left(1-\frac{1}{\varpi}\right)\right\}$$

$$+O\left\{\lambda'\log\left(1-\frac{1}{\lambda'}\right)+\lambda''\log\left(1-\frac{1}{\lambda''}\right)+\ldots+\varpi\log\left(1-\frac{1}{\varpi}\right)\right\}$$

$$=-\frac{\log p_1}{\log 2}\log\left\{\left(1-\frac{1}{\lambda'}\right)\left(1-\frac{1}{\lambda''}\right)\ldots\left(1-\frac{1}{\varpi}\right)\right\}+O\left\{\pi(\varpi)-\pi(\lambda)\right\}$$

$$=-\frac{\log p_1}{\log 2}\log\left\{\left(1-\frac{1}{\lambda'}\right)\left(1-\frac{1}{\lambda''}\right)\ldots\left(1-\frac{1}{\varpi}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right).$$

From (175), (178), and (179) it follows that

(180)

$$\log dd(N)=-\frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\ldots\left(1-\frac{1}{\lambda}\right)\right\}$$

$$+O\left\{\sqrt{(\log p_1\log\log p_1)}\log\log\log p_1\right\}$$

$$-\frac{\log p_1}{\log 2}\log\left\{\left(1-\frac{1}{\lambda'}\right)\left(1-\frac{1}{\lambda''}\right)\ldots\left(1-\frac{1}{\varpi}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right)$$

$$=-\frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\ldots\left(1-\frac{1}{\varpi}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right)$$

$$=\frac{\log p_1}{\log 2}\left\{\log\log\varpi+\gamma+O\left(\frac{1}{\log\varpi}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right)$$

$$=\frac{\log p_1}{\log 2}\left\{\log\log\log p_1+\gamma+O\left(\frac{1}{\log\log p_1}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right)$$

$$=\frac{\log\log N}{\log 2}\left\{\log\log\log\log N+\gamma+O\left(\frac{1}{\log\log\log N}\right)\right\},$$

in virtue of (177), (168), and (163). Hence, if N is a highly composite number, then

$$(181)\qquad dd(N)=(\log N)^{\frac{1}{\log 2}\left\{\log\log\log\log N+\gamma+O\left(\frac{1}{\log\log\log N}\right)\right\}}.$$

31. It may be interesting to note that, as far as the table is constructed,

$$2,\ 2^2,\ 2^3,\ \ldots,\ 2^{13},\quad 3,\ 3.2,\ 3.2^2,\ \ldots,\ 3.2^{11},\quad 5.2,\ 5.2^2,\ \ldots,\ 5.2^8,$$

$$7.2^5,\ 7.2^6,\ \ldots,\ 7.2^{10},\quad 9,\ 9.2,\ 9.2^2,\ \ldots,\ 9.2^{10},$$

and so on, occur as values of $d(N)$. But we know from § 29 that $k.2^m$ cannot be the value of $d(N)$ for sufficiently large values of m; and so numbers of the form $k.2^m$ which occur as the value of $d(N)$ in the table must disappear sooner or later when the table is extended.

Thus numbers of the form 5.2^m have begun to disappear in the table itself. The powers of 2 disappear at any rate from 2^{18} onwards. The least number having 2^{18} divisors is

$$2^7.3^3.5^3.7.11.13\ldots 41.43,$$

while the smaller number, viz.,

$$2^8.3^4.5^3.7^2.11.13\ldots 41$$

has a larger number of divisors, viz., 135.2^{11}. The numbers of the form 7.2^m disappear at least from 7.2^{13} onwards. The least number having 7.2^{13} divisors is

$$2^6.3^3.5^3.7.11.13\ldots 31.37,$$

while the smaller number, viz.,

$$2^9.3^4.5^2.7^2.11.13\ldots 31$$

has a larger number of divisors, viz., 225.2^8.

IV.

Superior Highly Composite Numbers.

32. A number N may be said to be a superior highly composite number if there is a positive number ϵ, such that

$$\frac{d(N)}{N^\epsilon} \geqslant \frac{d(N')}{(N')^\epsilon}, \tag{182}$$

for all values of N' less than N, and

$$\frac{d(N)}{N^\epsilon} > \frac{d(N')}{(N')^\epsilon} \tag{183}$$

for all values of N' greater than N.

All superior highly composite numbers are also highly composite. For, if $N' < N$, it follows from (182) that

$$d(N) \geqslant d(N')\left(\frac{N}{N'}\right)^\epsilon > d(N');$$

and so N is highly composite.

33. Now let us consider what must be the nature of N in order that it should be a superior highly composite number. In the first place it must be of the form

(184) $$2^{a_2}.3^{a_3}.5^{a_5}\dots p_1^{a_{p_1}},$$

or of the form
$$\begin{aligned} &2.3.5.7\dots\dots p_1\\ \times\,&2.3.5.7\dots p_2\\ \times\,&2.3.5\dots p_3\\ \times\,&\dots\dots\dots: \end{aligned}$$

i.e. must satisfy the conditions for a highly composite number. Now let

$$N' = N/\lambda,$$

where $\lambda \leqslant p_1$. Then from (182) it follows that

$$\frac{1+a_\lambda}{\lambda^{\epsilon a_\lambda}} \geqslant \frac{a_\lambda}{\lambda^{\epsilon(a_\lambda-1)}},$$

or

(185) $$\lambda^\epsilon \leqslant \left(1+\frac{1}{a_\lambda}\right).$$

Again let $$N' = N\lambda.$$

Then, from (183), we see that

$$\frac{1+a_\lambda}{\lambda^{\epsilon a_\lambda}} > \frac{2+a_\lambda}{\lambda^{\epsilon(a_\lambda+1)}},$$

or

(186) $$\lambda^\epsilon > \left(1+\frac{1}{1+a_\lambda}\right).$$

Now supposing that $\lambda = p_1$ in (185) and $\lambda = P_1$ in (186), we obtain

(187) $$\frac{\log 2}{\log P_1} < \epsilon \leqslant \frac{\log 2}{\log p_1}.$$

Now let us suppose that $\epsilon = 1/x$. Then, from (187), we have

(188) $$p_1 \leqslant 2^x < P_1.$$

That is, p_1 is the largest prime not exceeding 2^x. It follows from (185) that

(189) $$a_\lambda \leqslant (\lambda^{1/x}-1)^{-1}.$$

Similarly, from (186),

(190) $$a_\lambda > (\lambda^{1/x}-1)^{-1}-1.$$

From (189) and (190) it is clear that

$$a_\lambda = [(\lambda^{1/x} - 1)^{-1}]. \tag{191}$$

Hence N is of the form

$$2^{[(2^{1/x}-1)^{-1}]} . 3^{[(3^{1/x}-1)^{-1}]} . 5^{[(5^{1/x}-1)^{-1}]} \dots p_1, \tag{192}$$

where p_1 is the largest prime not exceeding 2^x.

34. Now let us suppose that $\lambda = p_r$ in (189). Then

$$a_{p_r} \leqslant (p_r^{1/x} - 1)^{-1}.$$

But we know that $r \leqslant a_{p_r}$. Hence

$$r \leqslant (p_r^{1/x} - 1)^{-1},$$

or

$$p_r \leqslant \left(1 + \frac{1}{r}\right)^x. \tag{193}$$

Similarly by supposing that $\lambda = P_r$ in (190), we see that

$$a_{P_r} > (P_r^{1/x} - 1)^{-1} - 1.$$

But we know that $r - 1 \geqslant a_{P_r}$. Hence

$$r > (P_r^{1/x} - 1)^{-1},$$

or

$$P_r > \left(1 + \frac{1}{r}\right)^x. \tag{194}$$

From (193) and (194) it is clear that p_r is the largest prime not exceeding $(1+1/r)^x$. Hence N is of the form

$$\begin{aligned} &2.3.5.7 \dots\dots p_1 \\ \times &2.3.5.7 \dots p_2 \\ \times &2.3.5 \dots p_3 \\ \times &\dots\dots\dots \end{aligned} \tag{195}$$

where p_1 is the largest prime not greater than 2^x, p_2 is the largest prime not greater than $(\frac{3}{2})^x$, and so on. In other words N is of the form

$$e^{\vartheta(2^x) + \vartheta(\frac{3}{2})^x + \vartheta(\frac{4}{3})^x + \dots}; \tag{196}$$

and $d(N)$ is of the form

$$2^{\pi(2^x)} \cdot (\tfrac{3}{2})^{\pi(\frac{3}{2})^x} \cdot (\tfrac{4}{3})^{\pi(\frac{4}{3})^x} \dots . \tag{197}$$

Thus to every value of x not less than 1 corresponds one, and only one, value of N.

35. Since
$$\frac{d(N)}{N^{1/x}} \geqslant \frac{d(N')}{(N')^{1/x}},$$
for all values of N', it follows from (196) and (197) that

$$d(N) \leqslant N^{1/x} \frac{2^{\pi(2^x)}}{e^{(1/x)\vartheta(2^x)}} \frac{(\frac{3}{2})^{\pi(\frac{3}{2})^x}}{e^{(1/x)\vartheta(\frac{3}{2})^x}} \frac{(\frac{4}{3})^{\pi(\frac{4}{3})^x}}{e^{(1/x)\vartheta(\frac{4}{3})^x}} \dots, \tag{198}$$

for all values of N and x; and $d(N)$ is equal to the right-hand side when

$$N = e^{\vartheta(2^x)+\vartheta(\frac{3}{2})^x+\vartheta(\frac{4}{3})^x+\dots}. \tag{199}$$

Thus, for example, putting $x = 2, 3, 4$ in (198), we obtain

$$\begin{cases} d(N) \leqslant \sqrt{(3N)}, \\ d(N) \leqslant 8(3N/35)^{\frac{1}{3}}, \\ d(N) \leqslant 96(3N/50050)^{\frac{1}{4}}, \end{cases} \tag{200}$$

for all values of N; and $d(N) = \sqrt{(3N)}$ when $N = 2^2 . 3$; $d(N) = 8(3N/35)^{\frac{1}{3}}$ when $N = 2^3 . 3^2 . 5 . 7$; $d(N) = 96(3N/50050)^{\frac{1}{4}}$ when

$$N = 2^5 . 3^3 . 5^2 . 7 . 11 . 13.$$

36. M and N are consecutive superior highly composite numbers if there are no superior highly composite numbers between M and N.

From (195) and (196) it is easily seen that, if M and N are any two superior highly composite numbers, and if $M > N$, then M is a multiple of N; and also that, if M and N are two consecutive superior highly composite numbers, and if $M > N$, then M/N is a prime number. From this it follows that consecutive superior highly composite numbers are of the form

$$\pi_1, \quad \pi_1\pi_2, \quad \pi_1\pi_2\pi_3, \quad \pi_1\pi_2\pi_3\pi_4, \quad \dots, \tag{201}$$

where $\pi_1, \pi_2, \pi_3, \dots$ are primes. In order to determine $\pi_1, \pi_2, \dots$, we proceed as follows. Let x_1' be the smallest value of x such that $[2^x]$ is prime, x_2' the smallest value of x such that $[(\frac{3}{2})^x]$ is prime, and so on; and let $x_1, x_2, \dots$ be the numbers $x_1', x_2', \dots$ arranged in order of magnitude.

Then π_n is the prime corresponding to x_n, and

$$N = \pi_1 \pi_2 \pi_3 \ldots \pi_n, \tag{202}$$

if $x_n \leqslant x < x_{n+1}$.

37. From the preceding results we see that the number of superior highly composite numbers not exceeding

$$e^{\vartheta(2^x) + \vartheta(\frac{3}{2})^x + \vartheta(\frac{4}{3})^x + \ldots} \tag{203}$$

is

$$\pi(2^x) + \pi(\tfrac{3}{2})^x + \pi(\tfrac{4}{3})^x + \ldots.$$

In other words if $x_n \leqslant x < x_{n+1}$ then

$$n = \pi(2^x) + \pi(\tfrac{3}{2})^x + \pi(\tfrac{4}{3})^x + \ldots. \tag{204}$$

It follows from (192) and (202) that, of the primes $\pi_1, \pi_2, \pi_3, \ldots, \pi_n$, the number of primes which are equal to a given prime ϖ is equal to

$$\left[(\varpi^{1/x} - 1)^{-1}\right]. \tag{205}$$

Further, the greatest of the primes $\pi_1, \pi_2, \pi_3, \ldots, \pi_n$ is the largest prime not greater than 2^x, and is asymptotically equivalent to the natural n-th prime, in virtue of (204).

The following table gives the values of π_n and x_n for the first 50 values of n, that is till x_n reaches very nearly 7.

$$\pi_1 = 2 \qquad x_1 = \frac{\log 2}{\log 2} = 1$$

$$\pi_2 = 3 \qquad x_2 = \frac{\log 3}{\log 2} = 1{\cdot}5849\ldots$$

$$\pi_3 = 2 \qquad x_3 = \frac{\log 2}{\log(\frac{3}{2})} = 1{\cdot}7095\ldots$$

$$\pi_4 = 5 \qquad x_4 = \frac{\log 5}{\log 2} = 2{\cdot}3219\ldots$$

$$\pi_5 = 2 \qquad x_5 = \frac{\log 2}{\log(\frac{4}{3})} = 2{\cdot}4094\ldots$$

$$\pi_6 = 3 \qquad x_6 = \frac{\log 3}{\log(\frac{3}{2})} = 2{\cdot}7095\ldots$$

$$\pi_7 = 7 \qquad x_7 = \frac{\log 7}{\log 2} = 2{\cdot}8073\ldots$$

$$\pi_8 = 2 \qquad x_8 = \frac{\log 2}{\log(\frac{5}{4})} = 3{\cdot}1062\ldots$$

$$\pi_9 = 11 \qquad x_9 = \frac{\log 11}{\log 2} = 3{\cdot}4594\ldots$$

$$\pi_{10} = 13 \qquad x_{10} = \frac{\log 13}{\log 2} = 3{\cdot}7004\ldots$$

$$\pi_{11} = 2 \qquad x_{11} = \frac{\log 2}{\log\left(\frac{6}{5}\right)} = 3{\cdot}8017\ldots$$

$$\pi_{12} = 3 \qquad x_{12} = \frac{\log 3}{\log\left(\frac{4}{3}\right)} = 3{\cdot}8188\ldots$$

$$\pi_{13} = 5 \qquad x_{13} = \frac{\log 5}{\log\left(\frac{3}{2}\right)} = 3{\cdot}9693\ldots$$

$$\pi_{14} = 17 \qquad x_{14} = \frac{\log 17}{\log 2} = 4{\cdot}0874\ldots$$

$$\pi_{15} = 19 \qquad x_{15} = \frac{\log 19}{\log 2} = 4{\cdot}2479\ldots$$

$$\pi_{16} = 2 \qquad x_{16} = \frac{\log 2}{\log\left(\frac{7}{6}\right)} = 4{\cdot}4965\ldots$$

$$\pi_{17} = 23 \qquad x_{17} = \frac{\log 23}{\log 2} = 4{\cdot}5235\ldots$$

$$\pi_{18} = 7 \qquad x_{18} = \frac{\log 7}{\log\left(\frac{3}{2}\right)} = 4{\cdot}7992\ldots$$

$$\pi_{19} = 29 \qquad x_{19} = \frac{\log 29}{\log 2} = 4{\cdot}8579\ldots$$

$$\pi_{20} = 3 \qquad x_{20} = \frac{\log 3}{\log\left(\frac{5}{4}\right)} = 4{\cdot}9233\ldots$$

$$\pi_{21} = 31 \qquad x_{21} = \frac{\log 31}{\log 2} = 4{\cdot}9541\ldots$$

$$\pi_{22} = 2 \qquad x_{22} = \frac{\log 2}{\log\left(\frac{8}{7}\right)} = 5{\cdot}1908\ldots$$

$$\pi_{23} = 37 \qquad x_{23} = \frac{\log 37}{\log 2} = 5{\cdot}2094\ldots$$

$$\pi_{24} = 41 \qquad x_{24} = \frac{\log 41}{\log 2} = 5{\cdot}3575\ldots$$

$$\pi_{25} = 43 \qquad x_{25} = \frac{\log 43}{\log 2} = 5{\cdot}4262\ldots$$

$$\pi_{26} = 47 \qquad x_{26} = \frac{\log 47}{\log 2} = 5{\cdot}5545\ldots$$

$$\pi_{27} = 5 \qquad x_{27} = \frac{\log 5}{\log\left(\frac{4}{3}\right)} = 5{\cdot}5945\ldots$$

$$\pi_{28} = 53 \qquad x_{28} = \frac{\log 53}{\log 2} = 5{\cdot}7279\ldots$$

$$\pi_{29} = 59 \qquad x_{29} = \frac{\log 59}{\log 2} = 5{\cdot}8826\ldots$$

$$\pi_{30} = 2 \qquad x_{30} = \frac{\log 2}{\log\left(\frac{9}{8}\right)} = 5{\cdot}8849\ldots$$

$$\pi_{31} = 11 \qquad x_{31} = \frac{\log 11}{\log\left(\frac{3}{2}\right)} = 5{\cdot}9139\ldots$$

$$\pi_{32} = 61 \qquad x_{32} = \frac{\log 61}{\log 2} = 5{\cdot}9307\ldots$$

$$\pi_{33} = 3 \qquad x_{33} = \frac{\log 3}{\log\left(\frac{6}{5}\right)} = 6{\cdot}0256\ldots$$

$$\pi_{34} = 67 \qquad x_{34} = \frac{\log 67}{\log 2} = 6{\cdot}0660\ldots$$

$$\pi_{35} = 71 \qquad x_{35} = \frac{\log 71}{\log 2} = 6{\cdot}1497\ldots$$

$$\pi_{36} = 73 \qquad x_{36} = \frac{\log 73}{\log 2} = 6{\cdot}1898\ldots$$

$$\pi_{37} = 79 \qquad x_{37} = \frac{\log 79}{\log 2} = 6{\cdot}3037\ldots$$

$$\pi_{38} = 13 \qquad x_{38} = \frac{\log 13}{\log\left(\frac{3}{2}\right)} = 6{\cdot}3259\ldots$$

$$\pi_{39} = 83 \qquad x_{39} = \frac{\log 83}{\log 2} = 6{\cdot}3750\ldots$$

$$\pi_{40} = 89 \qquad x_{40} = \frac{\log 89}{\log 2} = 6{\cdot}4757\ldots$$

$$\pi_{41} = 2 \qquad x_{41} = \frac{\log 2}{\log\left(\frac{10}{9}\right)} = 6{\cdot}5790\ldots$$

$$\pi_{42} = 97 \qquad x_{42} = \frac{\log 97}{\log 2} = 6{\cdot}5999\ldots$$

$$\pi_{43} = 101 \qquad x_{43} = \frac{\log 101}{\log 2} = 6{\cdot}6582\ldots$$

$$\pi_{44} = 103 \qquad x_{44} = \frac{\log 103}{\log 2} = 6{\cdot}6724\ldots$$

$$\pi_{45} = 107 \qquad x_{45} = \frac{\log 107}{\log 2} = 6{\cdot}7414 \ldots$$

$$\pi_{46} = 7 \qquad x_{46} = \frac{\log 7}{\log (\frac{4}{3})} = 6{\cdot}7641 \ldots$$

$$\pi_{47} = 109 \qquad x_{47} = \frac{\log 109}{\log 2} = 6{\cdot}7681 \ldots$$

$$\pi_{48} = 113 \qquad x_{48} = \frac{\log 113}{\log 2} = 6{\cdot}8201 \ldots$$

$$\pi_{49} = 17 \qquad x_{49} = \frac{\log 17}{\log (\frac{3}{2})} = 6{\cdot}9875 \ldots$$

$$\pi_{50} = 127 \qquad x_{50} = \frac{\log 127}{\log 2} = 6{\cdot}9886 \ldots$$

38. It follows from (17) and (198) that $\log d(N) \leqslant F(x)$, where

$$(206) \quad F(x) = \frac{1}{x}\log N + \frac{1}{x}\left\{\int_2^{2^x} \frac{\pi(t)}{t}\,dt + \int_2^{(\frac{3}{2})^x} \frac{\pi(t)}{t}\,dt + \int_2^{(\frac{4}{3})^x} \frac{\pi(t)}{t}\,dt + \ldots\right\},$$

for all values of N and x. In order to obtain the best possible upper limit for $\log d(N)$, we must choose x so as to make the right-hand side a minimum.

The function $F(x)$ is obviously continuous unless $(1+1/r)^x = p$, where r is a positive integer and p a prime. It is easily seen to be continuous even then, and so continuous without exception. Also

$$(207) \qquad F'(x) = -\frac{1}{x^2}\log N - \frac{1}{x^2}\left\{\int_2^{2^x} \frac{\pi(t)}{t}\,dt + \int_2^{(\frac{3}{2})^x} \frac{\pi(t)}{t}\,dt + \ldots\right\}$$
$$+\frac{1}{x}\left\{\pi(2^x)\log 2 + \pi(\tfrac{3}{2})^x \log \tfrac{3}{2} + \ldots\right\}$$
$$= \frac{1}{x^2}\left\{\vartheta(2^x) + \vartheta(\tfrac{3}{2})^x + \vartheta(\tfrac{4}{3})^x + \ldots - \log N\right\},$$

unless $(1+1/r)^x = p$, in virtue of (17).

Thus we see that $F(x)$ is continuous, and $F'(x)$ exists and is continuous except at certain isolated points. The sign of $F'(x)$, where it exists, is that of

$$\vartheta(2^x) + \vartheta(\tfrac{3}{2})^x + \vartheta(\tfrac{4}{3})^x + \ldots - \log N,$$

and

$$\vartheta(2^x) + \vartheta(\tfrac{3}{2})^x + \vartheta(\tfrac{4}{3})^x + \ldots,$$

is a monotonic function. Thus $F'(x)$ is first negative and then positive, changing sign once only, and so $F(x)$ has a unique minimum. Thus $F(x)$

is a minimum when x is a function of N defined by the inequalities

$$\vartheta(2^y)+\vartheta(\tfrac{3}{2})^y+\vartheta(\tfrac{4}{3})^y+\ldots \left\{\begin{matrix} < \log N & (y < x) \\ > \log N & (y > x) \end{matrix}\right\}. \tag{208}$$

Now let $D(N)$ be a function of N such that

$$D(N) = 2^{\pi(2\ )}(\tfrac{3}{2})^{\pi(\frac{3}{2})^x}(\tfrac{4}{3})^{\pi(\frac{4}{3})^x}\ldots, \tag{209}$$

where x is the function of N defined by the inequalities (208). Then, from (198), we see that

$$d(N) \leqslant D(N), \tag{210}$$

for all values of N; and $d(N) = D(N)$ for all superior highly composite values of N. Hence $D(N)$ is the maximum order of $d(N)$. In other words, $d(N)$ will attain its maximum order when N is a superior highly composite number.

V.

Application to the Order of d(N).

39. The most precise result known concerning the distribution of the prime numbers is that

$$\left\{\begin{aligned} \pi(x) &= Li(x)+O(xe^{-a\sqrt{\log x}}), \\ \vartheta(x) &= \quad x \ \ +O(xe^{-a\sqrt{\log x}}), \end{aligned}\right. \tag{211}$$

where $$Li(x) = \int^x \frac{dt}{\log t}$$

and a is a positive constant.

In order to find the maximum order of $d(N)$ we have merely to determine the order of $D(N)$ from the equations (208) and (209). Now, from (208), we have

$$\log N = \vartheta(2^x)+O(\tfrac{3}{2})^x = \vartheta(2^x)+o(2^{2x/3});$$

and so

$$\vartheta(2^x) = \log N+o(\log N)^{\frac{2}{3}}; \tag{212}$$

and similarly from (209) we have

$$\pi(2^x) = \frac{\log D(N)}{\log 2}+o(\log N)^{\frac{2}{3}}. \tag{213}$$

It follows from (211)-(213) that the maximum order of $d(N)$ is

$$2^{Li(\log N)+O[\log N e^{-a\sqrt{(\log\log N)}}]}. \tag{214}$$

It does not seem to be possible to obtain an upper limit for $d(N)$ notably more precise than (214) without assuming results concerning the distribution of primes which depend on hitherto unproved properties of the Riemann ζ-function.

40. We shall now assume that the "Riemann hypothesis" concerning the ζ-function is true, *i.e.*, that all the complex roots of $\zeta(s)$ have their real part equal to $\frac{1}{2}$. Then it is known that

$$\vartheta(x) = x - \sqrt{x} - \Sigma \frac{x^\rho}{\rho} + O(x^{\frac{1}{3}}), \tag{215}$$

where ρ is a complex root of $\zeta(s)$, and that

(216)

$$\begin{aligned}\pi(x) &= Li(x) - \tfrac{1}{2} Li(\sqrt{x}) - \Sigma\, Li(x^\rho) + O(x^{\frac{1}{3}}) \\ &= Li(x) - \frac{\sqrt{x}}{\log x} - \frac{2\sqrt{x}}{(\log x)^2} - \frac{1}{\log x} \Sigma \frac{x^\rho}{\rho} - \frac{1}{(\log x)^2} \Sigma \frac{x^\rho}{\rho^2} + O\left\{\frac{\sqrt{x}}{(\log x)^3}\right\},\end{aligned}$$

since $\Sigma \dfrac{x^\rho}{\rho^k}$ is absolutely convergent when $k > 1$. Also it is known that

$$\Sigma \frac{x^\rho}{\rho} = O\{\sqrt{x}(\log x)^2\}; \tag{217}$$

and so

$$\vartheta(x) - x = O\{\sqrt{x}(\log x)^2\}. \tag{218}$$

From (215) and (216) it is clear that

$$\pi(x) = Li(x) + \frac{\vartheta(x) - x}{\log x} - R(x) + O\left\{\frac{\sqrt{x}}{(\log x)^3}\right\}, \tag{219}$$

where

$$R(x) = \frac{2\sqrt{x} + \Sigma \dfrac{x^\rho}{\rho^2}}{(\log x)^2}. \tag{220}$$

But it follows from Taylor's theorem and (218) that

$$Li\,\vartheta(x) - Li(x) = \frac{\vartheta(x) - x}{\log x} + O(\log x)^2, \tag{221}$$

and from (219) and (221) it follows that

(222) $$\pi(x) = Li\,\vartheta(x) - R(x) + O\left\{\frac{\sqrt{x}}{(\log x)^3}\right\}.$$

41. It follows from the functional equation satisfied by $\zeta(s)$, viz.,

(223) $$(2\pi)^{-s}\,\Gamma(s)\,\zeta(s)\cos\tfrac{1}{2}\pi s = \tfrac{1}{2}\zeta(1-s),$$

that $$(1-s)\,\pi^{-\frac{1}{4}\sqrt{s}}\,\Gamma\left(\frac{1+\sqrt{s}}{4}\right)\zeta\left(\frac{1+\sqrt{s}}{2}\right)$$

is an integral function of s whose apparent order is less than 1, and hence is equal to

$$\Gamma(\tfrac{1}{4})\,\zeta(\tfrac{1}{2})\,\Pi\left\{1-\frac{s}{(2\rho-1)^2}\right\}.$$

From this we can easily deduce that

(224) $$s(1+s)\,\pi^{-\frac{1+s}{2}}\,\Gamma\left(\frac{1+s}{2}\right)\zeta(1+s) = \Pi\left(1+\frac{s}{\rho}\right).$$

Subtracting 1 from both sides, dividing the result by s, and then making $s \to 0$, we obtain

(225) $$\Sigma\frac{1}{\rho} = 1+\tfrac{1}{2}(\gamma-\log 4\pi),$$

where γ is the Eulerian constant. Hence we see that

(226) $$\left|\Sigma\frac{x^\rho}{\rho^2}\right| \leqslant \Sigma\left|\frac{x^\rho}{\rho^2}\right| = \sqrt{x}\,\Sigma\frac{1}{\rho(1-\rho)} = \sqrt{x}\,\Sigma\left(\frac{1}{\rho}+\frac{1}{1-\rho}\right)$$
$$= 2\sqrt{x}\,\Sigma\frac{1}{\rho} = \sqrt{x}\,(2+\gamma-\log 4\pi).$$

It follows from (220) and (226) that

(227) $$(\log 4\pi-\gamma)\sqrt{x} \leqslant R(x)(\log x)^2 \leqslant (4+\gamma-\log 4\pi)\sqrt{x}.$$

It can easily be verified that

(228) $$\begin{cases} \log 4\pi-\gamma = 1{\cdot}954, \\ 4+\gamma-\log 4\pi = 2{\cdot}046, \end{cases}$$

approximately.

42. Now $$R(x) = \frac{2\sqrt{x}+S(x)}{(\log x)^2},$$

where $$S(x) = \Sigma\frac{x^\rho}{\rho^2};$$

so that, considering $R(x)$ as a function of a continuous variable, we have

$$R'(x) = \frac{1}{\sqrt{x}\,(\log x)^2} - \frac{4\sqrt{x}+2S(x)}{x\,(\log x)^3} + \frac{S'(x)}{(\log x)^2}$$

$$= \frac{S'(x)}{(\log x)^2} + O\left\{\frac{1}{\sqrt{x}\,(\log x)^2}\right\},$$

for all values of x for which $S(x)$ possesses a differential coefficient.

Now the derived series of $S(x)$, viz.,

$$\bar{S}(x) = \frac{1}{x}\Sigma\frac{x^\rho}{\rho},$$

is uniformly convergent throughout any interval of positive values of x which does not include any value of x of the form $x = p^m$; and $S(x)$ is continuous for all values of x. It follows that

$$S(x_1) - S(x_2) = \int_{x_1}^{x_2} \bar{S}(x)\,dx,$$

for all positive values of x_1 and x_2, and that $S(x)$ possesses a derivative

$$S'(x) = \bar{S}(x),$$

whenever x is not of the form p^m. Also

$$\bar{S}(x) = O\left\{\frac{(\log x)^2}{\sqrt{x}}\right\}.$$

Hence

(229) $$R(x+h) = R(x) + \int_x^{x+h} O\left\{\frac{(\log t)^2}{\sqrt{t}}\right\} dt$$

$$= R(x) + O\left\{\frac{h\,(\log x)^2}{\sqrt{x}}\right\}.$$

43. Now $$\log N = \vartheta(2^x) + \vartheta(\tfrac{3}{2})^x + O(\tfrac{4}{3})^x$$

$$= \vartheta(2^x) + (\tfrac{3}{2})^x + O\left\{x^2(\tfrac{3}{2})^{\frac{1}{2}x}\right\} + O(\tfrac{4}{3})^x$$

$$= \vartheta(2^x) + (\tfrac{3}{2})^x + O(2^{5x/12}).$$

Similarly $$\log D(N) = \log 2\,.\,\pi(2^x) + \log(\tfrac{3}{2})\,Li(\tfrac{3}{2})^x + O(2^{5x/12}).$$

Writing X for 2^x, we have

(230) $$\begin{cases} \log N = \vartheta(X) + X^{\log(\frac{3}{2})/\log 2} + O(X^{\frac{5}{12}}), \\ \log D(N) = \log 2\,.\,\pi(X) + \log(\tfrac{3}{2})\,Li\left\{X^{\log(\frac{3}{2})/\log 2}\right\} + O(X^{\frac{5}{12}}). \end{cases}$$

It follows that $$\log N = X + O\left[X^{\log(\frac{3}{2})/\log 2}\right];$$

and so

$$X = \log N + O\left[(\log N)^{\log(\frac{3}{2})/\log 2}\right]. \tag{231}$$

Again, from (230) and (231), it follows that

$$\log N = \vartheta(X) + (\log N)^{\log(\frac{3}{2})/\log 2} + O\left\{(\log N)^{\frac{5}{12}}\right\}; \tag{232}$$

and

$$\begin{aligned} \log D(N) &= \log 2\,.\,\pi(X) + \log\left(\tfrac{3}{2}\right) Li(\log N)^{\log(\frac{3}{2})/\log 2} + O\left\{(\log N)^{\frac{5}{12}}\right\} \\ &= \log 2\left\{Li\,\vartheta(X) - R(X) + O\left[\frac{\sqrt{X}}{(\log X)^3}\right]\right\} \\ &\quad + \log\left(\tfrac{3}{2}\right) Li\left\{(\log N)^{\log(\frac{3}{2})/\log 2}\right\} + O\left\{(\log N)^{\frac{5}{12}}\right\}, \end{aligned} \tag{233}$$

in virtue of (222). From (231) and (233) it evidently follows that

$$\begin{aligned} \log D(N) &= \log 2\,.\,Li\,\vartheta(X) - \log 2\,.\,R(X) + \log\left(\tfrac{3}{2}\right) Li\left\{(\log N)^{\log(\frac{3}{2})/\log 2}\right\} \\ &\quad + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\} \\ &= \log 2\,.\,Li\left\{\log N - (\log N)^{\log(\frac{3}{2})/\log 2} + O(\log N)^{\frac{5}{12}}\right\} \\ &\quad - \log 2\,.\,R\left\{\log N + O(\log N)^{\log(\frac{3}{2})/\log 2}\right\} \\ &\quad + \log\left(\tfrac{3}{2}\right) Li\left\{(\log N)^{\log(\frac{3}{2})/\log 2}\right\} + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\} \end{aligned} \tag{234}$$

in virtue of (231) and (232). But

$$\begin{aligned} & Li\left\{\log N - (\log N)^{\log(\frac{3}{2})/\log 2} + O(\log N)^{\frac{5}{12}}\right\} \\ &= Li(\log N) - \frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N} + O\left\{\frac{(\log N)^{\frac{5}{12}}}{\log\log N}\right\} + O\left\{\frac{(\log N)^{\{2\log(\frac{3}{2})/\log 2\}-1}}{(\log\log N)^2}\right\} \\ &= Li(\log N) - \frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N} + O(\log N)^{\frac{5}{12}}; \end{aligned}$$

and

$$\begin{aligned} R\left\{\log N + O(\log N)^{\log(\frac{3}{2})/\log 2}\right\} &= R(\log N) + O\left\{(\log N)^{\{\log(\frac{3}{2})/\log 2\}-\frac{1}{2}}(\log\log N)^2\right\} \\ &= R(\log N) + O(\log N)^{\frac{1}{11}}, \end{aligned}$$

in virtue of (229). Hence (234) may be replaced by

$$\begin{aligned} \log D(N) &= \log 2\,.\,Li(\log N) + \log\left(\tfrac{3}{2}\right) Li\left\{(\log N)^{\log(\frac{3}{2})/\log 2}\right\} \\ &\quad - \log 2\frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N} - \log 2\,.\,R(\log N) + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}. \end{aligned} \tag{235}$$

That is to say the maximum order of $d(N)$ is

$$2^{Li(\log N)+\phi(N)}, \tag{236}$$

where

$$\phi(N)=\frac{\log(\frac{3}{2})}{\log 2}Li\left\{(\log N)^{\log(\frac{3}{2})/\log 2}\right\}-\frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N}-R(\log N)$$
$$+O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}.$$

This order is actually attained for an infinity of values of N.

44. We can now find the order of the number of superior highly composite numbers not exceeding a given number N. Let N' be the smallest superior highly composite number greater than N, and let

$$N'=e^{\vartheta(2^x)+\vartheta(\frac{3}{2})^x+\vartheta(\frac{4}{3})^x+\dots}.$$

Then, from § 37, we know that

$$2N\leqslant N'\leqslant 2^x N, \tag{237}$$

so that $N'=O(N\log N)$; and also that the number of superior highly composite numbers not exceeding N' is

$$n=\pi(2^x)+\pi(\tfrac{3}{2})^x+\pi(\tfrac{4}{3})^x+\dots.$$

By arguments similar to those of the previous section we can show that

$$n=Li(\log N)+Li(\log N)^{\log(\frac{3}{2})/\log 2}-\frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N}-R(\log N) \tag{238}$$
$$+O\left\{\frac{\sqrt{\log N}}{(\log\log N)^3}\right\}.$$

It is easy to see from § 37 that, if the largest superior highly composite number not exceeding N is

$$2^{a_2}.3^{a_3}.5^{a_5}\dots p^{a_p},$$

then the number of superior highly composite numbers not exceeding N is the sum of all the indices, viz.,

$$a_2+a_3+a_5+\dots+a_p.$$

45. Proceeding as in § 28, we can show that, if N is a superior highly

composite number, and m and n are any two positive integers such that

$$\log mn = o(\log\log N),$$

then

$$d\left(\frac{m}{n}N\right) = d(N)\, 2^{\frac{\log(m/n)}{\log\log N} + O\left(\frac{\log mn}{\log\log N}\right)^2}. \tag{239}$$

From this we can easily show that the next highly composite number is of the form

$$N + O\left\{\frac{N(\log\log\log N)^2}{\log\log N}\right\}, \tag{240}$$

Again, let S' and S be any two consecutive superior highly composite numbers, and let

$$S = e^{\vartheta(2^x) + \vartheta\left(\frac{3}{2}\right)^x + \vartheta\left(\frac{4}{3}\right)^x + \dots}.$$

Then it follows from § 35 that

$$d(N) < \left(\frac{N}{S}\right)^{1/x} d(S), \tag{241}$$

for all values of N except S and S'. Now, if S be the n-th superior highly composite number, so that

$$x_n \leqslant x < x_{n+1},$$

where x_n is the same as in § 36, we see that

$$d(N) < \left(\frac{N}{S}\right)^{1/x_n} d(S), \tag{241$'$}$$

for all values of N except S and S'. If N is S or S', then the inequality becomes an equality.

It follows from § 36 that $d(S) \leqslant 2d(S')$. Hence, if N be highly composite and $S' < N < S$, so that $d(S') < d(N) < d(S)$, then

$$\tfrac{1}{2}d(S) < d(N) < d(S), \quad d(S') < d(N) < 2d(S').$$

From this it is easy to see that the order (236) is actually attained by $d(N)$, whenever N is a highly composite number. But it may also be attained when N is not a highly composite number. For example, if

$$N = (2.3.5 \dots p_1) \times (2.3.5 \dots p_2),$$

where p_1 is the largest prime not greater than 2^x, and p_2 the largest prime not greater than $\left(\frac{3}{2}\right)^x$, it is easily seen that $d(N)$ attains the order (236): and N is not highly composite.

VI.

Special Forms of N.

46. In §§ 33–38 we have indirectly solved the following problem: to find the relations which must hold between x_1, x_2, x_3, ... in order that

$$2^{\pi(x_1)}.(\tfrac{3}{2})^{\pi(x_2)}.(\tfrac{4}{3})^{\pi(x_3)}\dots$$

may be a maximum, when it is given that

$$\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\dots$$

is a fixed number. The relations which we obtained are

$$\frac{\log 2}{\log x_1}=\frac{\log(\frac{3}{2})}{\log x_2}=\frac{\log(\frac{4}{3})}{\log x_3}=\dots.$$

This suggests the following more general problem. If N is an integer of the form

$$e^{c_1\vartheta(x_1)+c_2\vartheta(x_2)+c_3\vartheta(x_3)+\dots}, \tag{242}$$

where c_1, c_2, c_3, ... are any given positive integers, it is required to find the nature of N, that is to say the relations which hold between x_1, x_2, x_3, ..., when $d(N)$ is of maximum order. From (242) we see that

$$d(N)=(1+c_1)^{\pi(x_1)}\left(\frac{1+c_1+c_2}{1+c_1}\right)^{\pi(x_2)}\left(\frac{1+c_1+c_2+c_3}{1+c_1+c_2}\right)^{\pi(x_3)}\dots. \tag{242'}$$

If we define the "superior" numbers of the class (242) by the inequalities

$$\frac{d(N)}{N^{\epsilon}}\geqslant\frac{d(N')}{(N')^{\epsilon}},$$

for all values of N' less than N, and

$$\frac{d(N)}{N^{\epsilon}}>\frac{d(N')}{(N')^{\epsilon}},$$

for all values of N' greater than N, N and N' in the two inequalities being of the form (242), and proceed as in § 33, we can show that

$$d(N)\leqslant N^{1/x}\frac{(1+c_1)^{\pi(1+c_1)^{x/c_1}}}{e^{(c_1/x)\vartheta(1+c_1)^{x/c_1}}}\frac{\left(\frac{1+c_1+c_2}{1+c_1}\right)^{\pi\left(\frac{1+c_1+c_2}{1+c_1}\right)^{x/c_2}}}{e^{(c_2/x)\vartheta\left(\frac{1+c_1+c_2}{1+c_1}\right)^{x/c_2}}}\dots, \tag{243}$$

for all values of x, and for all values of N of the form (242). From this we can show, by arguments similar to those of § 38, that N must be of the form

$$e^{c_1\,\vartheta(1+c_1)^{x/c_1}+c_2\,\vartheta\left(\frac{1+c_1+c_2}{1+c_1}\right)^{x/c_2}+c_3\,\vartheta\left(\frac{1+c_1+c_2+c_3}{1+c_1+c_2}\right)^{x/c_3}+\dots}, \tag{244}$$

and $d(N)$ of the form

$$(1+c_1)^{\pi(1+c_1)^{x/c_1}}\left(\frac{1+c_1+c_2}{1+c_1}\right)^{\pi\left(\frac{1+c_1+c_2}{1+c_1}\right)^{x/c_2}}\left(\frac{1+c_1+c_2+c_3}{1+c_1+c_2}\right)^{\pi\left(\frac{1+c_1+c_2+c_3}{1+c_1+c_2}\right)^{x/c_3}}\dots \tag{244'}$$

From (244) and (244′) we can find the maximum order of $d(N)$, as in § 43.

47. We shall now consider the order of $d(N)$ for some special forms of N. The simplest case is that in which N is of the form

$$2.3.5.7\dots p;$$

so that

$$\log N = \vartheta(p),$$

and

$$d(N) = 2^{\pi(p)}.$$

It is easy to show that

$$d(N) = 2^{Li(\log N)-R(\log N)+O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}}. \tag{245}$$

In this case $d(N)$ is exactly a power of 2, and this naturally suggests the question: what is the maximum order of $d(N)$ when $d(N)$ is exactly a power of 2?

It is evident that, if $d(N)$ is a power of 2, the indices of the prime divisors of N cannot be any other numbers except 1, 3, 7, 15, 31, ...; and so in order that $d(N)$ should be of maximum order, N must be of the form

$$e^{\vartheta(x_1)+2\,\vartheta(x_2)+4\,\vartheta(x_3)+8\,\vartheta(x_4)+\dots},$$

and $d(N)$ of the form

$$2^{\pi(x_1)+\pi(x_2)+\pi(x_3)+\dots}.$$

It follows from § 46 that, in order that $d(N)$ should be of maximum order, N must be of the form

$$e^{\vartheta(x)+2\,\vartheta(\sqrt{x})+4\,\vartheta(x^{\frac{1}{4}})+8\,\vartheta(x^{\frac{1}{8}})+\dots}, \tag{246}$$

and $d(N)$ of the form

$$2^{\pi(x)+\pi(\sqrt{x})+\pi(x^{\frac{1}{4}})+\pi(x^{\frac{1}{8}})+\dots}. \tag{247}$$

Hence the maximum order of $d(N)$ can easily be shown to be

$$2^{Li(\log N)+\frac{4\sqrt{(\log N)}}{(\log\log N)^2}-R(\log N)+O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}}. \tag{248}$$

It is easily seen from (246) that the least number having 2^n divisors is

(249) $\quad 2.3.4.5.7.9.11.13.16.17.19.23.25.29\ldots$ to n factors,

where 2, 3, 4, 5, 7, ... are the natural primes, their squares, fourth powers, and so on, arranged according to order of magnitude.

48. We have seen that the last indices of the prime divisors of N must be 1, if $d(N)$ is of maximum order. Now we shall consider the maximum order of $d(N)$ when the indices of the prime divisors of N are never less than an integer n. In the first place, in order that $d(N)$ should be of maximum order, N must be of the form

$$e^{n\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\ldots},$$

and $d(N)$ of the form

$$(1+n)^{\pi(x_1)}\left(\frac{2+n}{1+n}\right)^{\pi(x_2)}\left(\frac{3+n}{2+n}\right)^{\pi(x_3)}\ldots.$$

It follows from § 46 that N must be of the form

$$e^{n\vartheta(1+n)^{x/n}+\vartheta\left(\frac{2+n}{1+n}\right)^{x}+\vartheta\left(\frac{3+n}{2+n}\right)^{x}+\ldots}, \tag{250}$$

and $d(N)$ of the form

$$(1+n)^{\pi(1+n)^{x/n}}\left(\frac{2+n}{1+n}\right)^{\pi\left(\frac{2+n}{1+n}\right)^{x}}\left(\frac{3+n}{2+n}\right)^{\pi\left(\frac{3+n}{2+n}\right)^{x}}\ldots. \tag{251}$$

Then, by arguments similar to those of § 43, we can show that the maximum order of $d(N)$ is

$$(n+1)^{Li\{(1/n)\log N\}+\phi(N)}, \tag{252}$$

where

$$\phi(N)=\left\{\frac{\log(n+2)}{\log(n+1)}-1\right\}Li\left\{\left(\frac{1}{n}\log N\right)^{n\frac{\log(n+2)}{\log(n+1)}-n}\right\}$$
$$-\frac{\left(\frac{1}{n}\log N\right)^{n\frac{\log(n+2)}{\log(n+1)}-n}}{n\log\left(\frac{1}{n}\log N\right)}-R\left(\frac{1}{n}\log N\right)+O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}.$$

If $n \geqslant 3$, it is easy to verify that

$$n\frac{\log(n+2)}{\log(n+1)}-n<\tfrac{1}{2};$$

and so (252) reduces to

$$(n+1)^{Li\{(1/n)\log N\}-R\{(1/n)\log N\}+O\left\{\frac{\surd(\log N)}{(\log\log N)^3}\right\}}, \tag{253}$$

provided that $n \geqslant 3$.

49. Let us next consider the maximum order of $d(N)$ when N is a perfect n-th power. In order that $d(N)$ should be of maximum order, N must be of the form

$$e^{n\vartheta(x_1)+n\vartheta(x_2)+n\vartheta(x_3)+\dots},$$

and $d(N)$ of the form

$$(1+n)^{\pi(x_1)}\left(\frac{1+2n}{1+n}\right)^{\pi(x_2)}\left(\frac{1+3n}{1+2n}\right)^{\pi(x_3)}\dots.$$

It follows from § 46 that N must be of the form

$$e^{n\vartheta(1+n)^x+n\vartheta\left(\frac{1+2n}{1+n}\right)^x+n\vartheta\left(\frac{1+3n}{1+2n}\right)^x+\dots}, \tag{254}$$

and $d(N)$ of the form

$$(1+n)^{\pi(1+n)^x}\left(\frac{1+2n}{1+n}\right)^{\pi\left(\frac{1+2n}{1+n}\right)^x}\left(\frac{1+3n}{1+2n}\right)^{\pi\left(\frac{1+3n}{1+2n}\right)^x}\dots. \tag{255}$$

Hence we can show that the maximum order of $d(N)$ is

$$(n+1)^{Li\{(1/n)\log N\}-R\{(1/n)\log N\}+O\left\{\frac{\surd(\log N)}{(\log\log N)^3}\right\}}, \tag{256}$$

provided that $n > 1$.

50. Let $l(N)$ denote the least common multiple of the first N natural numbers. Then it can easily be shown that

$$l(N) = 2^{[\log N/\log 2]}.3^{[\log N/\log 3]}.5^{[\log N/\log 5]}\dots p, \tag{257}$$

where p is the largest prime not greater than N. From this we can show that

$$l(N) = e^{\vartheta(N)+\vartheta(\surd N)+\vartheta(N^{\frac{1}{3}})+\vartheta(N^{\frac{1}{4}})+\dots}; \tag{258}$$

and so

$$d\{l(N)\} = 2^{\pi(N)}(\tfrac{3}{2})^{\pi(\surd N)}(\tfrac{4}{3})^{\pi(N^{\frac{1}{3}})}\dots. \tag{259}$$

From (258) and (259) we can show that, if N is of the form $l(M)$, then

$$d(N) = 2^{Li(\log N)+\phi(N)}, \tag{260}$$

where

$$\phi(N) = \frac{\log(\frac{9}{8})}{\log 2}\frac{\sqrt{(\log N)}}{\log\log N} + \frac{4\log(\frac{3}{2})}{\log 2}\frac{\sqrt{(\log N)}}{(\log\log N)^2} - R(\log N) + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}.$$

It follows from (258) that

$$(261) \qquad l(N) = e^{N+O\{\sqrt{N}(\log N)^2\}};$$

and from (259) that

$$(262) \qquad d\{l(N)\} = 2^{Li(N)+O(\sqrt{N}\log N)}.$$

51. Finally, we shall consider the number of divisors of $N!$. It is easily seen that

$$(263) \qquad N! = 2^{a_2}.3^{a_3}.5^{a_5}\ldots p^{a_p},$$

where p is the largest prime not greater than N, and

$$a_\lambda = \left[\frac{N}{\lambda}\right] + \left[\frac{N}{\lambda^2}\right] + \left[\frac{N}{\lambda^3}\right] + \ldots.$$

It is evident that the primes greater than $\frac{1}{2}N$ and not exceeding N appear once in $N!$, the primes greater than $\frac{1}{3}N$ and not exceeding $\frac{1}{2}N$ appear twice, and so on up to those greater than $N/[\sqrt{N}]$ and not exceeding $N/([\sqrt{N}]-1)$, appearing $[\sqrt{N}]-1$ times.* The indices of the smaller primes cannot be specified so simply. Hence it is clear that

$$(264) \qquad N! = e^{\vartheta(N)+\vartheta(\frac{1}{2}N)+\vartheta(\frac{1}{3}N)+\ldots+\vartheta\left(\frac{N}{[\sqrt{N}]-1}\right)} \times 2^{a_2}.3^{a_3}.5^{a_5}\ldots \varpi^{a_\varpi},$$

where ϖ is the largest prime not greater than $\sqrt{N}$, and

$$a_\lambda - 1 + [\sqrt{N}] = \left[\frac{N}{\lambda}\right] + \left[\frac{N}{\lambda^2}\right] + \left[\frac{N}{\lambda^3}\right] + \ldots.$$

From (264) we see that

$$\begin{aligned}(265) \qquad d(N!) &= 2^{\pi(N)}(\tfrac{3}{2})^{\pi(\frac{1}{2}N)}(\tfrac{4}{3})^{\pi(\frac{1}{3}N)}\ldots \text{ to } [\sqrt{N}]-1 \text{ factors} \\ &\qquad \times e^{O\{\log(1+a_2)+\log(1+a_3)+\ldots+\log(1+a_\varpi)\}} \\ &= 2^{\pi(N)}(\tfrac{3}{2})^{\pi(\frac{1}{2}N)}(\tfrac{4}{3})^{\pi(\frac{1}{3}N)}\ldots \text{ to } [\sqrt{N}]-1 \text{ factors} \\ &\qquad \times e^{O\{\varpi\log(1+a_2)\}} \\ &= 2^{Li(N)}(\tfrac{3}{2})^{Li(\frac{1}{2}N)}(\tfrac{4}{3})^{Li(\frac{1}{3}N)}\ldots \text{ to } [\sqrt{N}] \text{ factors} \\ &\qquad \times e^{O(\sqrt{N}\log N)}.\end{aligned}$$

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* Strictly speaking, this is true only when $N \geqslant 4$.

Since

$$Li(N) = \frac{N}{\log N} + O\left\{\frac{N}{(\log N)^2}\right\},$$

we see that

$$(266) \qquad d(N!) = C^{\frac{N}{\log N} + O\left\{\frac{N}{(\log N)^2}\right\}},$$

where

$$C = (1+1)^1 (1+\tfrac{1}{2})^{\frac{1}{2}} (1+\tfrac{1}{3})^{\frac{1}{3}} (1+\tfrac{1}{4})^{\frac{1}{4}} \dots .$$

From this we can easily deduce that, if N is of the form $M!$, then

$$(267) \qquad d(N) = C^{\frac{\log N}{(\log\log N)^2} + \frac{2\log N \log\log\log N}{(\log\log N)^3} + O\left\{\frac{\log N}{(\log\log N)^3}\right\}},$$

where C is the same constant as in (266).

52. It is interesting in this connection to show how, by considering numbers of certain special forms, we can obtain lower limits for the maximum orders of the iterated functions $dd(n)$ and $ddd(n)$. By supposing that

$$N = 2^{2-1}.3^{3-1}\dots p^{p-1},$$

we can show that

$$(268) \qquad dd(n) > 4^{\frac{\sqrt{(2\log n)}}{\log\log n}}$$

for an infinity of values of n. By supposing that

$$N = 2^{2^{a_2}-1}.3^{3^{a_3}-1}\dots p^{p^{a_p}-1},$$

where

$$a_\lambda = \left[\frac{\log p}{\log \lambda}\right] - 1,$$

we can show that

$$(269) \qquad ddd(n) > (\log n)^{\log\log\log\log n}$$

for an infinity of values of n.

A CLASS OF DIFFRACTION PROBLEMS

By H. M. MACDONALD.

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IN the solution of the problem of the diffraction of electric waves by a perfectly conducting right circular cone, the constants of the requisite series were determined from the fact that these constants must be identical with the constants in the series which represents the solution of the corresponding potential problem, where both series are expressed in terms of the appropriate harmonic functions.* The solution of the problem of the diffraction of electric waves by a perfectly conducting wedge was also obtained in the same way.† The object of the present communication is to exhibit in analytical form the argument which demonstrates that the constants are the same for the solution of the diffraction problem as for the corresponding potential problem, and also to apply the method to the solution of the diffraction of waves of sound by a rigid wedge. Solutions of the corresponding potential problems are obtained in the form suitable for comparison with the diffraction problem.

The solution of the diffraction problem depends on the solution of linear partial differential equations of the form

$$Ew+\kappa^2 w = 0, \tag{1}$$

where E is some operator, and w satisfies certain boundary conditions, while the solution of the corresponding potential problem depends on the solution of equations of the form

$$Ew = 0, \tag{2}$$

with the same boundary conditions. When the space for which the solutions of (1) and (2) are required is the space bounded by $\xi = \xi_1$, $\xi = \xi_2$; $\eta = \eta_1$, $\eta = \eta_2$; $\zeta = \zeta_1$, $\zeta = \zeta_2$; where ξ, η, ζ are coordinates determining three families of surfaces, if the equations (1) and (2), when they are transformed to the new variables, take the form

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* *Electric Waves*, p. 90, 1902.

† *Electric Waves*, p. 187.

$$D_1 w + D_2 w + D_3 w + \kappa^2 w\,[f_1(\xi)+f_2(\eta)+f_3(\zeta)] = 0, \tag{3}$$

$$D_1 w + D_2 w + D_3 w \qquad\qquad = 0, \tag{4}$$

where D_1 involves only ξ and operations with respect to ξ, D_2 involves only η and operations with respect to η, and D_3 involves only ζ and operations with respect to ζ, the solutions of (3), when the boundary condition is that w or $\partial w/\partial n$ is given over the boundary, will be of the form

$$w = \Sigma L'M'N', \tag{5}$$

and the solution of (4) will be of the form

$$w = \Sigma LMN, \tag{6}$$

where L', M', N', L, M, N are harmonic functions, and L', M', N' assume the values L, M, N when κ vanishes.

The function L' satisfies a linear differential equation involving ξ only, M' satisfies a linear differential equation involving η only, N' satisfies a linear differential equation involving ζ only, and when only one of these three equations involves κ the value of w which satisfies (3) can be immediately obtained from the value of w which satisfies (4) with the same boundary conditions. If the equation which is satisfied by L' is the one that involves κ, then

$$M' = M, \qquad N' = N$$

for the differential equations, and the boundary conditions are the same in both cases. Hence the solution of the problem is reduced to finding the solution of the linear differential equation satisfied by L', which satisfies the boundary conditions, and takes the value L when κ vanishes; therefore, when the solution of (3) and (4) are expressed in terms of the appropriate harmonic functions the constants in the two series are identical. This applies when the space is that bounded by two concentric spheres, two coaxial right circular cones, and two planes through the axis.

In order to apply the method to the solution of the problem of the diffraction of waves of sound by a wedge it is necessary to obtain the solution of the corresponding potential problem in the appropriate form.

Green's Function for a Wedge.

Choosing polar coordinates r, θ, ϕ, the potential V at any point satisfies the differential equation

$$\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial V}{\partial\mu}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}+4\pi\rho=0, \tag{7}$$

where μ is $\cos\theta$, ρ is the density of the inducing distribution at the point, and the condition to be satisfied at the boundary is that V vanishes. Writing $r=ae^{-\lambda}$, equation (7) becomes

$$\frac{\partial^2 V}{\partial\lambda^2}-\frac{\partial V}{\partial\lambda}+\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial V}{\partial\mu}+\frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}+4\pi\rho a^2e^{-2\lambda}=0,$$

and, substituting $$V=Ue^{\frac{1}{2}\lambda},$$

the equation satisfied by U is

$$\frac{\partial^2 U}{\partial\lambda^2}-\tfrac{1}{4}U+\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial U}{\partial\mu}+\frac{1}{\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}+4\pi\rho a^2e^{-\frac{5}{2}\lambda}=0. \tag{8}$$

When the space for which equation (8) is satisfied is the space bounded by the spheres

$$r=a, \qquad r=ae^{-\lambda_0},$$

and the planes $$\phi=0, \qquad \phi=\phi_0,$$

the harmonic functions of λ and ϕ which vanish at the boundary are

$$\sin\frac{\pi l\lambda}{\lambda_0}, \qquad \sin\frac{\pi m\phi}{\phi_0},$$

where l and m are positive integers, and the harmonic function of μ is the function which satisfies the equation

$$\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial y}{\partial\mu}-\frac{m^2\pi^2}{\phi_0^2}\frac{y}{1-\mu^2}+n(n+1)\,y=0, \tag{9}$$

where n is determined by the condition that y is finite for all values of μ in the space, that is, when

$$1\geqslant\mu\geqslant-1.$$

Writing $m\pi/\phi_0=m'$, the solution of (9), which is finite when $\mu=1$, is

$$y=P_n^{-m'}(\mu),$$

and n is restricted by the condition that $P_n^{-m'}(\mu)$ is to be finite when $\mu=-1$. Now

$$P_n^{-m'}(\mu)=\frac{1}{\Gamma(1+m')}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m'}F[-n,\, n+1,\, 1+m',\, \tfrac{1}{2}(1-\mu)],$$

and $F[-n,\, n+1,\, 1+m',\, \frac{1}{2}(1-\mu)]$

$$= \{\tfrac{1}{2}(1+\mu)\}^{m'} F[n+m'+1,\, m'-n,\, 1+m',\, \tfrac{1}{2}(1-\mu)];$$

whence $$P_n^{-m'}(\mu) = \frac{(1-\mu^2)^{\frac{1}{2}m'}}{2^{m'}\Gamma(1+m')} F[n+m'+1,\, m'-n,\, 1+m',\, \tfrac{1}{2}(1-\mu)],$$

therefore, if $P_n^{-m'}(-1)$ is finite, $F(n+m'+1,\, m'-n,\, 1+m',\, 1)$ must be finite, and since the series diverges, it must consist of a finite number of terms, that is, either $n+m'+1$ or $m'-n$ is a negative integer. Again, $P_{-n-1}^{-m'}(\mu)$ and $P_n^{-m'}(\mu)$ are not different solutions of (9), therefore it is sufficient to take the positive values of n for which $n-m'$ is a positive integer. Hence the solution of equation (8) is

$$U = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} A_{lmn} \sin\frac{l\pi\lambda}{\lambda_0} \sin m'\phi\, P_n^{-m'}(\mu),$$

where $n-m'$ is a positive integer, and

$$\tfrac{1}{4}\lambda_0\phi_0 A_{lmn} \int_{-1}^{1} \{P_n^{-m'}(\mu)\}^2 d\mu \left[(n+\tfrac{1}{2})^2 + \frac{l^2\pi^2}{\lambda_0^2}\right]$$

$$= 4\pi a^2 \int_0^{\lambda_0}\int_0^{\phi_0}\int_{-1}^{1} \rho e^{-\frac{5}{2}\lambda} \sin\frac{l\pi\lambda}{\lambda_0} \sin\frac{m\pi\phi}{\phi_0} P_n^{-m'}(\mu)\, d\lambda\, d\phi\, d\mu,$$

and therefore

$$U = \frac{16\pi a^2}{\lambda_0\phi_0} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} \sin\frac{l\pi\lambda}{\lambda_0} \sin\frac{m\pi\phi}{\phi_0} P_n^{-m'}(\mu) \int_0^{\lambda_0}\int_0^{\phi_0}\int_{-1}^{1} \rho e^{-\frac{5}{2}\lambda'}$$

$$\times \sin\frac{l\pi\lambda'}{\lambda_0} \sin\frac{m\pi\phi'}{\phi_0} P_n^{-m'}(\mu')\, d\lambda'\, d\phi'\, d\mu' \Big/ \left[(n+\tfrac{1}{2})^2 + \frac{l^2\pi^2}{\lambda_0^2}\right] \int_{-1}^{1} \{P_n^{-m'}(\mu')\}^2 d\mu', \tag{10}$$

and for the case of a point charge q at the point λ_1, ϕ_1, μ_1, this becomes

$$U = \frac{16\pi q e^{\frac{1}{2}\lambda_1}}{\lambda_0\phi_0 a} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} \sin\frac{l\pi\lambda}{\lambda_0} \sin\frac{l\pi\lambda_1}{\lambda_0} \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu) P_n^{-m'}(\mu_1)$$

$$\Big/ \left[(n+\tfrac{1}{2})^2 + \frac{l^2\pi^2}{\lambda_0^2}\right] \int_{-1}^{1} \{P_n^{-m'}(\mu')\}^2 d\mu'. \tag{11}$$

It is known that*

$$\int_{\mu_0}^{1} P_n^{-m'}(\mu) P_{n_1}^{-m'}(\mu)\, d\mu$$

$$= \frac{1-\mu_0^2}{(n_1-n)(n_1+n+1)} \left[P_n^{-m}(\mu_0) \frac{dP_{n_1}^{-m'}(\mu_0)}{d\mu_0} - P_{n_1}^{-m'}(\mu_0) \frac{dP_n^{-m'}(\mu_0)}{d\mu_0}\right],$$

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* *Proc. London Math. Soc.*, Vol. XXXI, p. 265, 1900.

therefore

$$\int_0^1 \{P_n^{-m'}(\mu)\}^2 d\mu = \frac{1}{2n+1}\left[P_n^{-m'}(\mu)\frac{d^2P_n^{-m'}(\mu)}{d\mu\,dn} - \frac{dP_n^{-m'}(\mu)}{d\mu}\,\frac{dP_n^{-m'}(\mu)}{dn}\right]_{\mu=0}.$$

Now

$$P_n^{-m'}(\mu) = 2^{-m'}\cos\tfrac{1}{2}(n-m')\pi\,\frac{\Pi\{\frac{1}{2}(n-m'-1)\}}{\Pi\{\frac{1}{2}(n+m')\}\,\Pi(-\frac{1}{2})}\,(1-\mu^2)^{-\frac{1}{2}m'}$$

$$\times F[\tfrac{1}{2}(n-m'+1),\ -\tfrac{1}{2}(n+m'),\ \tfrac{1}{2},\ \mu^2]$$

$$+2^{-m'}\sin\tfrac{1}{2}(n-m')\pi\,\frac{\Pi\{\frac{1}{2}(n-m')\}}{\Pi\{\frac{1}{2}(n+m'-1)\}\,\Pi(\frac{1}{2})}\,\mu(1-\mu^2)^{-\frac{1}{2}m'}$$

$$\times F[\tfrac{1}{2}(n-m'+2),\ -\tfrac{1}{2}(n+m'-1),\ \tfrac{3}{2},\ \mu^2],$$

whence, when $\mu = 0$,

$$P_n^{-m'}(0) = 2^{-m'}\cos\tfrac{1}{2}(n-m')\pi\,\frac{\Pi\{\frac{1}{2}(n-m'-1)\}}{\Pi\{\frac{1}{2}(n+m')\}\,\Pi(-\frac{1}{2})},$$

$$\frac{dP_n^{-m'}(\mu)}{d\mu} = 2^{-m'}\sin\tfrac{1}{2}(n-m')\pi\,\frac{\Pi\{\frac{1}{2}(n-m')\}}{\Pi\{\frac{1}{2}(n+m'-1)\}\,\Pi(\frac{1}{2})},$$

$$\frac{dP_n^{-m'}(\mu)}{dn} = -\tfrac{1}{2}\pi\,.\,2^{-m'}\sin\tfrac{1}{2}(n-m')\pi\,\frac{\Pi\{\frac{1}{2}(n-m'-1)\}}{\Pi\{\frac{1}{2}(n+m')\}\,\Pi(-\frac{1}{2})}$$

$$+2^{-m'}\cos\tfrac{1}{2}(n-m')\pi\,\frac{d}{dn}\,\frac{\Pi\{\frac{1}{2}(n-m'-1)\}}{\Pi\{\frac{1}{2}(n+m')\}\,\Pi(\frac{1}{2})},$$

$$\frac{d^2P_n^{-m'}(\mu)}{d\mu\,dn} = \tfrac{1}{2}\pi\,.\,2^{-m'}\cos\tfrac{1}{2}(n-m')\pi\,\frac{\Pi\{\frac{1}{2}(n-m')\}}{\Pi\{\frac{1}{2}(n+m'-1)\}\,\Pi(\frac{1}{2})}$$

$$+2^{-m'}\sin\tfrac{1}{2}(n-m')\pi\,\frac{d}{dn}\,\frac{\Pi\{\frac{1}{2}(n-m')\}}{\Pi\{\frac{1}{2}(n+m'-1)\}\,\Pi(\frac{1}{2})},$$

and, since $n-m'$ is an integer,

$$\left[P_n^{-m'}(\mu)\frac{d^2P_n^{-m'}(\mu)}{d\mu\,dn} - \frac{dP_n^{-m'}(\mu)}{d\mu}\,\frac{dP_n^{-m'}(\mu)}{dn}\right]_{\mu=0}$$

$$= \tfrac{1}{2}\pi\,.\,2^{-2m'}\,\frac{\Pi\{\frac{1}{2}(n-m')\}\,\Pi\{\frac{1}{2}(n-m'-1)\}}{\Pi\{\frac{1}{2}(n+m')\}\,\Pi\{\frac{1}{2}(n+m'-1)\}\,\Pi(-\frac{1}{2})\Pi(\frac{1}{2})},$$

hence
$$\int_0^1 \{P_n^{-m'}(\mu)\}^2 d\mu = \frac{1}{2n+1}\,\frac{\Pi(n-m')}{\Pi(n+m')}.$$

Again, when $n-m'$ is an integer,

$$P_n^{-m'}(-\mu) = \cos(n-m')\pi\,P_n^{-m}(\mu),$$

therefore $$\int_{-1}^{1}\{P_n^{-m'}(\mu)\}^2\,d\mu = 2\int_{0}^{1}\{P_n^{-m'}(\mu)\}^2\,d\mu,$$

whence $$\int_{-1}^{1}\{P_n^{-m'}(\mu)\}^2\,d\mu = \frac{2}{2n+1}\,\frac{\Pi(n-m')}{\Pi(n+m')}.$$

The value of V is therefore given by

$$V = \frac{8\pi q}{a\lambda_0\phi_0}\,e^{\frac{1}{2}(\lambda+\lambda_1)}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=m'}^{\infty}\frac{2n+1}{(n+\frac{1}{2})^2+\dfrac{l^2\pi^2}{\lambda_0^2}}\,\frac{\Pi(n+m')}{\Pi(n-m')}$$
$$\times\sin\frac{l\pi\lambda}{\lambda_0}\sin\frac{l\pi\lambda_1}{\lambda_0}\sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0}\,P_n^{-m'}(\mu)P_n^{-m'}(\mu_1);$$

and effecting the summation for the values of l, this becomes

$$V = \frac{4\pi q}{a\phi_0}\,e^{\frac{1}{2}(\lambda+\lambda_1)}\sum_{m=0}^{\infty}\sum_{n=m'}^{\infty}\frac{\Pi(n+m')}{\Pi(n-m')}\,\frac{\cosh(n+\frac{1}{2})(\lambda_0-\lambda+\lambda_1)-\cosh(n+\frac{1}{2})(\lambda_0-\lambda-\lambda_1)}{\sinh(n+\frac{1}{2})\lambda_0}$$
$$\times\sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0}\,P_n^{-m'}(\mu)\,P_n^{-m'}(\mu_1),$$

when $\lambda > \lambda_1$, and

$$V = \frac{4\pi q}{a\phi_0}\,e^{\frac{1}{2}(\lambda+\lambda_1)}\sum_{m=0}^{\infty}\sum_{n=m'}^{\infty}\frac{\Pi(n+m')}{\Pi(n-m')}\,\frac{\cosh(n+\frac{1}{2})(\lambda_0-\lambda_1+\lambda)-\cosh(n+\frac{1}{2})(\lambda_0-\lambda-\lambda_1)}{\sinh(n+\frac{1}{2})\lambda_0}$$
$$\times\sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0}\,P_n^{-m'}(\mu)\,P_n^{-m'}(\mu_1),$$

when $\lambda < \lambda_1$.

When the space, for which the solution is required, is the space inside the sphere $r = a$, bounded by the planes $\phi = 0$, $\phi = \phi_0$, the value of λ_0 is infinite, and the corresponding expressions for V are

$$V = \frac{4\pi q}{\phi_0}\sum_{m=0}^{\infty}\sum_{n=m'}^{\infty}\frac{\Pi(n+m')}{\Pi(n-m')}\left\{\frac{r^n}{r_1^{n+1}}-\frac{r^n r_1^n}{a^{2n+1}}\right\}$$
$$\times\sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0}\,P_n^{-m'}(\mu)\,P_n^{-m'}(\mu_1)\quad(r < r_1),$$

$$V = \frac{4\pi q}{\phi_0}\sum_{m=0}^{\infty}\sum_{n=m'}^{\infty}\frac{\Pi(n+m')}{\Pi(n-m')}\left\{\frac{r_1^n}{r^{n+1}}-\frac{r^n r_1^n}{a^{2n+1}}\right\}$$
$$\times\sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0}\,P_n^{-m'}(\mu)\,P_n^{-m'}(\mu_1)\quad(r > r_1).$$

For the wedge, the space outside it being bounded by the planes

$\phi = 0$, $\phi = \phi_0$, the value of V is obtained by making $a = \infty$ in the above, and therefore

$$V = \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} \frac{\Pi(n+m')}{\Pi(n-m')} \frac{r^n}{r_1^{n+1}} \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu)\, P_n^{-m'}(\mu_1) \quad (r<r_1),$$

$$V = \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} \frac{\Pi(n+m')}{\Pi(n-m')} \frac{r_1^n}{r^{n+1}} \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu)\, P_n^{-m'}(\mu_1) \quad (r>r_1).$$

These expressions can be simplified by observing that the origin of the coordinates can be chosen so that the point charge is in the plane $\theta = \frac{1}{2}\pi$, then $\mu_1 = 0$, and the expressions for V become

$$V = \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} 2^{m'} \frac{\Pi\left\{\frac{1}{2}(n+m'-1)\right\}}{\Pi\left\{\frac{1}{2}(n-m')\right\}\Pi(-\frac{1}{2})} \cos\tfrac{1}{2}(n-m')\pi \, \frac{r^n}{r_1^{n+1}}$$
$$\times \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu) \quad (r<r_1),$$

$$V = \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} 2^{m'} \frac{\Pi\left\{\frac{1}{2}(n+m'-1)\right\}}{\Pi\left\{\frac{1}{2}(n-m')\right\}\Pi(-\frac{1}{2})} \cos\tfrac{1}{2}(n-m')\pi \, \frac{r_1^n}{r^{n+1}}$$
$$\times \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu) \quad (r>r_1);$$

or

$$\left.\begin{aligned}
V &= \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} 2^{m'} \frac{\Pi(m'+k-\frac{1}{2})}{\Pi(k)\,\Pi(-\frac{1}{2})} \cos k\pi \, \frac{r^{m'+2k}}{r_1^{m'+2k+1}} \\
&\qquad \times \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_{m'+2k}^{-m'}(\mu) \quad (r<r_1) \\
V &= \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} 2^{m'} \frac{\Pi(m'+k-\frac{1}{2})}{\Pi(k)\,\Pi(-\frac{1}{2})} \cos k\pi \, \frac{r_1^{m'+2k}}{r^{m'+2k+1}} \\
&\qquad \times \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_{m'+2k}^{-m'}(\mu) \quad (r>r_1)
\end{aligned}\right\} . \quad (12)$$

These expressions give V in terms of the harmonic functions appropriate to the space; the results can be obtained from them in the form previously given by the writer* as follows.

It is known that

$$P_{m'+2k}^{-m'}(\mu) = \frac{2^{m'}\Pi(m'-\frac{1}{2})\,\Pi(2k)}{\Pi(2m'+2k)\,\Pi(-\frac{1}{2})} (1-\mu^2)^{\frac{1}{2}m}\, C_{2k}^{m'+\frac{1}{2}}(\mu),$$

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* *Proc. London Math. Soc.*, Vol. XXVI, p. 160, 1895.

where $C_{2k}^{m'+\frac{1}{2}}(\mu)$ is the coefficient of x^{2k} in the development of

$$(1-2\mu x+x^2)^{m'+\frac{1}{2}}$$

in a series of powers of x, hence

$$V=\frac{4q}{r_1\phi_0}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{r}{r_1}\right)^{m'+2k}(1-\mu^2)^{\frac{1}{2}m'}\cos k\pi\,\frac{\Pi(m'-\frac{1}{2})\,\Pi(k-\frac{1}{2})}{\Pi(m'+k)}$$

$$\times\sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0}C_{2k}^{m'+\frac{1}{2}}(\mu)\quad(r<r_1),$$

where r and r_1 are interchanged, when $r>r_1$. Writing

$$S_m=\sum_{k=0}^{\infty}\left(\frac{r}{r_1}\right)^{2k}\frac{\Pi(m'-\frac{1}{2})\,\Pi(k-\frac{1}{2})}{\Pi(m'+k)}\cos k\pi\,.\,C_{2k}^{m'+\frac{1}{2}}(\mu),$$

it follows that

$$S_m=\sum_{k=0}^{\infty}\int_{-1}^{1}(1-u^2)^{m'-\frac{1}{2}}u^{2k}x^{2k}\cos k\pi\,.\,C_{2k}^{m'+\frac{1}{2}}(\mu)\,du,$$

where $x=r/r_1$, hence

$$S_m=\int_{-1}^{1}\frac{(1-u^2)^{m'-\frac{1}{2}}\,du}{(1-2\iota\mu xu-x^2u^2)^{m'+\frac{1}{2}}},$$

and, substituting

$$1-2\iota\mu xu-x^2u^2=(1-u^2)\,x\sin\theta\,.\,e^{-\iota\zeta},$$

this becomes

$$S_m=\frac{\iota}{2}(x\sin\theta)^{-m'}\int_{\iota\infty-\gamma}^{\iota\infty+\gamma}\frac{e^{m'\zeta\iota}\,d\zeta}{\sqrt{(2x\sin\theta\cos\zeta-1-x^2)}},$$

where the path of integration lies wholly in the upper part of the ζ-plane and passes between the origin and the branch point $\iota\zeta_0$, where

$$\cosh\zeta_0=(1+x^2)/2x\sin\theta$$

of the integrand which lies on the imaginary axis in the upper half, but none of the other branch points $\pm\iota\zeta_0\pm 2n\pi$. Now

$$V=\frac{4q}{r_1\phi_0}\sum_{m=0}^{\infty}\left(\frac{r}{r_1}\right)^{m'}S_m\sin^{m'}\theta\sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0}\quad(r<r_1),$$

therefore

$$V=\frac{2q\iota}{\phi_0}\sum_{m=0}^{\infty}\int_{\iota\infty-\gamma}^{\iota\infty+\gamma}\frac{e^{m\pi\zeta\iota/\phi_0}\,d\zeta}{\sqrt{(2rr_1\sin\theta\cos\zeta-r^2-r_1^2)}}\sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi}{\phi_1},$$

or $$V = \frac{q}{2\phi_0}\int_{\iota\infty-\gamma}^{\iota\infty+\gamma} \frac{d\zeta}{\sqrt{(2rr_1\sin\theta\cos\zeta - r^2 - r_1^2)}}$$

$$\times\left\{\frac{\sin\dfrac{\pi\zeta}{\phi_0}}{\cos\dfrac{\pi\zeta}{\phi_0} - \cos\dfrac{\pi}{\phi_0}(\phi-\phi_1)} - \frac{\sin\dfrac{\pi\zeta}{\phi_0}}{\cos\dfrac{\pi\zeta}{\phi_0} - \cos\dfrac{\pi}{\phi_0}(\phi+\phi_1)}\right\},$$

that is, writing for ζ, $\iota\zeta$ and $(r^2+r_1^2)/2rr_1\sin\theta = \cosh\eta$,

$$V = \frac{q}{\phi_0\sqrt{(2rr_1\sin\theta)}}\int_\eta^\infty \frac{d\zeta}{\sqrt{(\cosh\zeta - \cosh\eta)}}$$

$$\times\left\{\frac{\sinh\dfrac{\pi\zeta}{\phi_0}}{\cosh\dfrac{\pi\zeta}{\phi_0} - \cos\dfrac{\pi}{\phi_0}(\phi-\phi_1)} - \frac{\sinh\dfrac{\pi\zeta}{\phi_0}}{\cosh\dfrac{\pi\zeta}{\phi_0} - \cos\dfrac{\pi}{\phi_0}(\phi+\phi_1)}\right\}; \quad (13)$$

the result in its previous form.

The velocity potential of the motion of liquid, in the space bounded by the planes $\phi = 0$, $\phi = \phi_0$, due to a source of strength q at the point r_1, $\frac{1}{2}\pi$, ϕ_1, is immediately obtained in a similar way, and is given by

$$\frac{2\pi q}{r_1\phi_0}\sum_{k=0}^{\infty}\left(\frac{r}{r_1}\right)^{2k}\frac{\Pi(k-\frac{1}{2})}{\Pi(k)\,\Pi(-\frac{1}{2})}\cos k\pi\,P_{2k}(\mu) + \frac{4\pi q}{r_1\phi_0}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}2^{m'}$$

$$\times\cos k\pi\frac{\Pi(m'+k-\frac{1}{2})}{\Pi(k)\,\Pi(-\frac{1}{2})}\left(\frac{r}{r_1}\right)^{m'+2k}P_{m'+2k}^{-m'}(\mu)\cos\frac{m\pi\phi}{\phi_0}\cos\frac{m\pi\phi_1}{\phi_0} \quad (r < r_1), \quad (14)$$

and the corresponding expression with r and r_1 interchanged, when $r > r_1$. These expressions can, as above, be transformed into

$$\frac{q}{\phi_0\sqrt{(2rr_1\sin\theta)}}\int_\eta^\infty \frac{d\zeta}{\sqrt{(\cosh\zeta - \cosh\eta)}}$$

$$\times\left\{\frac{\sinh\dfrac{\pi\zeta}{\phi_0}}{\cosh\dfrac{\pi\zeta}{\phi_0} - \cos\dfrac{\pi}{\phi_0}(\phi-\phi_1)} + \frac{\sinh\dfrac{\pi\zeta}{\phi_0}}{\cosh\dfrac{\pi\zeta}{\phi_0} - \cos\dfrac{\pi}{\phi_0}(\phi+\phi_1)}\right\},$$

where $$\cosh\eta = (r^2+r_1^2)/2rr_1\sin\theta.$$

The solution of the problem of the diffraction of waves of sound due to a source at the point r_1, $\frac{1}{2}\pi$, ϕ_1 is obtained as follows. The linear differential equation that involves κ, where $2\pi/\kappa$ is the wave length, is

$$\frac{d^2L}{dr^2} + \frac{2}{r}\frac{dL}{dr} + \left\{\kappa^2 - \frac{n(n+1)}{r^2}\right\}L = 0,$$

and the solutions of this equation are required which tend to the value r^n/r_1^{n+1}, when $r < r_1$, and to the value r_1^n/r^{n+1}, when $r > r_1$. Since L is finite when $r = 0$, the solution, when $r < r_1$, is

$$L = Ar^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa r),$$

and, since the waves are all travelling outwards at infinity, the solution, when $r > r_1$, is

$$L = Br^{-\frac{1}{2}}K_{n+\frac{1}{2}}(\iota\kappa r);$$

and, since the two solutions are identical, when $r = r_1$,

$$L = Cr^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa r)\,K_{n+\frac{1}{2}}(\iota\kappa r_1) \quad (r < r_1),$$

$$L = Cr^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa r_1)K_{n+\frac{1}{2}}(\iota\kappa r) \quad (r > r_1),$$

Now, when $\kappa = 0$,

$$L = \frac{C}{2n+1}\,\frac{r^n}{r_1^{n+\frac{1}{2}}}\,e^{-(n+\frac{1}{2})\frac{1}{2}\pi\iota} \quad (r < r_1),$$

therefore
$$\frac{C}{2n+1}\,\frac{r^n}{r_1^{n+\frac{1}{2}}}\,e^{-(n+\frac{1}{2})\frac{1}{2}\pi\iota} = \frac{r^n}{r_1^{n+1}},$$

whence
$$C = (2n+1)\,e^{(n+\frac{1}{2})\frac{1}{2}\pi\iota}\,r_1^{-\frac{1}{2}},$$

and, if the velocity potential due to the source alone is $e^{\iota\kappa(at-R)}/R$ at a distance R from it, where a is the velocity of sound, the velocity potential of the wave motion is $Ue^{\iota\kappa at}$, where

$$U = \frac{2\pi}{\phi_0}\sum_{k=0}^{\infty}(4k+1)e^{(2k+\frac{1}{2})\frac{1}{2}\pi\iota}r^{-\frac{1}{2}}r_1^{-\frac{1}{2}}J_{2k+\frac{1}{2}}(\kappa r)K_{2k+\frac{1}{2}}(\iota\kappa r_1)\cos k\pi\frac{\Pi(k-\frac{1}{2})}{\Pi(k)\Pi(-\frac{1}{2})}P_{2k}(\mu)$$

$$+\frac{4\pi}{\phi_0}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}2^{m'}(2m'+4k+1)\,e^{(m'+2k+\frac{1}{2})\frac{1}{2}\pi\iota}r^{-\frac{1}{2}}r_1^{-\frac{1}{2}}J_{m'+2k+\frac{1}{2}}(\kappa r)\,K_{m'+2k+\frac{1}{2}}(\iota\kappa r_1)$$

$$\times\cos k\pi\frac{\Pi(m'+k-\frac{1}{2})}{\Pi(k)\,\Pi(-\frac{1}{2})}\,P_{m'+2k}^{-m'}(\mu)\cos\frac{m\pi\phi}{\phi_0}\cos\frac{m\pi\phi_1}{\phi_0} \quad (\text{when } r < r_1), \quad (15)$$

and the corresponding expression with r and r_1 interchanged, when $r > r_1$. Now*

$$J_{m'+2k+\frac{1}{2}}(\kappa r)\,K_{m'+2k+\frac{1}{2}}(\iota\kappa r_1)$$
$$= -\tfrac{1}{2}e^{-(m'+2k+\frac{1}{2})\frac{1}{2}\pi\iota}\int_{c-\infty\iota}^{0} e^{\frac{1}{2}s-\kappa^2(r^2+r_1^2)/2s}\,I_{m'+2k+\frac{1}{2}}\left(\frac{\kappa^2 rr_1}{s}\right)\frac{ds}{s},$$

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* *Proc. London Math. Soc.*, Vol. XXXII, p. 155, 1900.

and*
$$I_{m'+2k+\frac{1}{2}}\left(\frac{\kappa^2 r r_1}{s}\right) = -\frac{1}{2\pi}\int_{\infty\iota+\gamma}^{\infty\iota+\gamma'} e^{\kappa^2 r r_1/s\,\cdot\,\cos\zeta+(m'+2k+\frac{1}{2})\zeta\iota}\,d\zeta,$$

where
$$2\pi > \gamma > \pi, \qquad 0 > \gamma' > -\pi,$$

hence, writing

$$S_m = 2^{m'} \sum_{k=0} (2m'+4k+1)\, e^{(m'+2k+\frac{1}{2})\frac{1}{2}\pi\iota}$$
$$\times \cos k\pi\, J_{m'+2k+\frac{1}{2}}(\kappa r)\, K_{m'+2k+\frac{1}{2}}(\iota\kappa r_1)\, \frac{\Pi(m'+k-\frac{1}{2})}{\Pi(k)\,\Pi(-\frac{1}{2})}\, P_{m'+2k}^{-m'}(\mu),$$

that is,

$$S_m = \frac{1}{\pi}\sin^{m'}\theta \sum_{k=0}^{\infty} (2m'+4k+1)\, e^{(m'+2k+\frac{1}{2})\frac{1}{2}\pi\iota}$$
$$\times \cos k\pi\, J_{m'+2k+\frac{1}{2}}(\kappa r)\ K_{m'+2k+\frac{1}{2})}(\iota\kappa r_1)\, \frac{\Pi(m'-\frac{1}{2})\,\Pi(k-\frac{1}{2})}{\Pi(m'+k)}\, C_{2k}^{m'+\frac{1}{2}}(\mu),$$

it follows that

$$S_m = \frac{1}{4\pi^2}\sin^{m'}\theta \sum_{k=0}^{\infty} (2m'+4k+1)$$
$$\times \cos k\pi\, \frac{\Pi(m'-\frac{1}{2})\,\Pi(k-\frac{1}{2})}{\Pi(m'+k)} \int_{c-\infty\iota}^{0} \int_{\infty\iota+\gamma}^{\infty\iota+\gamma'} e^{\frac{1}{2}s-\kappa^2/2s\,(r^2+r_1^2-2rr_1\cos\zeta)+(m'+2k+\frac{1}{2})\zeta\iota}$$
$$\times s^{-1}\,d\zeta\,ds\; C_{2k}^{m'+\frac{1}{2}}(\mu),$$

or, integrating by parts with respect to ζ,

$$S_m = \frac{1}{2\pi^2\iota}\sin^{m'}\theta \sum_{k=0}^{\infty} \cos k\pi\, \frac{\Pi(m'-\frac{1}{2})\,\Pi(k-\frac{1}{2})}{\Pi(m'+k)}\, C_{2k}^{m'+\frac{1}{2}}(\mu)\, \kappa^2 r r_1 \int_{\iota-\infty\iota}^{0} \int_{\infty\iota+\gamma}^{\infty\iota+\gamma'}$$
$$\times e^{\frac{1}{2}s-\kappa^2\,2s\,(r^2+r_1^2-2rr_1\cos\zeta)+(m'+2k+\frac{1}{2})\zeta\iota}\sin\zeta\,d\zeta\,\frac{ds}{s^2}.$$

Now, by the above,

$$\sum_{k=0}^{\infty} \cos k\pi\, \frac{\Pi(m'-\frac{1}{2})\,\Pi(k-\frac{1}{2})}{\Pi(m'+k)}\, e^{2k\zeta\iota}\, C_{2k}^{m'+\frac{1}{2}}(\mu)$$
$$= \tfrac{1}{2}\iota\,(e^{\zeta\iota}\sin\theta)^{-m'} \int_{\infty\iota-\gamma_1}^{\infty\iota+\gamma_1'} \frac{e^{m'\zeta_1\iota}\,d\zeta_1}{(2e^{\zeta\iota}\sin\theta\cos\zeta_1-1-e^{2\zeta\iota})^{\frac{1}{2}}},$$

where the extremities of the path of integration lie in the upper half of the plane, and the path encloses the branch point of the integrand that is continuous with the branch point of the integrand on the positive imaginary axis when ζ is a pure imaginary, therefore

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* *Electric Waves*, p. 191, 1902.

$$S_m = \frac{\kappa^2 r r_1}{4\pi^2} \int_{c-\infty\iota}^{0} \int_{\infty\iota+\gamma}^{\infty\iota+\gamma'} \int_{\infty\iota-\gamma_1}^{\infty\iota+\gamma_1'} e^{\frac{1}{2}s-\kappa^2/2s\,(r^2+r_1^2-2rr_1\cos\zeta)+m'\zeta_1\iota}$$
$$\times\, 2^{-\frac{1}{2}} (\sin\theta\cos\zeta_1 - \cos\zeta)^{-\frac{1}{2}} \sin\zeta \, d\zeta_1 \, d\zeta \, \frac{ds}{s^2},$$

whence, changing the order of integration, and integrating with respect to ζ,

$$S_m = -(2\pi)^{-\frac{3}{2}} \kappa (rr_1)^{\frac{1}{2}} \int_{c-\infty\iota}^{0} \int_{\infty\iota-\gamma_1}^{\infty\iota+\gamma_1'} e^{\frac{1}{2}s-\kappa^2/2s\,(r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)+m'\zeta_1\iota} \, s^{-\frac{3}{2}} \, d\zeta_1 \, ds,$$

where $$2\pi > \gamma_1' > \pi, \quad \pi > \gamma_1 > 0;$$

or, replacing the integral with respect to s by its known value,

$$S_m = \frac{1}{2\pi} (rr_1)^{\frac{1}{2}} \int_{\infty\iota-\gamma_1}^{\infty\iota+\gamma_1'} e^{-\iota\kappa\sqrt{(r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)}+m'\zeta_1\iota} (r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)^{-\frac{1}{2}} d\zeta_1,$$

and, substituting this result in the expression for U, it becomes

$$U = -\frac{1}{\phi_0} \int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1} e^{-\iota\kappa\sqrt{(r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)}} (r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)^{-\frac{1}{2}}$$
$$\times \left(1 + 2\sum_{1}^{\infty} e^{m\pi\zeta_1\iota/\phi_0} \cos\frac{m\pi\phi}{\phi_0} \cos\frac{m\pi\phi_1}{\phi_0}\right) d\zeta_1,$$

that is,

$$U = \frac{\iota}{2\phi_0} \int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1} e^{-\iota\kappa\sqrt{(r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)}} (r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)^{-\frac{1}{2}}$$
$$\times \left\{ \frac{\sin\frac{\pi\zeta_1}{\phi_0}}{\cos\frac{\pi\zeta_1}{\phi_0} - \cos\frac{\pi}{\phi_0}(\phi-\phi_1)} + \frac{\sin\frac{\pi\zeta_1}{\phi_0}}{\cos\frac{\pi\zeta_1}{\phi_0} - \cos\frac{\pi}{\phi_0}(\phi+\phi_1)} \right\}.^*$$

When $\phi_0 = \pi/n$, where n is an integer, the result agrees with that obtained by the method of images, for

$$\frac{n\sin n\zeta}{\cos n\zeta - \cos n(\phi\pm\phi_1)} = \sum_{k=0}^{n-1} \frac{\sin\zeta}{\cos\zeta - \cos(\phi\pm\phi_1 - 2k\pi/n)},$$

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* In the particular case when $\phi_0 = 2\pi$, this result is equivalent to that given by Carslaw *Proc. London Math. Soc.*, Vol. xxx, p. 139, 1898.

and each term of the sum is immediately integrable, giving

$$U = \sum_{k=0}^{n-1} \Big[e^{-\iota\kappa\sqrt{\{r^2+r_1{}^2-2rr_1 \sin\theta \cos(\phi-\phi_1-2k\pi/n)\}}} \times \{r^2+r_1^2-2rr_1 \sin\theta \cos(\phi-\phi_1-2k\pi/n)\}^{-\frac{1}{2}} \Big]$$

$$+ e^{-\iota\kappa\sqrt{\{r^2+r_1{}^2-2rr_1 \sin\theta \cos(\phi+\phi_1-2k\pi/n)\}}} \times \{r^2+r_1^2-2rr_1 \sin\theta \cos(\phi+\phi_1-2k\pi/n)\}^{-\frac{1}{2}}.$$

For the case of the straight edge or half plane $\phi_0 = 2\pi$, and

$$U = \frac{\iota}{4\pi} \int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1} e^{-\iota\kappa\sqrt{(r^2+r_1{}^2-2rr_1 \sin\theta \cos\zeta)}} (r^2+r_1^2-2rr_1 \sin\theta \cos\zeta)^{-\frac{1}{2}} \times \left\{ \frac{\sin\frac{1}{2}\zeta}{\cos\frac{1}{2}\zeta-\cos\frac{1}{2}(\phi-\phi_1)} + \frac{\sin\frac{1}{2}\zeta}{\cos\frac{1}{2}\zeta-\cos\frac{1}{2}(\phi+\phi_1)} \right\} d\zeta,$$

which can be expressed in terms of integrals between real limits in a form suitable for approximation as follows. Writing

$$I = \frac{\iota}{4\pi} \int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1} e^{-\iota\kappa\sqrt{(r^2+r_1{}^2-2rr_1 \sin\theta \cos\zeta)}} \times (r^2+r_1^2-2rr_1 \sin\theta \cos\zeta)^{-\frac{1}{2}} \frac{\sin\frac{1}{2}\zeta}{\cos\frac{1}{2}\zeta-\cos\frac{1}{2}(\phi-\phi_1)} d\zeta,$$

it follows that

$$I = -\tfrac{1}{2}\iota\kappa(2\pi)^{-\frac{3}{2}} \int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1} \int_{c-\infty\iota}^{0} e^{\frac{1}{2}s-\kappa^2/2s\,(r^2+r_1{}^2-2rr_1 \sin\theta \cos\zeta)} \frac{\sin\frac{1}{2}\zeta \, d\zeta}{\cos\frac{1}{2}\zeta-\cos\frac{1}{2}(\phi-\phi_1)} \frac{ds}{s^{\frac{3}{2}}},$$

and writing

$$W = \int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1} e^{\kappa^2 rr_1/s\, \sin\theta \cos\zeta} \frac{\sin\frac{1}{2}\zeta}{\cos\frac{1}{2}\zeta-\cos\frac{1}{2}(\phi-\phi_1)} d\zeta,$$

this becomes, on substituting

$$\cos\tfrac{1}{2}\zeta = t \cos\tfrac{1}{2}(\phi-\phi_1),$$

$$W = 2\int_{t_0}^{t_1} e^{\kappa^2 rr_1/s\, \sin\theta\, [2t^2 \cos^2\frac{1}{2}(\phi-\phi_1)-1]} \frac{dt}{t-1},$$

where, when $\cos\frac{1}{2}(\phi-\phi_1)$ is positive,

$$t_0 = c'+\infty\iota, \qquad t_1 = -c'-\infty\iota,$$

and the path of integration crosses the positive real axis to the right of the point $t = 1$, since, when ζ is a pure imaginary t is greater than unity;

when $\cos\frac{1}{2}(\phi-\phi_1)$ is negative,

$$t_0 = -c'-\infty\iota, \qquad t_1 = c'+\infty\iota,$$

and the path of integration cuts the real axis to the left of the origin since t is negative when ζ is a pure imaginary. Hence, when $\cos\frac{1}{2}(\phi-\phi_1)$ is positive,

$$W = 2e^{-\kappa^2 rr_1/s\sin\theta}\int_{c'+\infty\iota}^{-c'-\infty\iota} e^{2\kappa^2 rr_1/s\, t^2\sin\theta\cos^2\frac{1}{2}(\phi-\phi_1)}\frac{dt}{t-1},$$

that is, deforming the path so as to make it coincide with the imaginary axis, since it passes over the singularity $t=1$,

$$W = -4\pi\iota\, e^{-\kappa^2 rr_1/s\sin\theta\cos(\phi-\phi_1)} + 2e^{-\kappa^2 rr_1/s\sin\theta}\int_{\infty\iota}^{-\infty\iota} e^{2\kappa^2 rr_1/s\, t^2\sin\theta\cos^2\frac{1}{2}(\phi-\phi_1)}\frac{dt}{t-1};$$

now writing $$W' = \int_{\infty\iota}^{-\infty\iota} e^{ht^2}\frac{dt}{t-1},$$

it follows that $$W' = -2\int_0^{\infty\iota}\frac{e^{ht^2}}{t^2-1}dt = 2\iota\int_0^\infty\frac{e^{-ht^2}dt}{t^2+1},$$

or $$W' = 2\iota\int_0^\infty\int_0^\infty e^{-(h+u)t^2-u}\,dt\,du = \iota\sqrt{\pi}\int_0^\infty e^{-u}(h+u)^{-\frac{1}{2}}du,$$

that is, writing $$h+u = w^2/2s,$$

since the real part of s is always positive,

$$W' = \iota(2\pi)^{\frac{1}{2}}\int_{w_0}^\infty e^{-w^2/2s}s^{-\frac{1}{2}}dw,$$

where $$w_0 = (2sh)^{\frac{1}{2}};$$

therefore

$$W = -4\pi\iota e^{-\kappa^2 rr_1/s\sin\theta\cos(\phi-\phi_1)} + 2\iota(2\pi)^{\frac{1}{2}}s^{-\frac{1}{2}}e^{\kappa^2 rr_1/s\sin\theta\cos(\phi-\phi_1)}\int_{w_0}^\infty e^{-w^2/2s}dw.$$

where $$w_0 = 2\kappa(rr_1\sin\theta)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi-\phi_1).$$

Hence

$$I = -(2\pi)^{-\frac{1}{2}}\kappa\int_{c-\infty\iota}^{0} e^{\frac{1}{2}s-\kappa^2/2s\{r^2+r_1^2-2rr_1\sin\theta\cos(\phi-\phi_1)\}}s^{-\frac{3}{2}}ds$$

$$+(2\pi)^{-1}\kappa\int_{c-\infty\iota}^{0}\int_{w_0}^\infty e^{\frac{1}{2}s-\kappa^2/2s\{r^2+r_1^2-2rr_1\sin\theta\cos(\phi-\phi_1)\}-w^2/2s}s^{-2}ds\,dw,$$

and writing $$R^2 = r^2+r_1^2-2rr_1 \sin\theta \cos(\phi-\phi_1),$$

where R is the distance of the point (r, θ, ϕ) from the source, this becomes

$$I = \frac{e^{-\iota\kappa R}}{R} - \frac{\iota\kappa}{\pi}\int_{w_0}^{\infty} \frac{K_1\{\iota\sqrt{(\kappa^2R^2+w^2)}\}}{\sqrt{(\kappa^2R^2+w^2)}}\,dw,$$

that is $$I = \frac{e^{-\iota\kappa R}}{R} - \frac{\iota\kappa}{\pi}\int_{\xi_0}^{\infty} K_1(\iota\kappa R\cosh\xi)\,d\xi,$$

where $$\sinh\xi_0 = 2R^{-1}(rr_1\sin\theta)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi-\phi_1)\;;$$

now $$\frac{\iota\kappa}{\pi}\int_{-\infty}^{\infty} K_1(\iota\kappa R\cosh\xi)\,d\xi = \frac{e^{-\iota\kappa R}}{R},$$

therefore the result can be written in the form

$$I = \frac{\iota\kappa}{\pi}\int_{-\infty}^{\xi_0} K_1(\iota\kappa R\cosh\xi)\,d\xi.$$

Again, when $\cos\frac{1}{2}(\phi-\phi_1)$ is negative,

$$W = 2e^{-\kappa^2rr_1/s\sin\theta}\int_{-c'-\infty\iota}^{c'+\infty\iota} e^{2\kappa^2rr_1/s\sin\theta\cos^2\frac{1}{2}(\phi-\phi_1)\,t^2}\,\frac{dt}{t-1},$$

and, deforming the path of integration so as to make it coincide with the imaginary axis, since it does not pass over the pole of the integrand

$$W = 2\int_{-\infty\iota}^{\infty\iota} e^{2\kappa^2rr_1/s\sin\theta\cos^2\frac{1}{2}(\phi-\phi_1)\,t^2}\,\frac{dt}{t-1},$$

whence, by the above,

$$W = -2\iota(2\pi)^{\frac{1}{2}}s^{-\frac{1}{2}}e^{\kappa^2rr_1/s\sin\theta\cos(\phi-\phi_1)}\int_{w_0}^{\infty} e^{-w^2/2s}\,dw,$$

where $$w_0 = -2\kappa(rr_1\sin\theta)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi-\phi_1)\;;$$

and therefore $$I = \frac{\iota\kappa}{\pi}\int_{\xi_0'}^{\infty} K_1(\iota\kappa R\cosh\xi)\,d\xi,$$

where $$\sinh\xi_0' = -2R^{-1}(rr_1\sin\theta)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi-\phi_1),$$

and this can be written

$$I = \frac{\iota\kappa}{\pi}\int_{-\infty}^{\xi_0} K_1(\iota\kappa R\cosh\xi)\,d\xi,$$

where $$\sinh\xi_0 = 2R^{-1}(rr_1\sin\theta)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi-\phi_1).$$

Similarly, writing

$$I_1 = \frac{\iota}{4\pi}\int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1} e^{-\iota\kappa\sqrt{(r^2+r_1^2-2rr_1\sin\theta\cos\zeta)}}\,(r^2+r_1^2-2rr_1\sin\theta\cos\zeta)^{-\frac{1}{2}} \times \frac{\sin\frac{1}{2}\zeta}{\cos\frac{1}{2}\zeta-\cos\frac{1}{2}(\phi-\phi_1)}\,d\zeta,$$

it follows that
$$I_1 = \frac{\iota\kappa}{\pi}\int_{-\infty}^{\xi_1} K_1(\iota\kappa R'\cosh\xi)\,d\xi,$$

where
$$R'^2 = r^2+r_1^2-2rr_1\sin\theta\cos(\phi+\phi_1),$$
$$\sinh\xi_1 = 2R'^{-1}(rr_1\sin\theta)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi+\phi_1),$$

and R' is the distance of the point (r, θ, ϕ) from the image of the source in the plane $\phi = 0$. Hence

$$U = \frac{\iota\kappa}{\pi}\int_{-\infty}^{\xi_0} K_1(\iota\kappa R\cosh\xi)\,d\xi + \frac{\iota\kappa}{\pi}\int_{-\infty}^{\xi_1} K_1(\iota\kappa R'\cosh\xi)\,d\xi.$$

The corresponding expression for the velocity potential of the liquid motion in the space due to a source of unit strength is obtained from this by substituting $\kappa = 0$, and gives

$$\frac{1}{\pi R}\int_{-\infty}^{\xi_0}\frac{d\xi}{\cosh\xi} + \frac{1}{\pi R'}\int_{-\infty}^{\xi_1}\frac{d\xi}{\cosh\xi},$$

that is
$$\frac{1}{R}\left\{\frac{1}{2}+\frac{2}{\pi}\tan^{-1}(\tanh\tfrac{1}{2}\xi_0)\right\} + \frac{1}{R'}\left\{\frac{1}{2}+\frac{2}{\pi}\tan^{-1}(\tanh\tfrac{1}{2}\xi_1)\right\},$$

and the corresponding expression for Green's function is

$$\frac{1}{R}\left\{\frac{1}{2}+\frac{2}{\pi}\tan^{-1}(\tanh\tfrac{1}{2}\xi_0)\right\} - \frac{1}{R'}\left\{\frac{1}{2}+\frac{2}{\pi}\tan^{-1}(\tanh\tfrac{1}{2}\xi_1)\right\},$$

a known result.

The expression given above for U is in a form suitable for approximation. When R is small, that is, when the point (r, θ, ϕ) is near to the source, ξ_0 is very large and tends to infinity as R tends to zero, hence the value of the first integral tends to $e^{-\iota\kappa R}/R$. When the point (r, θ, ϕ) is not near to the source, R is finite and $R\cosh\xi$ is not small within the range of integration, hence $K_1(\iota\kappa R\cosh\xi)$ can be replaced by its asymptotic expansion, and the principal part of the first integral is given by

$$I = \frac{\iota\kappa}{\pi}\int_{-\infty}^{\xi_0} K_1(\iota\kappa R\cosh\xi)\,d\xi = \kappa^{\frac{1}{2}}(2\pi R)^{-\frac{1}{2}}e^{\frac{1}{4}\pi\iota}\int_{-\infty}^{\xi_0} e^{-\iota\kappa R\cosh\xi}(\cosh\xi)^{-\frac{1}{2}}\,d\xi,$$

that is, writing
$$(2\kappa R)^{\frac{1}{2}}\sinh\tfrac{1}{2}\xi = u,$$

by $$\pi^{-\frac{1}{2}}R^{-1}e^{\frac{1}{4}\pi\iota-\iota\kappa R}\int_{-\infty}^{u_0}\left(1+\frac{u^2}{\kappa R}\right)^{-\frac{1}{2}}\left(1+\frac{u^2}{2\kappa R}\right)^{-\frac{1}{2}}e^{-\iota u^2}\,du,$$

where $$u_0=\pm\sqrt{\{(r^2+r_1^2+2rr_1\sin\theta)^{\frac{1}{2}}-R\}}\,\kappa^{\frac{1}{2}},$$

and the upper or lower sign of the radical is taken according as $\cos\frac{1}{2}(\phi-\phi_1)$ is positive or negative, hence the principal value of I is $\bar{I}$, where

$$\bar{I}=\pi^{-\frac{1}{2}}R^{-1}e^{\frac{1}{4}\pi\iota-\iota\kappa R}\int_{-\infty}^{u_0}e^{-\iota u^2}\,du.$$

In the second integral the Bessel's function can always be replaced by its asymptotic expansion, since as R' tends to zero, $R'\cosh\xi$ is always finite within the range of integration, being always not less than

$$\sqrt{(r^2+r_1^2+2rr_1\sin\theta)},$$

its value at the upper limit ξ_1; hence the principal part of I_1 is

$$\pi^{-\frac{1}{2}}\kappa e^{\frac{1}{4}\pi\iota-\iota\kappa R'}\int_{-\infty}^{u_1}(u^2+\kappa R')^{-\frac{1}{2}}(\tfrac{1}{2}u^2+\kappa R')^{-\frac{1}{2}}e^{-\iota u^2}\,du,$$

where $$u_1=\pm\sqrt{\{(r^2+r_1^2+2rr_1\sin\theta)^{\frac{1}{2}}-R'\}}\,\kappa^{\frac{1}{2}},$$

and the upper or lower sign of the radical is taken according as $\cos\frac{1}{2}(\phi-\phi_1)$ is positive or negative, therefore the value of the principal part of I_1 is $\bar{I}_1$, where

$$\bar{I}_1=\pi^{-\frac{1}{2}}R'^{-1}e^{\frac{1}{4}\pi\iota-\iota\kappa R'}\int_{-\infty}^{u_1}e^{-\iota u^2}\,du,$$

when R' is not small, and $\bar{I}_1$ tends to the value

$$2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\kappa e^{\frac{1}{4}\pi\iota}\int_{-\infty}^{-u_1'}u^{-2}e^{-\iota u^2}\,du,$$

where $$u_1'=\sqrt{(2\kappa r_1)},$$

when R' tends to zero. When r_1 is very large, u_0 and u_1 tend to the values

$$\sqrt{(2\kappa r\sin\theta)}\cos\tfrac{1}{2}(\phi-\phi_1),\qquad\sqrt{(2\kappa r\sin\theta)}\cos\tfrac{1}{2}(\phi+\phi_1),$$

and the result becomes that for plane waves agreeing with the known result.

Similar forms of the solution can be obtained for the case of a line source parallel to the edge of the half plane at a finite distance from it; the method of reduction of the integrals in the known solution* is identical with the preceding investigation, and the results are, when the electric force due to the line source in the line (r_1, ϕ_1), is

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* *Electric Waves*, pp. 192, 195.

$$K_0 \{ \iota\kappa\sqrt{[r^2+r_1^2-2rr_1 \cos(\phi-\phi_1)]} \} e^{\iota\kappa Vt},$$

the resulting electric force is given by $Ze^{\iota\kappa Vt}$, where

$$Z = \tfrac{1}{2}\int_{-\infty}^{\xi_0} e^{-\iota\kappa R \cosh\xi}\, d\xi - \tfrac{1}{2}\int_{-\infty}^{\xi_1} e^{-\iota\kappa R' \cosh\xi}\, d\xi_1,$$

where

$$R \sinh \xi_0 = 2(rr_1)^{\frac{1}{2}} \cos \tfrac{1}{2}(\phi-\phi_1), \qquad R' \sinh \xi_1 = 2(rr_1)^{\frac{1}{2}} \cos \tfrac{1}{2}(\phi+\phi_1),$$

$$R^2 = r^2+r_1^2-2rr_1 \cos(\phi-\phi_1), \qquad R'^2 = r^2+r_1^2-2rr_1 \cos(\phi+\phi_1);$$

when the magnetic force due to the line source on the line (r_1, ϕ_1) is

$$K_0 \{ \iota\kappa\sqrt{[r^2+r_1^2-2rr_1 \cos(\phi-\phi_1)]} \} e^{\iota\kappa Vt},$$

the resulting magnetic force is given by $\gamma e^{\iota\kappa Vt}$, where

$$\gamma = \tfrac{1}{2}\int_{-\infty}^{\xi_0} e^{-\iota\kappa R \cosh\xi}\, d\xi + \tfrac{1}{2}\int_{-\infty}^{\xi_1} e^{-\iota\kappa R' \cosh\xi}\, d\xi.$$

ON CERTAIN THEOREMS IN THE THEORY OF SERIES OF NORMAL ORTHOGONAL FUNCTIONS

By E. W. HOBSON.

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IF $\{\phi_t(x)\}$ denotes a set of normal orthogonal functions for an interval which may be taken to be (0, 1), it is known that if $\{c_t\}$ denotes a set of numbers such that $\sum\limits_{t=1} c_t^2$ converges, a function $f(x)$ can be determined such that

$$c_t = \int_0^1 f(x)\,\phi_t(x)\,dx$$

for every value of t, and such that the square of the function is summable, and that

$$\int_0^1 \{f(x)\}^2\,dx = \sum_{t=1}^{\infty} c^2.$$

This function is uniquely determinate if sets of points of measure zero be neglected.

The question of the convergence of series of the type $\sum\limits_{t=1} c_t\phi_t(x)$ has attracted some attention. It was first shewn by Weyl* that the convergence of the series $\sum\limits_{t=1} t^{\frac{1}{2}} c_t^2$ is sufficient to ensure that the series $\sum\limits_{t=1} c_t\phi_t(x)$ is convergent almost everywhere in the interval (0, 1). It was further shewn by Weyl that, when $|\phi_t(x)|$ is less than some fixed number for all the values of t and x, the convergence of $\sum\limits_{t=1} t^{\frac{1}{3}} c_t^2$ is sufficient. I proved† the more general theorem that the convergence of $\sum\limits_{t=1} t^k c_t^2$, for any value of $k > 0$, is sufficient to ensure the convergence of $\sum\limits_{t=1} c_t\phi_t(x)$ almost everywhere in the interval, without any restriction on the functions $\phi_t(x)$.

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* *Math. Annalen*, Vol. 67.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 12.

It was next proved by Plancherel[*] that the convergence of $\sum\limits_{t=1}^{\infty} (\log t)^3 c_t^2$ is sufficient; this of course includes the above-mentioned results as particular cases. It seems improbable that this theorem is the final one of the kind. Both before and after the publication of Plancherel's result, I have endeavoured, without success, to shew that the convergence of $\sum\limits_{t=1} (\log t)^{2+k} c_t^2$, for a value of $k > 0$, is sufficient, or that in the alternative the convergence of $\sum\limits_{t=1} (\log t)^2 c_t^2$ is sufficient. The probability of the correctness of one or other of these theorems is increased by the fact that the latter holds good in the case of the ordinary Fourier's series. It has, in fact, been proved by Hardy[†] that, if $\Sigma (a_n \cos nx + b_n \sin nx)$ is any Fourier's series, the series $\sum\limits_{n=2}^{\infty} \dfrac{a_n \cos nx + b_n \sin nx}{\log n}$ is convergent almost everywhere; in the case of those series that correspond to functions whose squares are summable, this is equivalent to the theorem that the Fourier's series converges almost everywhere if $\sum\limits_{n=1} (\log n)^2 (a_n^2 + b_n^2)$ is convergent.

W. H. Young[‡] had already published theorems for Fourier's series which would include the result that the convergence of $\Sigma (\log n)^{2+k} (a_n^2 + b_n^2)$ for $k > 0$ is sufficient.

In accordance with a known result due to Haar and Mercer, these properties of Fourier's series also hold as regards series of Sturm-Liouville functions.

In § 1 of the present communication I have given a proof of Plancherel's theorem which is, I think, simpler in form than the proof given by Plancherel himself.

Another question relating to these series is that of their summability in accordance with Cesàro's method. Weyl (*loc. cit.*, p. 242) has stated that he had obtained a proof that the convergence of the series $\sum\limits_{t=1} (\log t) c_t^2$ is sufficient to ensure that the series $\sum\limits_{t=1} c_t \phi_t(x)$ is summable, in accordance with Cesàro's procedure, for almost all values of x in the interval. He gives only some indications as to the nature of his proof, which appears to involve the use of the somewhat complicated apparatus employed in the other parts of his memoir. I have thought it might be of interest to give, in § 2, a proof of this result, free from the intricacy of Weyl's own method.

It is of interest to observe that the gap between Weyl's theorem that the convergence of $\Sigma c_t^2 \log t$ is sufficient for summability of $\Sigma c_t \phi_t(x)$, and the known result that for the particular case of Fourier's trigonometrical series the convergence of Σc_t^2 is sufficient, is the same as the gap between Plancherel's result that the convergence of $\Sigma c_t^2 (\log t)^3$ is sufficient for the convergence of $\Sigma c_t \phi_t(x)$, and Hardy's result that for the trigonometrical series the convergence of $\Sigma c_t^2 (\log t)^2$ is sufficient.

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[*] *Comptes Rendus*, October 6th, 1913.

[†] *Proc. London Math. Soc.*, Ser. 2, Vol. 12.

[‡] *Comptes Rendus*, December 23rd, 1912.

1. Let $s(x, n)$ denote the partial sum $\sum\limits_{t=1}^{t=n} c_t \phi_t(x)$, where $\{\phi(x)\}$ denote a system of normal orthogonal functions for the interval (0, 1).

It will first be shewn that, if the series $\sum\limits_{t=1}^{\infty} c_t^2 \log t$ is convergent, $s(x, 2^t)$ converges for almost all values of x, as $t \sim \infty$. As a matter of convenience the logarithms will throughout be taken to have the base 2.

We have

$$\int_0^1 \left[\{s(x, 2^{r+m}) - s(x, 2^r)\}^2 + \{s(x, 2^{r+m}) - s(x, 2^{r+1})\}^2 + \dots \right.$$

$$\left. + \{s(x, 2^{r+m}) - s(x, 2^{r+m-1})\}^2\right] dx$$

$$= \sum_{2^r+1}^{2^{r+m}} c_t^2 + \sum_{2^{r+1}+1}^{2^{r+m}} c_t^2 + \dots + \sum_{2^{r+m-1}+1}^{2^{r+m}} c_t^2$$

$$= \sum_{2^r+1}^{2^{r+1}} c_t^2 + 2 \sum_{2^{r+1}+1}^{2^{r+2}} c_t^2 + \dots + m \sum_{2^{r+m-1}+1}^{2^{r+m}} c_t^2$$

$$< \sum_{2^r+1}^{2^{r+m}} c_t^2 \log t.$$

If we choose r so large that

$$\sum_{2^r+1 \leq t}^{\infty} c^2 \log t < \delta^3,$$

we have
$$\int_0^1 \sum_{p=0}^{m-1} \{s(x, 2^{r+m}) - s(x, 2^{r+p})\}^2 \, dx < \delta^3.$$

It follows that in a set of points H_m of measure $> 1 - \delta$,

$$|s(x, 2^{r+m}) - s(x, 2^{r+p})| < \delta,$$

for $p = 0, 1, 2, \dots, m-1$.

Thus, in H_m,
$$|s(x, 2^q) - s(x, 2^{q'})| < 2\delta,$$

for all pairs of values of q and q', such that

$$r \leqslant q \leqslant r+m, \qquad r \leqslant q' \leqslant r+m.$$

Consider the sets $H_m, H_{m+1}, H_{m+2}, \ldots$, each of which is of measure $> 1-\delta$. It is clear that each set contains the next one, and therefore a set H of points common to all the sets $H_m, H_{m+1}, \ldots$ exists, and is of measure $\geqslant 1-\delta$.

In the set H, we have

$$|s(x, 2^q)-s(x, 2^{q'})| < 2\delta,$$

for all pairs of values of q and q', such that $r \leqslant q$, $r \leqslant q'$.

Let η be an arbitrarily chosen positive number, and let $\delta_1, \delta_2, \ldots$ denote a sequence such that $\delta_1+\delta_2+\delta_3+\ldots$ converges to the value η. Corresponding to each of the values δ_n of δ, a value r_n of r can be chosen, and a set $H^{(n)}$ exists, such that

$$|s(x, 2^q)-s(x, 2^{q'})| < 2\delta_n,$$

in that set for all pairs of values of q and q' such that $r_n \leqslant q$, $r_n \leqslant q'$. The measure of $H^{(n)}$ is $\geqslant 1-\delta_n$.

All the sets $H^{(n)}$ have in common a set K of measure

$$\geqslant 1-\sum_{n=1}^{\infty} \delta_n, \quad \text{or} \quad \geqslant 1-\eta.$$

In the set K, we have

$$|s(x, 2^q)-s(x, 2^{q'})| < 2\delta_n,$$

provided $r_n \leqslant q$, $r_n \leqslant q'$, for all values of n; therefore in the set K, $s(x, 2^t)$ is uniformly convergent as $t \sim \infty$.

Since η is arbitrarily small, we see that $s(x, 2^t)$ is convergent for almost all values of x, as $t \sim \infty$.

A fortiori the result holds if $\sum\limits_{t=1} c_t^2(\log t)^3$ is convergent.

Next let us consider a number N_p expressed in the dyad scale, and of the form

$$N_p = 2^r+k_1.2^{r-1}+k_2.2^{r-2}+\ldots+k_p.2^{r-p};$$

each of the numbers $k_1, k_2, \ldots, k_p$ is either zero or 1. We have

$$\int_0^1 \{s(x, N_p)-s(x, N_p+2^{r-p-1})\}^2 dx = \sum_{N_p+1}^{N_p+2^{r-p-1}} c_t^2.$$

If all the numbers N_p be arranged in ascending order of magnitude, 2^r, 2^r+2^{r-p}, 2^r+2^{r-p+1}, $2^r+2^{r-p+1}+2^{r-p}$, 2^r+2^{r-p+2}, ..., we see that the difference between two consecutive numbers is always $> 2^{r-p-1}$, hence

$$\int_0^1 \Sigma \{ s(x, N_p) - s(x, N_p + 2^{r-p-1}) \}^2 \, dx < \sum_{2^r+1}^{2^{r+1}} c_t^2,$$

where the summation in the integrand is taken for all the possible values of N_p.

We now see that in a set of measure greater than $1 - \frac{r^2}{\delta^2} \sum_{2^r+1}^{2^{r+1}} c_t^2$,

$$| s(x, N_p) - s(x, N_p + 2^{r-p-1}) | < \frac{\delta}{r},$$

for all the numbers N_p.

The number p may have the values $0, 1, 2, \ldots, r-1$, and for each of these such a set exists. By taking the set common to all these r sets, we have a set of points of measure

$$> 1 - \frac{r^3}{\delta^2} \sum_{2^r+1}^{2^{r+1}} c_t^2,$$

such that $$| s(x, N_p) - s(x, N_p + 2^{r-p-1}) | < \frac{\delta}{r},$$

for all the values of p, and for all the numbers N_p.

Now any number n between 2^r and 2^{r+1} can be expressed as a number N_p, where p has one of the values $0, 1, 2, \ldots, r-1$, or else is of the form $N_{r-1}+1$. Hence $s(x, 2^r) - s(x, n)$ is of the form

$$\{ s(x, 2^r) - s(x, N_1) \} + \{ s(x, N_1) - s(x, N_2) \} + \ldots + \{ s(x, N_{p-1}) - s(x, N_p) \},$$

where the last bracket may be

$$\{ s(x, N_{r-1}) - s(x, N_{r-1}+1) \}.$$

We have clearly

$$N_1 = 2^r + 2^{r-1}, \text{ or } 2^r; \quad N_2 = N_1 + 2^{r-2}, \text{ or } N_1, \quad N_3 = N_2 + 2^{r-3}, \text{ or } N_2, \text{ \&c.}$$

Each of the expressions in the brackets is either zero or is an absolute value less than δ/r in the set of points described above.

We have therefore

$$| s(x, 2^r) - s(x, n) | < \delta,$$

at every point x in a set of measure

$$> 1 - \frac{r^3}{\delta^2} \sum_{2^r+1}^{2^{r+1}} c_t^2,$$

and for every number n, such that

$$2^r \leqslant n \leqslant 2^{r+1}.$$

Let it now be assumed that the series $\sum\limits_{n=1} c_t^2 (\log t)^3$ is convergent; then r may be so chosen that

$$\sum_{2^r+1}^{\infty} c_t^2 (\log t)^3 < \delta^3,$$

and thus that

$$r^3 \sum_{2^r+1}^{2^{r+1}} c_t^2 + (r+1)^3 \sum_{{}^{r+1}+1}^{2^{r+2}} c_t^2 + \ldots + (r+m-1)^3 \sum_{2^{r+m}+1}^{2^{r+m}} c_t^2 < \delta^3,$$

for every value of m.

We now see that in a set G_m of measure $> 1-\delta$, for every number n, such that $2^r \leqslant n \leqslant 2^{r+m}$,

$$| s(x, 2^{r+q}) - s(x, n) | < \delta,$$

where q is that integer ($< m$), such that

$$2^{r+q} \leqslant n \leqslant 2^{r+q+1}.$$

The sets G_m, G_{m+1}, G_{m+2}, ... are all of measure $> 1-\delta$, and each set contains the next; hence there is a set G of measure $\geqslant 1-\delta$, such that

$$| s(x, 2^{r+q}) - s(x, n) | < \delta,$$

for every n such that $n \geqslant 2^r$, where q is such that

$$2^{r+q} \leqslant n \leqslant 2^{r+q+1}.$$

We may further suppose that r is so chosen that

$$| s(x, 2^{r+q}) - s(x, 2^{r+q'}) | < 2\delta,$$

in a set H of measure $\geqslant 1-\delta$, for $0 \leqslant q$, $0 \leqslant q'$.

In the set L_δ of measure $\geqslant 1-2\delta$, of points common to H and G, we have clearly

$$| s(x, n) - s(x, n') | < 4\delta,$$

for every pair of numbers n and n', such that $n \geqslant r$, $n' \geqslant r$.

If we now ascribe to δ a set of decreasing values δ_1, δ_2, ..., such that $\delta_1 + \delta_2 + \ldots$ converges to $\frac{1}{2}\epsilon$, the sets L_{δ_1}, L_{δ_2}, .. have in common a set L of measure $\geqslant 1-\epsilon$. In this set L we have

$$| s(x, n) - s(x, n') | < 4\delta_t,$$

provided n and n' are both $\geqslant r_{\delta_t}$; and this holds for all the values of t, the number r_{δ_t} increasing indefinitely as t does so. Therefore $s(x, n)$ converges uniformly in the set L; and since ϵ is arbitrary, we see that $s(x, n)$ converges in a set of measure 1, *i.e.*, almost everywhere in the interval (0, 1). We thus have Plancherel's theorem :—

If the series $\sum\limits_{t=1} c_t^2(\log t)^3$ *is convergent, the series* $\sum\limits_{t=1} c_t\phi_t(x)$, *where* $\{\phi_t(x)\}$ *are normal orthogonal functions for the interval* (0, 1), *converges almost everywhere in the interval.*

Since the series of normal functions such that $\sum\limits_{t=1} c_t^2$ is convergent is always the series of which the coefficients c_t are given by

$$c_t = \int_0^1 f(x)\,\phi_t(x)\,dx,$$

where $f(x)$ is a function such that $\{f(x)\}^2$ is summable, the theorem may be stated as follows :—

If the series $\sum\limits_{t=1} c_t\phi_t(x)$ *correspond to a function of which the square is summable, the series* $\sum \dfrac{c_t}{(\log t)^{\frac{3}{2}}}\phi_t(x)$ *converges almost everywhere in the interval.*

2. It will now be shewn that the convergence of $\sum\limits_{t=1} c_t^2 \log t$ is sufficient to ensure that the series $\sum\limits_{t=1} c_t\phi_t(x)$ is summable $(C, 1)$ almost everywhere.

We shall denote the expression

$$\frac{s(x, 1)+s(x, 2)+\ldots+s(x, m)}{m},$$

formed in accordance with Cesàro's method, by $S(x, m)$, and sometimes simply by $S(m)$, or S_m.

(1) We have

$$\int_0^1 \{S(x, m)-s(x, m)\}^2 dx$$
$$= \frac{1}{m^2}\{c_2^2+(2c_3)^2+\ldots+(\overline{r-1}\,c_r)^2\} + \frac{1}{m^2}\{(rc_{r+1})^2+\ldots+(\overline{m-1}\,c_m)^2\}$$
$$< \frac{1}{m^2}\{c_2^2+(2c_3)^2+\ldots+(\overline{r-1}\,c_r)^2\} + \sum_{r+1}^{m} c_t^2.$$

Let $r = 2^\rho$, $m = 2^p$, where $p > \rho$; we have then

$$\int_0^1 \{S(x, 2^p) - s(x, 2^p)\}^2\, dx < \frac{A(\rho)}{2^{2p}} + \sum_{2^\rho+1}^{2p} c_t^2,$$

where $A(\rho)$ denotes $c_2^2 + (2c_3)^2 + \ldots + \{(2^\rho - 1)\, c(2^\rho)\}^2$,

$c(2^\rho)$ being written for c_{2^ρ}.

Now let $\rho\; (= \rho_1)$ be so chosen that

$$\sum_{2^{\rho_1}+1}^{\infty} c_t^2 + \sum_{2^{\rho_1+1}+1}^{\infty} c_t^2 + \ldots < \epsilon;$$

this can be done in virtue of the convergence of the series $\Sigma c_t^2 \log t$. Then let $p\; (= p_1)$ be so chosen that $\dfrac{A(\rho_1)}{2^{2p_1}} < \epsilon$. We have

$$\frac{A(\rho_1+s)}{2^{2(p_1+2s)}} = \frac{A(\rho_1) + \{2^{\rho_1} c(2^{\rho_1}+1)\}^2 + \ldots + \{(2^{\rho_1+s}-1)\, c(2^{\rho_1+s})\}^2}{2^{2(p_1+2s)}}$$

$$< \frac{A(\rho_1)}{2^{2(p_1+2s)}} + \frac{1}{2^{2(p_1-\rho_1+s)}} \sum_{2^{\rho_1}+1}^{\infty} c_t^2$$

$$< \frac{\epsilon}{2^{4s}} + \frac{\epsilon}{2^{2s}}\, \frac{1}{2^{2(p_1-\rho_1)}}.$$

It follows that

$$\sum_{s=0}^{\infty} \frac{A(\rho_1+s)}{2^{2(p_1+2s)}} < \frac{\epsilon}{1-\dfrac{1}{2^4}} + \frac{\epsilon}{1-\dfrac{1}{2^2}}\, \frac{1}{2^{2(p_1-\rho_1)}} < 4\epsilon.$$

We have now, by giving p the values $p_1, p_1+2, p_1+4, \ldots, p_1+2q$, and ρ the values $\rho_1, \rho_1+1, \rho_1+2, \ldots, \rho_1+q$,

$$\int_0^1 \sum_{s=0}^{s=q} \{S(x, 2^{p_1+2s}) - s(x, 2^{p_1+2s})\}^2\, dx < 5\epsilon,$$

whatever value q may have.

In this inequality we may change p_1 into p_1+1, and hence by addition, we have

$$\int_0^1 \sum_{s=0}^{s=q} \{S(x, 2^{p_1+s}) - s(x, 2^{p_1+s})\}^2\, dx < 10\epsilon,$$

for all values of q. If we write $10\epsilon = \delta^3$, we see that in a set of measure $> 1-\delta$, we have

$$| S(x,\, 2^{p_1+s}) - s(x,\, 2^{p_1+s}) | < \delta,$$

for $s = 0, 1, 2, 3, \ldots, q$. By considering the sets corresponding to $q, q+1, q+2, \ldots$, we see that in a set of measure $\geqslant 1-\delta$, the result holds for all values of s.

By considering the sets corresponding to a sequence of values of δ, such that $\Sigma\delta = \eta$, and considering the values of p_1 which correspond to $\delta_1, \delta_2, \ldots$, we see that a set of points of measure $\geqslant 1-\eta$ exists in which $| S(x, 2^n) - s(x, 2^n) |$ converges uniformly to zero. Therefore, since η is arbitrary, $S(x, 2^n) - s(x, 2^n)$ converges almost everywhere to zero.

It has been shewn in § 1 that $s(x, 2^n)$ is convergent as $n \sim \infty$, for almost all values of x, in virtue of the convergence of $\Sigma c_t^2 \log t$. It follows that $S(x, 2^n)$ converges for almost all values of x, and that almost everywhere $S(x, 2^n)$ and $s(x, 2^n)$ converge to the same limit.

(2) If $n \geqslant m$, we have

$$\int_0^1 (S_n - s_m)^2\, dx = \int_0^1 \left\{ \frac{1}{n} (nc_1\phi_1 + \overline{n-1}\, c_2\phi_2 + \ldots + c_n\phi_n) - (c_1\phi_1 + \ldots + c_m\phi_m) \right\}^2 dx$$

$$< \frac{1}{n^2} \{ c_2^2 + (2c_3)^2 + \ldots + (\overline{m-1}\, c_m)^2 \} + (c_{m+1}^2 + \ldots + c_n^2).$$

Hence, if $p \leqslant m \leqslant 2p$, we have

$$\int_0^1 (S_n - s_m)^2\, dx < \frac{1}{n^2} \{ c_2^2 + (2c_3)^2 + \ldots + (2p-1)^2 c_{2p}^2 \} + \sum_{p+1}^{\infty} c_t^2 ;$$

and thence, if $n \geqslant 2p$,

$$\int_0^1 [(S_n - s_{p+1})^2 + (S_n - s_{p+2})^2 + \ldots + (S_n - s_{2p})^2]\, dx$$

$$< \frac{p}{n^2} \{ c_2^2 + (2c_3)^2 + \ldots + (2p-1)^2 c_{2p}^2 \} + p \sum_{p+1}^{\infty} c_t^2 .$$

Since $\left\{ \dfrac{| S_n - s_{p+1} | + | S_n - s_{p+2} | + \ldots + | S_n - s_{2p} |}{p} \right\}^2$

$$< \frac{1}{p} \{ (S_n - s_{p+1})^2 + \ldots + (S_n - s_{2p})^2 \},$$

we have

$$\int_0^1 \left[\frac{|S_n-s_{p+1}|+|S_n-s_{p+2}|+\ldots+|S_n-s_{2p}|}{p}\right]^2 dx$$

$$< \frac{1}{n^2}\left\{c_2^2+(2c_3)^2+\ldots+(2p-1)^2 c_{2p}^2\right\}+\sum_{p+1}^{\infty} c_t^2.$$

Let $p=2^\rho$, $n=2^q$;

$$c_2^2+(2c_3)^2+\ldots+(2^{\rho+1}-1)^2\, c^2\,(2^{\rho+1}) = A\,(\rho+1).$$

Choose ρ so large that

$$\sum_{2^\rho+1}^{\infty} c_t^2+\sum_{2^{\rho+1}+1}^{\infty} c_t^2+\ldots<\epsilon,$$

and then choose q so large that

$$\frac{A\,(\rho+1)}{2^{2q}}<\epsilon.$$

Remembering that
$$\sum_{s=0}^{\infty}\frac{A\,(\rho+1+s)}{2^{2(q+2s)}}<4\epsilon,$$

we then have

$$\int_0^1 \sum_{s=0}^{s=r}\left[\frac{|S\,(2^{q+2s})-s\,(2^{\rho+s}+1)|+\ldots+|S\,(2^{q+2s})-s\,(2^{\rho+s+1})|}{2^{\rho+s}}\right]^2 dx<5\epsilon.$$

We thus find that if $5\epsilon=\eta^3$, there exists a set of points x of measure $>1-\eta$, such that, when ρ and q are properly chosen

$$\frac{|S\,(2^{q+2s})-s\,(2^{\rho+s}+1)|+\ldots+|S\,(2^{q+2s})-s\,(2^{\rho+s+1})|}{2^{\rho+s}}<\eta,$$

in the set. The number s does not exceed the arbitrarily fixed number r. By considering, as before, the sets corresponding to the values r, $r+1$, $r+2$, ... of r, we see that there exists a set of measure $\geqslant 1-\eta$, in which the above inequality holds for every value of s.

(3) We have at once

$$(S_n-s_{p+1})+(S_n-s_{p+2})+\ldots+(S_n-s_m) = m\,(S_n-S_m)-p\,(S_n-S_p),$$

where $p\leqslant m$. From this we find that

$$|S_n-S_m|<|S_n-S_p|+\frac{|S_n-s_{p+1}|+\ldots+|S_n-s_m|}{p}.$$

If $p \leqslant m \leqslant 2p$, we then have

$$|S_n - S_m| < |S_n - S_p| + \frac{|S_n - s_{p+1}| + \ldots + |S_n - s_{2p}|}{p}.$$

Let $p = 2^{\rho+s}$, $n = 2^{q+2s}$, we then have

$$|S(2^{q+2s}) - S_m| < |S(2^{q+2s}) - S(2^{\rho+s})| + \frac{|S(2^{q+2s}) - s(2^{\rho+s}+1)| + \ldots + |S(2^{q+2s}) - s(2^{\rho+s+1})|}{2^{\rho+s}}$$

for every value of s, and for every value of m between $2^{\rho+s}$ and $2^{\rho+s+1}$.

Employing the result obtained in (2), we see that in a certain set of points of measure $\geqslant 1-\eta$,

$$|S(2^{q+2s}) - S_m| < |S(2^{q+2s}) - S(2^{\rho+s})| + \eta$$

for every value of s, and for every value of m between $2^{\rho+s}$ and $2^{\rho+s+1}$.

$S(2^n)$ has been shewn to converge uniformly in a certain set of points of measure $> 1-\eta$. The number ρ can therefore be, if necessary, increased, until it is so large that

$$|S(2^{q+2s}) - S(2^{\rho+s})| < \eta,$$

in that set, for all values of s. It follows that, in a certain set of points of measure $> 1-2\eta$, we have

$$|S(2^{q+2s}) - S_m| < 2\eta,$$

for every value of m between $2^{\rho+s}$ and $2^{\rho+s+1}$; and for every value of s. If m_1 and m_2 be two values of m, both greater than 2^ρ, we have

$$|S_{m_1} - S_{m_2}| < |S(2^{q+2s_1}) - S_{m_1}| + |S(2^{q+2s_2}) - S_{m_2}| + |S(2^{q+2s_1}) - S(2^{q+2s_2})|,$$

where m is between $2^{\rho+s_1}$ and $2^{\rho+s_1+1}$, and m_2 is between $2^{\rho+s_2}$ and $2^{\rho+s_2+1}$.

We see now that $|S_{m_1} - S_{m_2}| < 6\eta$ in a certain set L_η of measure $> 1-2\eta$, for all values of m_1 and m_2 that are greater than a fixed number N_η.

Considering the sets $L_{\eta_1}, L_{\eta_2}, \ldots$, where $\eta_1, \eta_2, \ldots$ are so chosen that $\eta_1 + \eta_2 + \ldots$ converges to an arbitrarily chosen number ζ, and remarking that the sets $L_{\eta_1}, L_{\eta_2}, \ldots$, have in common a set W_ζ of measure $\geqslant 1-2\zeta$, we see that in the set W_ζ,

$$|S_{m_1}(x) - S_{m_2}(x)| < 6\eta_r,$$

if m_1 and m_2 are both $> N_{\eta_r}$. Since η_r converges to zero as r is indefinitely increased, we see that, in the points of the set W_ζ, $S_m(x)$ is uniformly convergent. Since ζ is arbitrary, we then see that $S_m(x)$ is convergent almost everywhere in the interval (0, 1).

The following theorem has now been established :—

If $\sum\limits_{t=1} c_t^2 \log t$ *is convergent, the series* $\sum\limits_{t=1} c_t \phi_t(x)$ *is summable in accordance with Cesàro's method at almost every point of the interval* (0, 1) *for which the functions* $\phi(x)$ *form a normal orthogonal set of functions.*

THE INFLUENCE OF THE OCEANIC WATERS ON THE LAW OF VARIATION OF LATITUDES

By Sir Joseph Larmor.

[Received April 14th, 1915.—Read April 22nd, 1915.]

If the Earth's permanent axis of steady rotation—which must be a principal axis of inertia of its mass, and to ensure *secular** stability must be that of maximum moment of inertia, say C—is slightly disturbed, then according to the Eulerian analysis afterwards transformed into geometrical shape by Poinsot, the pole of the axis of terrestrial rotation ought to describe on the Earth an ellipse around the pole of inertia in $\sqrt{\dfrac{AB}{(C-A)(C-B)}}$ sidereal days. One of the significant facts relating to the Earth's constitution and origin is that A and B are equal, so far as investigation has detected: thus the period in this free Eulerian polar orbit, now circular when undisturbed, and in the same direction as the diurnal motion by Poinsot's representation, is $A/(C-A)$ sidereal days. The value of this quantity is directly determined by the amount of the ordinary forced Precession of the Equinoxes arising from the known solar and lunar attractions. The free period thus precisely estimated is 304 sidereal days: and accordingly the records of the slight observed fluctuations of the latitudes of observatories were examined long ago, by Peters and by Maxwell, in search for a recurring component of this period, but without success. In 1892, however, S. C. Chandler announced an actual period of about 428 days: and following on this Newcomb showed briefly† by general reasoning that the elastic yielding of the Earth to centrifugal force, as it changes with axis of rotation, would, in fact, lengthen the period, and conceivably to the order of magnitude thus announced, while, of course, irregular meteorological or other transfers of mass would disturb the movement.* A mathematical investigation by Hough made shortly afterwards,† involving a determination of the yielding of an elastic sphere, taken as incompressible, to centrifugal or tidal forces, gave definite formulas which would agree with this conclusion if the Earth were about as rigid as steel. He remarked the result, on his special investigation, that the effect of this varying centrifugal force on the free precession is the same as if the centrifugal elastic bulging were annulled by removing the diurnal rotation, and the resulting form of the Earth were treated as invariable and absolutely rigid. Actually, it is small changes in the total centrifugal protuberance that are concerned, arising from change of axis of rotation, so that the calculation which thus estimates the whole of it is to be effected on the hypothesis of perfect linear elasticity. This result was proved by the writer‡ to express a general dynamical principle, whatever be the nature of the centrifugal deformation, elastic or otherwise, to which the Earth, however heterogeneous, may be subject. A method was later worked out, and applied in conjunction with E. H. Hills,§ for deducing and mapping from the observed path of the pole, very irregular as it is and far from circular, the character of the meteorological and other disturbances, of the nature of shift of surface terrestrial material, that mask the circularity of the free precession. Finally, generalising the result of an important investigation by A. E. H. Love, it was shown* that the amount of the change in period of the free precession determines the factor (k) by which the potential of a tidal or centrifugal bodily force, of the type of a harmonic of the second order, is to be multiplied in order to give the change in the potential of the Earth's gravitational attraction resulting from it—or, what is the same thing, $1+k$ is the factor by which the total height of the statical oceanic tide is increased by the yielding of the Earth as a whole; and that the further knowledge of the ratio in which the observed part of the long-period equilibrium oceanic tides is reduced owing to elastic solid tides elevating the coasts of the Earth, which is $1+k-h$, allows us to determine roughly the amount (h) of this elastic tidal yielding over the surface relative to the true oceanic equilibrium tide on an invariable Earth, viz., the height of the solid tide, and so to compare the average or effective elastic quality of the solid Earth with that of a globe, say, of glass or steel. If the distribution of mass and elastic quality in the Earth were known, one of these data would suffice to determine the other: but as the constitution of the Earth's interior remains unknown, the two data together suffice without further knowledge to determine the information here recounted—just as, for example, the form of the sea-level suffices by itself, without any reference to the constitution of the Earth, to determine the distribution of gravity over its surface. One of the aspects in which mathematical analysis of natural phenomena appears most to advantage is in its power of suppression and adaptation, through general revision, of its own unnecessarily restricted initial hypotheses and computations.

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* Cf. *Monthly Notices Roy. Astro. Soc.*, Nov. 1906, p. 33. The Poinsot representation is an ellipsoid $Ax^2+By^2+Cz^2 = (2T)^{-1}$ rolling on a plane at distance G^{-1} from its fixed centre, where G is the angular momentum which is invariable, while T is the energy which is subject to dissipation and to renewal by disturbances, but always lies between $G^2/2A$ and $G^2/2C$.

† *Monthly Notices Roy. Astro. Soc.*, 1892.

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* Cf. also Lord Kelvin, *Brit. Assoc. Address*, 1876, in "Popular Lectures . . . .," Vol. II, p. 262.

† *Phil. Trans.*, 1896.

‡ *Proc. Camb. Phil. Soc.*, 1896; cf. also *Monthly Notices Roy. Astro. Soc.*, 1906; *Proc. Roy. Soc.*, 1909. The extension of Hough's principle is also inferred, as I find, by a somewhat complex argument, in Klein and Sommerfeld, *Theorie des Kreisels*, Heft 3, 1903, p. 607; where pp. 663–730 are devoted to a dynamical and historical discussion of the whole subject.

§ *Monthly Notices Roy. Astro. Soc.*, 1906.

A part of the displacement of terrestrial material which disturbs the free precession of the pole is that of the waters of the oceans; and it is easy to see, from the formula for x determining the ocean level *infra*, that an oscillation of the axis of rotation, of amplitude one second of arc, would produce a concomitant tidal fluctuation with rise and fall of the order of two inches, such as was detected early by Bakhuysen in the Dutch records, and also recently extricated by General Madsen from the Danish Survey Observations of variation of coast level. But this oceanic disturbance is synchronous with the oscillation of the pole, and so its effect cannot be estimated by the methods appropriate to irregular disturbing influences. As Clairaut was compelled to recognise in the Lunar Theory, following Newton, the effect in such cases is not merely to disturb the existing oscillation, but to alter its period, thus producing fundamental change in the type of motion. Lord Rayleigh has shown how the cognate refined lunar investigations of J. C. Adams and G. W. Hill are applicable to a general class of prominent physical phenomena,[†] *e.g.*, to trace the profound influence, on the persistence of acoustical or other vibrations, that may arise from forces that are operative only indirectly by slightly changing the elastic or other constants of the system, provided this alteration occurs in a periodic time half that of the free vibrations in question.

The argument above referred to, for estimating change of free period of terrestrial precession, implying the presence of disturbances harmonic only of the second order, is valid effectively for elastic deformation of the Earth, surface irregularities practically not counting; but it does not

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[*] *Roy. Soc. Proc.*, Dec. 1908.

[†] *Phil. Mag.*, 1887; *Collected Papers*, Vol. III, p. 1.

directly include these oceanic displacements owing to the irregular form of the oceans. We are thus prompted to estimate independently how much of the lengthening of the period of the free precession the oceanic movement will account for: if it proves to be a substantial proportion, only what remains after it is deducted can be used in the geophysical argument directed towards the Earth's degree of internal rigidity. The general principle which determines directly its influence on the phenomena is that the momentum of the centrifugal protuberance is allowed time to follow closely the axis of rotation. Thus, if I is the moment of inertia of the centrifugal effect around the axis of moment C, which remains close to the wandering axis of the Earth's rotation—the latter with components ω_1, ω_2, and ω_3, and resultant the practically constant axial velocity Ω—then the angular momentum of the system has components

$$A'\omega_1+I\omega_1, \quad B'\omega_2+I\omega_2, \quad C'\omega_3+I\omega_3.$$

In other words, the effective principal moments of inertia are $A'+I$, $B'+I$, $C'+I$, where A', B', C' are what they would be if the rotation were stopped and the centrifugal change thus removed. These are sensibly constant when the free precession is of small amplitude; and $I = C-C'$, where C is the actual principal moment of inertia with which the Precession of the Equinoxes is concerned.

But in our present case, owing to the continental masses, the change in the distribution of oceanic waters is not symmetrical around the axes of rotation. The momentum belonging to oceanic shift has three components, specified in terms of its moments and products of inertia; one around the axis of rotation as above, which possesses a permanent configuration; and two others around equatorial axes which travel round with the diurnal motion, and, in so far as they are unsymmetrical with respect to the polar axis, have no sensible influence on phenomena of period long in comparison with a sidereal day. It is thus the change of C and the averaged change of A, produced by the heaping up of oceanic waters by the Earth's rotation, that we have to estimate.

The form of the ocean level is given by

$$E\gamma r^{-1}+\tfrac{1}{2}\omega^2 r^2 \sin^2\theta = \text{constant};$$

so that, if $r = a+x$, we have

$$\begin{aligned} x &= \frac{\omega^2 a^4}{2E\gamma}\sin^2\theta+\text{constant} \\ &= \frac{1}{2.289}\, a \sin^2\theta+\text{constant}, \end{aligned}$$

on substituting $$g = \frac{E\gamma}{a^2}, \qquad \frac{\omega^2 a}{g} = \frac{1}{289}.$$

The moments of inertia C_0, A_0 of a spheroid of water $(a, a+h)$ are, $\frac{11}{2}$ being the Earth's mean density,

$$C_0 = \tfrac{1}{5} \cdot \tfrac{2}{11} E \,.\, 2(a+h)^2, \quad A_0 = \tfrac{1}{5} \cdot \tfrac{2}{11} E \,\{a^2+(a+h)^2\}.$$

Thus for a centrifugal protuberance of water covering the whole Earth

$$\frac{C_0-A_0}{C} = \frac{\frac{1}{5} \cdot \frac{2}{11} E \,.\, 4ha}{E \,.\, \frac{2}{5} a^2} = \frac{4}{11}\,\frac{h}{a} = \frac{4}{11}\,\frac{1}{2.289} = \frac{1}{1590}.$$

If we take it that the ocean covers $\frac{2}{3}$ of the Earth's surface in low latitudes, we may apply roughly the factor $\frac{2}{3}$;* and we have to remember that $\frac{1}{3}$ of the protuberance belongs to the elastic Earth, requiring another factor $\frac{2}{3}$. In all we obtain for $(C_0-A_0)/A$ a value $\frac{1}{3600}$. If some possible free oceanic tide of similar type had a period in the neighbourhood of 428 days, this estimate would, of course, be exceeded owing to resonance.

The actual value of $(C-A)/A$ is $\frac{1}{304}$. The influence of the mobile oceanic waters on the free precessional oscillation of the poles is the same as if $(C_0-A_0)/A$ were subtracted from it, which by itself would reduce the effect by about $\frac{1}{12}$. Thus the period of the free precession for the Earth supposed rigid but covered by the actual ocean would be about $\frac{12}{11}$ of 304, which is 332 sidereal days. The actual period disentangled from the observations of latitudes is about 428 days: and this remaining difference is attributable to elastic yielding of the solid Earth. Of the total increase from 304 days, about $\frac{24}{31}$ is thus ascribable to elastic yielding: thus we have $k = \frac{24}{31} \cdot \frac{4}{15} = \frac{1}{5}$ say. And observations of tides of long period give $1+k-h = \frac{2}{3}$. Hence roughly

$$h = \tfrac{8}{15}, \qquad k = \tfrac{1}{5},$$

in place of the values† $$h = \tfrac{3}{5}, \qquad k = \tfrac{4}{15},$$

deduced by Prof. Love from considerations of the elastic centrifugal deformation alone.‡ The elastic terrestrial tide is thus of height $\frac{8}{15}$ of that to which the tidal force would draw the ocean if it lay on an invariable

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* But see *addendum infra*, which computes the actual value $\frac{1}{3}$, so that the oceanic effects which follow in the next paragraph must be halved, with result as on p. 449.

† Cf. *Roy. Soc. Proc.*, Dec. 1908, p. 94.

‡ *Proc. Roy. Soc.*, Dec. 1908; or *Problems in Geodynamics*, § 58.

Earth : and the yielding of the Earth increases the absolute tidal protuberance by $\frac{1}{5}$:—similar statements holding as regards centrifugal forces arising from axial rotation.

The inferences to be drawn as regards the measure of the Earth's resistance to stress are thus not very substantially altered by oceanic mobility.

It may be observed that, according to the principle which forms the basis of these estimates, a planet without an ocean could not rotate permanently about a principal axis, in face of internal friction, if when the centrifugal strain of the rotation is supposed taken off, the planet were no longer dynamically oblate, *i.e.*, if that axis ceased to be the axis of maximum moment of inertia. This conclusion harmonises with what we would in fact have been compelled to anticipate, from comparison with other possible axes. But we are required also to infer that, when there are oceanic areas, secular axial permanence requires that, after the strain arising from rotation is taken off, the planet shall still be dynamically oblate to an extent sufficient to compensate the influence of the free waters. For the Eulerian equations subsist in modified form, and so does their Poinsot representation. Thus, if the planet were dynamically nearly spherical, but with oceanic waters, there would appear to be no axis about which rotation could persist secularly; once it is disturbed, the pole of rotation would wander over the surface of the planet, keeping however the same direction in space, so that glacial periods of the resulting type and other climatic changes would be normal events, though at long intervals of time.* What effective ellipticity would be required to prevent the imperfectly anchored material from producing this loosening of the position of the axis of rotation, in the case of the Earth? One third of the present ellipticity of $\frac{1}{300}$, and therefore roughly of the existing value of $(C-A)/A$, is due to rotational strain, and we have seen that an additional part in the value of $(C-A)/A$, equal to $\frac{1}{2}\cdot\frac{1}{3600}$, is required to compensate the mobility of the actual oceanic waters. The atmosphere contributes a part depending on the mean difference of the weight of air on the equator and elsewhere, for this is mainly a temperature effect attached to the axis of rotation: but it is practically nothing, and in any case it would be largely compensated by barometric influence on ocean level. Adding these components we find that a value of $(C-A)/A$ at least equal to $\frac{1}{810}$ is needed in order to prevent the pole of rotation wandering over the Earth's surface, whereas the actual value of this measure of dynamical oblateness is as much as $\frac{1}{304}$. In other planets it might readily be different: but the path towards the explanation of glacial local climatic changes on the Earth in this way still remains firmly closed. The estimate may be expressed otherwise by the statement that with the actual ellipticities, in order for wide polar mobility to be attained, the density of the ocean would have to be as much as 16, instead of 1: while we know that it were more than $5\frac{1}{2}$ the Earth must emerge and so to speak float gravitationally on the ocean, the water being heaped up on one side as Laplace's formula showed.

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* The Poinsot representation by a rolling ellipsoid teaches that when a position of secular stability does exist, the falling away from steady motion around the axis of least effective inertia, owing to dissipation of energy, would if unchecked be far from a gentle process : it must involve free precessions of wider and wider radius and corresponding ocean deluges, ultimately again attaining quiescence with rotation around the axis of greatest effective inertia.

We can follow graphically the first stages of secular falling away from steadiness. For so long as the displacement of the axis of rotation remains small enough for A', B', C' to be treated as constants, the Eulerian equations are of type

$$(A'+I)\,\dot{\omega}_1+(B'-C')\,\omega_2\omega_3 = 0,$$

leading readily to

$$(A'+I)\,\omega_1^2+(B'+I)\,\omega_2^2+(C'+I)\,\omega_3^2 = 2T,$$

$$(A'+I)^2\omega_1^2+(B'+I)^2\omega_2^2+(C'+I)^2\omega_3^2 = G^2,$$

where T is the kinetic energy and G the constant angular momentum of the system. Thus, as in footnote, p. 440, as T is slowly dissipated, the motion continues to be represented by an invariable ellipsoid

$$(A'+I)x^2+(B'+I)y^2+(C'+I)z^2 = 1,$$

rolling with centre fixed on a plane, whose distance from the centre is $(2T)^{\frac{1}{2}}/G$ and so is slowly changing.

Added May 20th, 1915.—

In the above estimate the factor $\frac{2}{3}$ has been taken to represent the effect of the limited extent of the ocean, that fraction having been presumed to be as exact as the factor $\frac{2}{3}$ which reduces the accumulation of oceanic waters, when the tidal rise of the bottom is deducted and separately allowed for by a calculation of elastic deformation. As remarked above, if the solid earth were rigid, the influence of the ocean could be estimated exactly, in so far as an equilibrium theory applies so that its surface assumes always the statical form corresponding to the changing instantaneous axis of the diurnal rotation. It seems worth while to record here the actual terrestrial data and the result of such an estimate, for which I am indebted to my brother J. S. B. Larmor.

The elevation of the ocean in latitude λ due to the whole centrifugal effect of the diurnal rotation is

$$h = ka^2 \cos^2 \lambda - H,$$

where $k = \omega^2/2g$, and H is a uniform fall required to maintain the constancy of volume and so is given by

$$H \int dS = ka^2 \int \cos^2 \lambda \, dS.$$

The changes thereby produced in the polar and other moments of inertia are given by

$$\delta C = \int ha^2 \cos^2 \lambda \, dS,$$

$$\delta A - \tfrac{1}{2} \delta C = \int ha^2 \sin^2 \lambda \, dS.$$

Thus $$\delta (C - A) = \int ha^2 (\tfrac{1}{2} \cos^2 \lambda - \sin^2 \lambda) \, dS.$$

To evaluate this we require the values of $\int dS$, $\int \cos^2 \lambda \, dS$, $\int \cos^4 \lambda \, dS$, integrated over the ocean surface, from which the values of $\int \sin^2 \lambda \, dS$ and $\int \sin^2 \lambda \cos^2 \lambda \, dS$ are readily inferred. These quantities have been estimated by quadratures in the following table, which is determined from the number of parts of water out of 24 parts in all representing each zone of 10° breadth, as taken off from a Mercator map of the world. The entries on the table have to be multiplied by $2\pi a^2$ to obtain absolute values.

Northern Hemisphere.

Latitude.	Water. (24)	δS (area)	$\cos^2 \lambda \, \delta S$	$\cos^4 \lambda \, \delta S$
0°–10°	19	·137	·1367	·1356
10°–20°	17	·119	·1110	·1035
20°–30°	15	·098	·0810	·0665
30°–40°	$13\frac{1}{2}$	·080	·0539	·0362
40°–50°	11	·056	·0281	·0140
50°–60°	10	·041	·0137	·0045
60°–70°	6	·018	·0033	·0006
70°–80°	18	·034	·0022	·0001
80°–90°	0	—	—	—
		·583	·4299	·3610

Southern Hemisphere.

0°–10°	18	·130	·1295	·1285
10°–20°	18½	·129	·1207	·1126
20°–30°	18½	·122	·0999	·0820
30°–40°	21	·125	·0839	·0563
40°–50°	23	·118	·0589	·0294
50°–60°	24	·100	·0330	·0108
60°–70°	24	·074	·0132	·0023
70°–80°	24	·045	·0030	·0002
80°–90°	0	—	—	—
		·843	·5421	·4221
half sum		·713	·486	·391
if all submerged		1·000	·666	·533

the last line corresponding to an ocean covering the whole Earth.

It may be noted in passing that the preponderance of water over the Southern hemisphere involves greater abstraction from it when the ocean level for any reason falls, and in consequence some displacement of the Earth's centre of mass. But such an effect would not be sensible astronomically: for to alter latitudes to the order of one-tenth of a second of arc the shift would have to be of the order of 15 feet, so that changes of ocean level of the order of 50 feet would be needed.

From the table above we obtain

$$H = ka^2 \frac{972}{1426};$$

and therefore for the actual limited ocean

$$\delta(C-A) = 2\pi a^6 k \int \left(\cos^2\lambda - \frac{972}{1426}\right)\left(\tfrac{3}{2}\cos^2\lambda - 1\right) dS,$$

where $$k = \frac{\omega^2}{2g} = \frac{1}{2.289a},$$

while $$C = \tfrac{4}{3}\pi a^3 . \tfrac{11}{2} . \tfrac{2}{5} a^2,$$

leading by aid of the tabular data to

$$\frac{\delta(C-A)}{C} = \frac{1}{4700}.$$

If the ocean covered the whole Earth, H would not count in $\delta(C-A)$, and we would have for that case as above

$$\frac{\delta(C-A)}{C} = \tfrac{4}{11}ka = \frac{1}{1590}.$$

Thus the value of this quantity is reduced owing to limitation of oceanic area by a factor slightly over $\frac{1}{3}$, instead of $\frac{2}{3}$ as was too hastily assumed above.

This correction amends the final result (with the previous meanings of h and k) to

$$h-k = \tfrac{1}{3}, \qquad k = \tfrac{3}{13};$$

so that, for example, for tidal action of long period, the absolute tidal deformation of the solid Earth is the fraction $h/(1+k)$, here $\frac{11}{24}$, of the observed oceanic tidal elevation, and would not be far from $\frac{1}{2}$ for any reasonable value of k.

June 9th, 1915.—The effect has to be reduced still further. For the oceanic tidal load itself compresses the solid Earth, and so diminishes its momental difference $C-A$, to a degree that may be calculated from the formulæ in Prof. Love's *Treatise on Elasticity*, § 177. It is easy to recognize that this reduction is considerable; for a vertical bar of steel of the length of the Earth's radius, with 2.10^{12} c.g.s. for Young's modulus, would be compressed $\frac{1}{4}H$ by an end-load of water of depth H. Bearing in mind that the Earth's mean density is $5\frac{1}{2}$, it would seem that the reduction in the oceanic precessional influence arising on this account might well, if the Earth were like steel, be 20 or 30 per cent. Also $h-k$ might well be changed substantially, even from $\frac{1}{3}$ to $\frac{1}{2}$, when allowance is made for this sinking of the ocean-bed owing to the tidal load, while k would remain at about $\frac{1}{4}$.

DIFFRACTION OF WAVES BY A WEDGE

By T. J. I'A. BROMWICH.

[Received January 14th, 1915.—Read May 13th, 1915.]

THE following treatment of the wedge-problem starts from the idea of generalizing the familiar process of taking images, as used in electrostatics. It will be remembered that in the ordinary elementary problems, success depends on the fact that the angle outside the conductor is of the type π/n, where n is a positive integer: so that the angle of the wedge is then $2\pi-\pi/n$.

The method of generalization is to replace the sum of the effects of n images by a complex integral; and then to extend the integral so as to obtain a formula valid for any positive value of n (that is, for any angle of the wedge). The extension required is found to be a modification of the path of integration; the subject of integration remains unchanged. The discussion of the various points involved in this generalization occupies the greater part of the paper (§ 2).

Starting from § 2 it is possible to solve (§§ 3, 4) the diffraction-problem for impact on a wedge (i) of sound waves from a source, (ii) of electromagnetic waves from a Hertzian oscillator with its axis parallel to the edge of the wedge, (iii) of electromagnetic waves from an oscillating magnet, also parallel to the edge.

The problem (i) has been solved by Prof. Macdonald for a simple harmonic source; it is here extended to a source of any type. The problems (ii) and (iii) are new, so far as I know.

1. *A General Integral of the Wave-Equation.*

It is evident that the complex integral

$$U=\int\frac{f(R-ct)}{R}\,g(\xi)\,d\xi$$

is a solution of the fundamental wave-equation, where

$$R^2 = r^2 + r_0^2 + (z - z_0)^2 - 2rr_0 \cos(\theta - \xi),$$

and (r, θ, z) are the cylindrical coordinates of a variable point, and (r_0, θ_0, z_0) are those of a fixed point.

In order to make R single-valued in the plane of the complex variable ξ, it will be convenient to make cuts as shown, parallel to the imaginary axis, from the branch-points of R; these branch-points are given by

$$\xi - \theta = 2k\pi \pm \iota a,$$

where $$2rr_0 \cosh a = r^2 + r_0^2 + (z - z_0)^2,$$

and k is any integer (positive or negative) or zero.

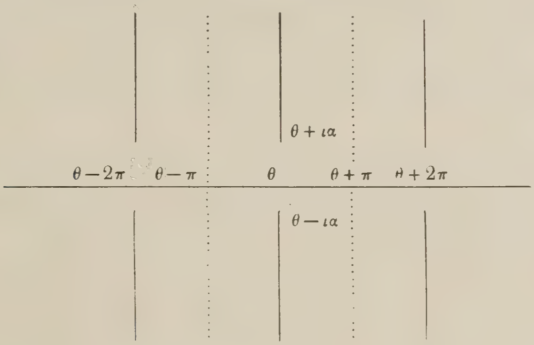

FIG. 1.

The value of R is fixed if we agree that R is to be positive, say, on the real axis; it is easy to see that R is then also positive along the lines (indicated by dots) which are midway between the cuts.

The phase of R will be $+\frac{1}{2}\pi$ on the right-hand edge of each of the upper cuts, and $-\frac{1}{2}\pi$ on the left-hand edge; for the lower cuts these signs will be reversed.

For the special cases of the potential equation and of simple harmonic waves,* this solution was first given by Sommerfeld.† Using the other methods developed by Sommerfeld in the paper quoted, we could proceed to the solution of § 2, for the special case $n = \frac{1}{2}$ (a half-plane or straight-edge) with very slight additional work: but the process fails for other values of n.

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* Given by taking $f(R - ct) =$ const. and $f(R - ct) = e^{\iota\kappa(ct - R)}$, respectively.

† *Proc. London Math. Soc.*, Vol. XXVIII, 1897, pp. 405, 429.

2. *Solution of the Wave-Equation, corresponding to a Source outside a Wedge.*

Consider now the question of finding a solution V of the wave-equation corresponding to a source placed at the point (r_0, θ_0, z_0) between the two planes $\theta = 0$, $\theta = \pi/n$, subject to the condition that the solution is to vanish on both the boundary planes.

When n is an integer, the solution is given by the familiar image process of electrostatics, and can be written in the form

$$V = V_1 - V_2,$$

where
$$V_1 = \sum_{m=0}^{n-1} \frac{f(R_1-ct)}{R_1}, \qquad V_2 = \sum_{m=1}^{n} \frac{f(R_2-ct)}{R_2},$$

and
$$R_1^2 = r^2+r_0^2-2rr_0\cos\left(\theta-\theta_0-\frac{2m\pi}{n}\right)+(z-z_0)^2,$$

while
$$R_2^2 = r^2+r_0^2-2rr_0\cos\left(\theta+\theta_0-\frac{2m\pi}{n}\right)+(z-z_0)^2,$$

and the nature of the function f is given by the character of the source.

We shall now express the two functions V_1, V_2 by means of integrals of the type U defined above (§ 1). It is evident that

$$V_1 = \frac{1}{2\pi\iota}\int \frac{f(R-ct)}{R} g_1(\xi)\,d\xi,$$

provided that the path of integration is a closed loop surrounding the points
$$\xi = \theta_0+2m\pi/n \quad \text{(where } m = 0, 1, 2, \ldots, n-1\text{)},$$

and that $g_1(\xi)$ is a function which has a pole of residue unity at each of these n points. It may be noticed that, since $0 < \theta_0 < \pi/n$, these n points all lie between $\xi = 0$ and $\xi = 2\pi$, on the real axis of our complex diagram; and so the path of integration may be *any* simple loop cutting the real axis at the points $\xi = 0$ and $\xi = 2\pi$.

In the subsequent argument we suppose that $g_1(\xi)$ has a period 2π in ξ, and further that $g_1(\xi)$ remains finite when ξ tends to infinity in either direction parallel to the imaginary axis (see p. 455 below). These two conditions, together with the specification of the poles, determine the function $g_1(\xi)$ completely, save for an arbitrary additive constant C: and, according to a known theorem on periodic functions, we have

$$g_1(\xi) = \tfrac{1}{2}\sum_{m=0}^{n-1} \cot\tfrac{1}{2}(\xi-\theta_0-2m\pi/n)+C = \tfrac{1}{2}n\cot\tfrac{1}{2}n(\theta-\xi_0)+C.$$

It is easy to see that the value of C does not affect the value of the integral for V_1: and so we may put $C = 0$.

Since the two functions R and $g_1(\xi)$ both have the period 2π in ξ, the path of integration may be any simple loop, provided that it cuts the real axis in two points whose distance apart is 2π.

We now change the variable of integration to $\zeta = \xi - \theta$, and then

$$V_1 = \frac{1}{2\pi\iota}\int \frac{f(R-ct)}{R}\, G_1(\zeta)\, d\zeta,$$

where $$R^2 = r^2 + r_0^2 + (z-z_0)^2 - 2rr_0 \cos\zeta,$$

and $$G_1(\zeta) = \tfrac{1}{2}n \cot \tfrac{1}{2}n\,(\zeta+\theta-\theta_0),$$

the path of integration still cutting the real axis in two points at a distance 2π apart. Of course, in the ζ-plane, the cuts extend from the branch-points $2k\pi \pm \iota\alpha$ parallel to the imaginary axis.

In order to have a standard diagram we suppose that the path of integration cuts the real axis at $\zeta = -\pi$, $\zeta = +\pi$ as indicated in Fig. 2.

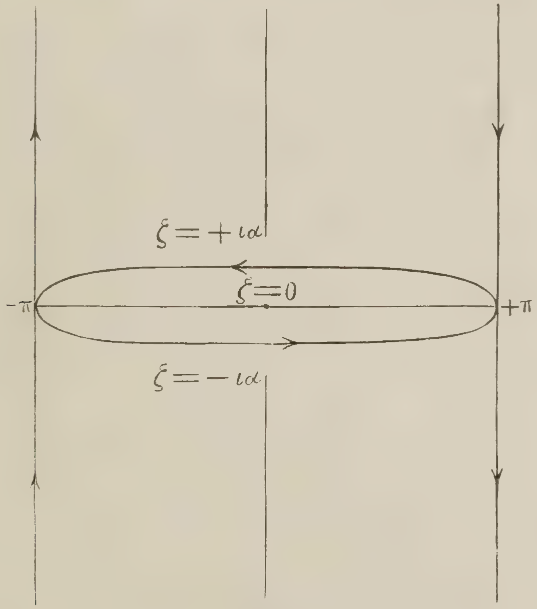

FIG. 2.

In like manner we see that

$$V_2 = \frac{1}{2\pi\iota}\int \frac{f(R-ct)}{R}\, G_2(\zeta)\, d\zeta,$$

where $$G_2(\zeta) = \tfrac{1}{2}n \cot \tfrac{1}{2}n(\zeta+\theta+\theta_0),$$

the path of integration being the same as for V_1.

We have now expressed V_1 and V_2 by means of integrals which contain n only through the two functions $G_1(\zeta)$, $G_2(\zeta)$: and it may therefore seem unnecessary to retain any longer the assumption that n is an integer. That is, we may be tempted to suppose that the corresponding solution for a wedge of *any* angle π/n, may be obtained by taking

$$V = V_1 - V_2,$$

as expressed above. It will be seen, in fact, that this function does satisfy the necessary boundary conditions, and that it has the correct infinity at the source. But, as a matter of fact, *the function V is no longer a continuous function of θ.* To explain the reason for this phenomenon, it is necessary to consider the poles of $G_1(\zeta)$, $G_2(\zeta)$, which are still given by the formulæ

$$\left.\begin{aligned} \zeta &= \theta_0 - \theta + 2k\pi/n, \\ \zeta &= -\theta_0 - \theta + 2k\pi/n, \end{aligned}\right\} \quad \begin{aligned} 0 &< \theta_0 < \pi/n, \\ 0 &< \theta < \pi/n, \end{aligned}$$

k being any integer.

Now, as θ varies, these poles travel along the real axis; but the number of them contained in the interval $(-\pi, +\pi)$ no longer remains *fixed.* Thus, when θ increases through such a value as brings one of the poles of $G_1(\zeta)$ to the point $\zeta = +\pi$, the effect on the integral V_1 is to introduce an abrupt change in its value: this is due to the presence of an additional pole in the interior of the path of integration. Similarly, when a pole comes to $\zeta = -\pi$, it will disappear from the interior of the loop, and again a discontinuity will occur in the value of V.

To illustrate the difficulty let us consider in detail the simple case $n = \frac{2}{3}$. The poles of $G_1(\zeta)$ are then given by

$$\zeta = \theta_0 - \theta + 3k\pi,$$

and those of $G_2(\zeta)$ are given by

$$\zeta = -\theta_0 - \theta + 3k\pi,$$

where k is any integer, and θ, θ_0 both lie between the limits 0 and $\frac{3}{2}\pi$.

It will be easily seen that

(i) If $0 < \theta_0 < \frac{1}{2}\pi$, $G_1(\zeta)$ has a pole ($k = 0$) at $\zeta = -\pi$, when $\theta = \theta_0 + \pi$; and $G_2(\zeta)$ has a pole ($k = 0$) at $\zeta = -\pi$, corresponding to $\theta = \pi - \theta_0$. Thus V_1 will have a discontinuity corresponding to $\theta = \theta_0 + \pi$, and V_2 will have one at $\theta = \pi - \theta_0$.

(ii) If $\frac{1}{2}\pi < \theta_0 < \pi$, $G_1(\zeta)$ has no pole at either limit; but $G_2(\zeta)$ has a pole ($k = 1$) at $\zeta = +\pi$, when $\theta = 2\pi - \theta_0$, and a second pole ($k = 0$) at $\zeta = -\pi$, when $\theta = \pi - \theta_0$. Thus here V_2 has two points of discontinuity.

(iii) If $\pi < \theta_0 < \frac{3}{2}\pi$, $G_1(\zeta)$ has a pole ($k = 0$) at $\zeta = +\pi$, when $\theta = \theta_0 - \pi$: and $G_2(\zeta)$ has a pole ($k = 1$) at $\zeta = +\pi$, when $\theta = 2\pi - \theta_0$. Thus again both functions V_1, V_2 are discontinuous.

In order then to solve our problem for any value of the angle π/n, it will be necessary to choose a path of integration which does not meet the real axis; and further the path chosen must be equivalent to the loop already used when n is an integer.

Such a path can be obtained by adding on to the loop the two vertical lines of the last figure (the directions of integration being indicated by the arrows); for, when n is an integer, the functions G_1, G_2, and R, have each a period 2π in ζ, and consequently the contributions to V_1 and V_2 from the two vertical lines will cancel each other.

Of course care must be taken to see that the integrals taken along the vertical lines are convergent. Now along these two lines, R is real and positive, and tends to infinity like $\sqrt{(2rr_0 \cosh \eta)}$, if η is the imaginary part of ζ. Thus $\int d\zeta/R$ is absolutely convergent: and $|G_1|$, $|G_2|$ tend to the common limit $\frac{1}{2}n$. Hence, provided that $f(R-ct)$ remains numerically less than some fixed value as R tends to $+\infty$ by real values, the integrals V_1 and V_2 will be absolutely convergent along the vertical lines.*

Now, keeping the ends of the path fixed at infinity, we can, by Cauchy's theorem, deform the path of integration into the shape given in Fig. 3. Care must be taken that neither portion of the path comes in contact with a cut: and both parts must be kept clear of the real axis, so as to avoid the poles which traverse the interval $(-\pi, +\pi)$ as θ varies.

We have still to prove that the solution $V = V_1 - V_2$ satisfies the prescribed conditions. It is evident that it satisfies the fundamental differential equation: and that it corresponds to the case of *divergent* waves, which have not been reflected back to the source. Also we have just seen that by using our new path of integration the function V will be in general continuous in the interior of the space bounded by $\theta = 0$, $\theta = \pi/n$.

We have now to see that the necessary boundary conditions are satisfied; at $\theta = 0$, we have

$$G_1(\zeta) = \tfrac{1}{2}n \cot \tfrac{1}{2}n(\zeta - \theta_0), \quad \text{and} \quad G_2(\zeta) = \tfrac{1}{2}n \cot \tfrac{1}{2}n(\zeta + \theta_0),$$

and so

$$G_1(\zeta) = -G_2(-\zeta).$$

But we can choose our two halves of the path so as to be symmetrical about the origin; and then, by changing the sign of ζ, we change one half into the other, *described in the proper sense.* Thus it is seen that $V_1 = +V_2$ at $\theta = 0$, because R does not alter when the sign of ζ is changed. Hence $V = 0$ at $\theta = 0$.

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* It is at this stage that we utilise the condition imposed on the functions $G_1(\zeta)$, $G_2(\zeta)$, that they must remain finite as η tends to either $+\infty$ or $-\infty$ (see p. 452).

Again, at $\theta = \pi/n$, we have

$$G_1(\zeta) = -\tfrac{1}{2}n \tan \tfrac{1}{2}n(\zeta - \theta_0), \qquad G_2(\zeta) = -\tfrac{1}{2}n \tan \tfrac{1}{2}n(\zeta + \theta_0),$$

so that

$$G_1(\zeta) = -G_2(-\zeta);$$

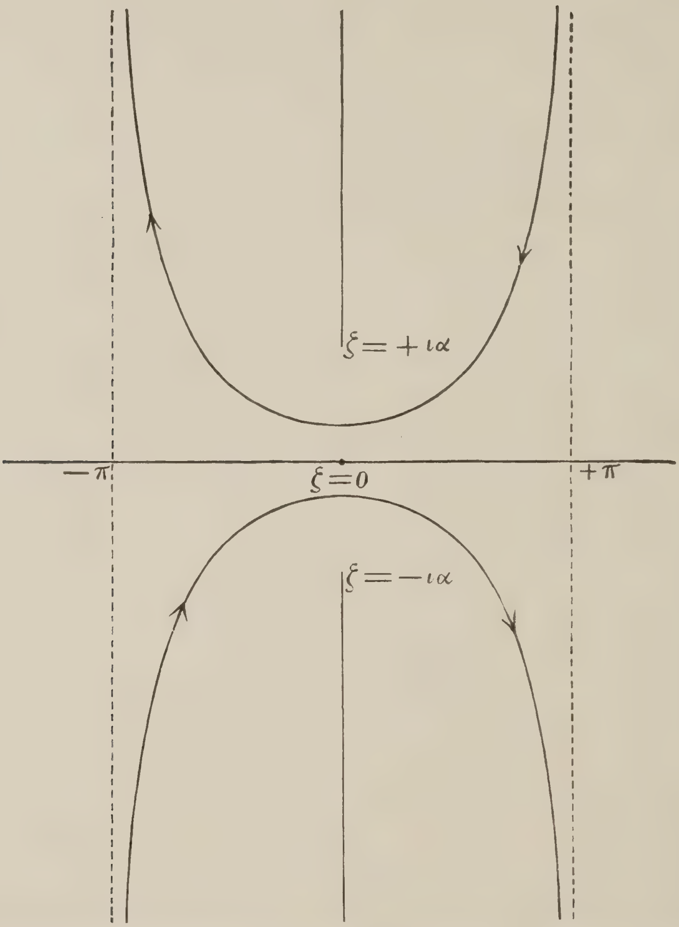

FIG. 3.—The curves are at present restricted to be asymptotic to the vertical lines; when more information as to the nature of the function f is available, we can often remove this restriction.

and thus $V_1 = V_2$ and $V = 0$ at both the planes $\theta = 0$ and $\theta = \pi/n$. That is, *the required boundary conditions are satisfied.*

The only point remaining is to prove that V has the right form near the source. When the point (r, θ, z) approaches (r_0, θ_0, z_0) it is evident that α will tend to zero because

$$2rr_0 \cosh \alpha = r^2 + r_0^2 + (z - z_0)^2.$$

Thus the two cuts of our diagram will tend to join; and since our path of integration must pass between the cuts (twice), we may anticipate that a special discussion will be needed. We can replace our path by the form given in Fig. 4. Then, as $\theta_0-\theta$ tends to zero, we can take the inner loop smaller and smaller. Now $G_1(\zeta)$ has the pole $\zeta=\theta_0-\theta$ inside the loop, and the residue there is unity; but $G_2(\zeta)$ has no pole there. Thus, by Cauchy's theorem, the contributions from the loop to V_1 and V_2 are respectively

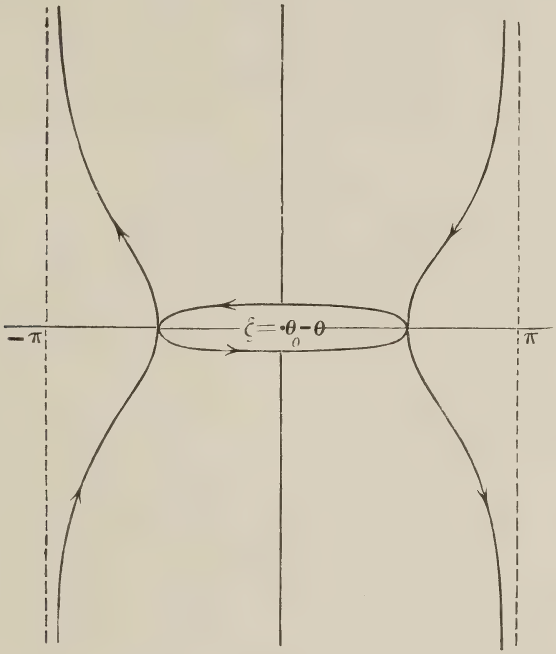

FIG. 4.

$$\frac{1}{R_0}f(R_0-ct) \quad \text{and} \quad 0,$$

where
$$R_0^2=r^2+r_0^2-2rr_0\cos(\theta-\theta_0)+(z-z_0)^2.$$

The remainder of the path indicated in Fig. 4 will contribute finite amounts to V_1 and V_2.

Thus in all we see that the character of V near the source is of the type $\frac{1}{R_0}f(R_0-ct)$: and by a proper choice of the function f, this will represent any assigned type of source (having an infinity of the *first* order only).

The proof that the function V satisfies all the prescribed conditions is now complete: and *our problem is accordingly solved, for all angles* π/n, *by the formula*

$$V = V_1 - V_2,$$

where $V_1 = \dfrac{1}{2\pi\iota}\displaystyle\int \frac{f(R-ct)}{R} G_1(\zeta)\,d\zeta, \qquad V_2 = \dfrac{1}{2\pi\iota}\displaystyle\int \frac{f(R-ct)}{R} G_2(\zeta)\,d\zeta,$

and
$$R^2 = r^2 + r_0^2 - 2rr_0 \cos\zeta + (z-z_0)^2,$$

$$G_1(\zeta) = \tfrac{1}{2}n \cot \tfrac{1}{2}n(\zeta+\theta-\theta_0), \qquad G_2(\zeta) = \tfrac{1}{2}n \cot \tfrac{1}{2}n(\zeta+\theta+\theta_0),$$

the path of integration being that drawn in Fig. 3.

We may suppose that the two halves of the path are symmetrical about $\zeta = 0$; then the lower path changes into the upper (taken in the proper sense), by changing the sign of ζ; and so we may replace the integral along the two paths by the sum of two integrals along one path. Thus *we find*

$$V_1 = \frac{1}{2\pi\iota}\int \frac{f(R-ct)}{R} H_1(\zeta)\,d\zeta, \qquad V_2 = \frac{1}{2\pi\iota}\int \frac{f(R-ct)}{R} H_2(\zeta)\,d\zeta,$$

where
$$H_1(\zeta) = G_1(\zeta) - G_1(-\zeta) = -\frac{n \sin n\zeta}{\cos n\zeta - \cos n(\theta-\theta_0)},$$

and
$$H_2(\zeta) = G_2(\zeta) - G_2(-\zeta) = -\frac{n \sin n\zeta}{\cos n\zeta - \cos n(\theta+\theta_0)},$$

the path of integration being now reduced to the upper path only in Fig. 3.

The second formula has the advantage of making more obvious the fact that the boundary condition $V=0$ is satisfied at the two planes $\theta = 0$, $\theta = \pi/n$; for clearly at each of these planes $H_1(\zeta) = H_2(\zeta)$, and consequently $V_1 = V_2$ also.

3. *The Diffraction-Problem for Sound-Waves from a Source, impinging on a Wedge.*

We can now write down without difficulty the solution ϕ of the wave equation which satisfies the boundary condition

$$\frac{\partial\phi}{\partial\nu} = 0,$$

at the planes $\theta = 0$, $\theta = \pi/n$ (for any value of n), corresponding to a source between the planes.

As is well known, when n is an integer, the solution is given by

$$\phi = V_1 + V_2,$$

where V_1 and V_2 are the functions obtained above. Thus, by a corresponding generalization, we see that this formula will continue to represent the required solution for all values of n, provided that V_1, V_2 have the values given by the complex integrals of p. 458.

We note as a verification that in the second form of the solution

$$\frac{\partial H_1}{\partial \theta} + \frac{\partial H_2}{\partial \theta} = 0,$$

both for $\theta = 0$ and π/n.

For a simple harmonic source, we may take

$$f(R-ct) = e^{\iota\kappa(ct-R)},$$

the real or imaginary part of the results being finally selected, as may be convenient. In this case the solution reduces to that found by Prof. H. M. Macdonald.* We can then also, if we wish, suppose that the ends of the path are displaced so that the imaginary part of R is *negative;* that is the path may start from any point at infinity of the type $\iota\infty + \lambda$, where $\pi \leqslant \lambda < 2\pi$, and it may finish at any point of the type $\iota\infty - \lambda'$, where $0 < \lambda' \leqslant \pi$; instead of being asymptotic to the dotted lines $\pm\pi$, as indicated in Fig. 3.

Prof. Macdonald has indicated how the complex integrals can be expressed in other forms which lend themselves to numerical calculation when $n = \frac{1}{2}$ (the case of a *half-plane*); his formulæ may be written (in the notation used here),

$$V_1 = \frac{\iota\kappa}{\pi} \int_{-\infty}^{v_1} K_1(\iota\kappa R_0 \cosh v)\, dv,$$

where $$R_0^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta-\theta_0) + (z-z_0)^2,$$

and $$\sinh v_1 = \frac{2}{R_0} \sqrt{(rr_0)} \cos \tfrac{1}{2}(\theta-\theta_0).$$

An alternative form, suitable when v_1 is positive, is given by

$$V_1 = \frac{e^{-\iota\kappa R_0}}{R_0} - \frac{\iota\kappa}{\pi} \int_{v_1}^{\infty} K_1(\iota\kappa R_0 \cosh v)\, dv.$$

The value of V_2 is found by similar formulæ in which $\theta+\theta_0$ replaces $\theta-\theta_0$.

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[*] *Proc. London Math. Soc.*, Ser. 2, Vol. 14, 1915, p. 418 above; the results quoted are given on p. 424, and the reduction of the integrals to Fresnel's form on p. 426.

4. *The Diffraction-Problem for Waves from a Hertzian Oscillator impinging on a Wedge.*

It is known that if V denotes a solution of the fundamental wave-equation, and if

$$W = l\frac{\partial V}{\partial x} + m\frac{\partial V}{\partial y} + n\frac{\partial V}{\partial z},$$

then the electromagnetic equations are satisfied by the solution

$$X = -\frac{\partial W}{\partial x} + \frac{l}{c^2}\frac{\partial^2 V}{\partial t^2}, \qquad \alpha = \frac{1}{c^2}\frac{\partial}{\partial t}\left(m\frac{\partial V}{\partial z} - n\frac{\partial V}{\partial y}\right),$$

$$Y = -\frac{\partial W}{\partial y} + \frac{m}{c^2}\frac{\partial^2 V}{\partial t^2}, \qquad \beta = \frac{1}{c^2}\frac{\partial}{\partial t}\left(n\frac{\partial V}{\partial x} - l\frac{\partial V}{\partial z}\right),$$

$$Z = -\frac{\partial W}{\partial z} + \frac{n}{c^2}\frac{\partial^2 V}{\partial t^2}, \qquad \gamma = \frac{1}{c^2}\frac{\partial}{\partial t}\left(l\frac{\partial V}{\partial y} - m\frac{\partial V}{\partial x}\right),$$

where (l, m, n) are any fixed direction-cosines, (X, Y, Z) is the electric force, and (α, β, γ) the magnetic force.

Assuming the wedge to be perfectly conducting, the condition to be satisfied at the boundaries is that the electric force shall be normal to the boundary. Now the boundaries are planes: say that one of them is the plane of yz. Then the conditions to be satisfied at the plane $x = 0$ are $Y = 0$, $Z = 0$; and both these conditions will clearly be satisfied provided that

$$V = 0, \qquad W = 0.$$

If we choose $V = 0$ at the plane $x = 0$, the value of W at that plane reduces to

$$l\frac{\partial V}{\partial x},$$

which can only be zero if $l = 0$: or if the axis of the oscillator is parallel to the plane boundary.

Similar reasoning applies at the second face of the wedge: and we conclude that our condition is satisfied by $V = 0$, provided that (l, m, n) is parallel to *both* planes; that is, provided that (l, m, n) is parallel to the edge of the wedge.

We can accordingly solve our problem for a Hertzian oscillator with its axis parallel to the edge of the wedge, by taking V to be the solution determined in § 2 (see p. 458), and $l = 0$, $m = 0$, $n = 1$; so that

$$X = -\frac{\partial^2 V}{\partial x \partial z}, \qquad \alpha = -\frac{1}{c^2}\frac{\partial^2 V}{\partial y \partial t},$$

$$Y = -\frac{\partial^2 V}{\partial y \partial z}, \qquad \beta = +\frac{1}{c^2}\frac{\partial^2 V}{\partial x \partial t},$$

$$Z = -\frac{\partial^2 V}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}, \qquad \gamma = 0.$$

In cylindrical coordinates this solution can be written*

$$X' = -\frac{\partial^2 V}{\partial r \partial z}, \qquad \alpha' = -\frac{1}{c^2 r}\frac{\partial^2 V}{\partial \theta \partial t},$$

$$Y' = -\frac{1}{r}\frac{\partial^2 V}{\partial \theta \partial z}, \qquad \beta' = +\frac{1}{c^2}\frac{\partial^2 V}{\partial r \partial t},$$

Z and γ remaining unchanged.

If we assume that the source is an oscillating magnet, instead of a Hertzian oscillator, it is easy to see that the corresponding solution is

$$\alpha' = -\frac{\partial^2 V}{\partial r \partial z}, \qquad X' = +\frac{1}{r}\frac{\partial^2 V}{\partial \theta \partial t},$$

$$\beta' = -\frac{1}{r}\frac{\partial^2 V}{\partial \theta \partial z}, \qquad Y' = -\frac{\partial^2 V}{\partial r \partial t},$$

$$\gamma = -\frac{\partial^2 V}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}, \qquad Z = 0,$$

where now the value of V is of the form $V_1 + V_2$ (as in § 3). For then it is evident that $X' = 0$ at the faces of the wedge; and consequently the electric force is normal to the boundary.

These two solutions can be adapted to numerical calculation by means of Prof. Macdonald's formulæ (quoted above, p. 459) for the case of a half-plane or straight-edge, the source being supposed simple harmonic.

It is not without interest to deduce from our solutions the known results for the case of simple harmonic *plane* waves. If we make r_0 tend to infinity, and take

$$\frac{f(ct-R)}{R} = \frac{Ar_0}{R} e^{\iota\kappa(ct+r_0-R)},$$

it will be easily seen that $R - r_0$ tends to $-r\cos\zeta$ and that $R_0 - r_0$ tends to the form $-r\cos(\theta-\theta_0)$; consequently the limiting form corresponds to a value of

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* Here X', Y' are the components of electric force along the directions of r, θ respectively; while α', β' are the corresponding components of the magnetic force.

$$V = A\,e^{\iota\kappa\,[ct+r\cos(\theta-\theta_0)]}$$

in the incident wave. Thus the incident wave is given by

$$X' = 0, \qquad Y' = 0, \qquad Z = e^{\iota\kappa\,[ct+r\cos(\theta-\theta_0)]},$$

provided that

$$A = -1/\kappa^2.$$

The complete solution is then given by

$$Z = Z_1 - Z_2,$$

where

$$Z_1 = \frac{1}{2\pi\iota}\int e^{\iota\kappa r\cos\zeta}\,H_1(\zeta)\,d\zeta, \qquad Z_2 = \frac{1}{2\pi\iota}\int e^{\iota\kappa r\cos\zeta}\,H_2(\zeta)\,d\zeta,$$

$H_1(\zeta)$, $H_2(\zeta)$ being the functions defined above (p. 458), and the path of integration being the upper curve in Fig. 3, or its extension defined in § 3.

Similarly, by starting from the solution when the source is an oscillating magnet, we can deduce the solution corresponding to an incident wave

$$\alpha' = 0, \qquad \beta' = 0, \qquad \gamma = e^{\iota\kappa\,[ct+r\cos(\theta-\theta_0)]}.$$

The result is seen to be

$$\gamma = \gamma_1 + \gamma_2,$$

where γ_1, γ_2 are expressed by the same integrals as Z_1, Z_2 above.

These results are due to Macdonald,* and forms suitable for numerical calculation have been deduced from them by W. H. Jackson.†

5. *Electrostatic and Hydrodynamical Problems.*

The problem of electrostatics may be briefly mentioned here: it corresponds to the case in which $f(R-ct)$ is a constant, say unity. The solution then gives Green's function for the space defined by the limits

$$0 \leqslant \theta \leqslant \pi/n.$$

Under these conditions the ends of the path in Fig. 3 can be displaced to any extent (between adjacent cuts), without affecting the convergence of the integrals; and so the upper path may be reduced to two integrals along the edges of the upper cut. On the right-hand edge we then find

$$R = +\iota\sqrt{\{2rr_0(\cosh v - \cosh \alpha)\}},$$

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* *Electric Waves*, 1902, pp. 192, 195.

† *Proc. London Math. Soc.*, Ser. 2. Vol. 1, 1903, p. 393.

where $\zeta = \iota v$, and on the left-hand edge, the sign of R is changed; also we find on either edge

$$H_1(\zeta) = -\frac{n\iota \sinh nv}{\cosh nv - \cos n(\theta-\theta_0)}, \qquad H_2(\zeta) = -\frac{n\iota \sinh nv}{\cosh nv - \cos n(\theta+\theta_0)}.$$

Hence the formula reduces to $V = V_1 - V_2$, where

$$V_1 = \frac{n}{\pi}\int_a^\infty \frac{dv}{\sqrt{\{2rr_0(\cosh v - \cosh a)\}}} \frac{\sinh nv}{\cosh nv - \cos n(\theta-\theta_0)},$$

and V_2 is a corresponding integral with $\theta+\theta_0$ in place of $\theta-\theta_0$.

These formulæ were given by Prof. H. M. Macdonald,[*] and require no further development now.

It may be worth while to add the remark that the corresponding hydrodynamical problem (for a source in an incompressible fluid occupying the space between two planes) is given by the solution

$$\phi = V_1 + V_2,$$

the integrals for V_1 and V_2 being those given above. For it is then evident that $\partial\phi/\partial\theta = 0$ both at $\theta = 0$ and at $\theta = \pi/n$.

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[*] *Proc. London Math. Soc.*, Vol. XXVI, 1895, p. 160, and Ser. 2, Vol. 14, 1915, p. 412 above. See also Sommerfeld's paper, quoted above, for the special case $n = \frac{1}{2}$.

DIVISION OF THE LEMNISCATE INTO SEVEN EQUAL PARTS

By G. B. MATHEWS.

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IN this note $\wp u$ stands for the function defined by

$$\wp'^2 u = 4\wp^3 u - 4\wp u,$$

and ω is the real half period given by

$$\omega = \int_0^1 \frac{dt}{\sqrt{(1-t^4)}} = \int_1^\infty \frac{dt}{\sqrt{(4t^3-4t)}}.$$

Taking μ to be any complex integer $(m+ni)$ not divisible by 7 (real values of μ are, of course, included), the quantity x_μ defined by

$$x_\mu = \wp\,\frac{2\mu\omega}{7},$$

is one of 24 values, which are the roots of the equation $\psi_7 = 0$, where

$$\psi_7 = 7x^{24} - 308x^{22} - 2954x^{20} + 19852x^{18} - 35231x^{16} + 82264x^{14} - 111916x^{12}$$
$$+ 42168x^{10} + 15673x^8 - 14756x^6 + 1302x^4 - 196x^2 - 1.$$

This polynomial is irreducible in the rational field; but by a series of adjunctions it can be broken up into the product of eight cubics in the following manner.

Every coefficient of ψ_7, except the last, is divisible by 7, and we have identically

$$\tfrac{1}{7}\psi_7 = u^2 - 7v^2,$$

where $\quad u = x^{12} - 22x^{10} - 229x^8 + 1308x^6 - 633x^4 - 614x^2 - 3,$

$$v = 8x^{10} + 88x^8 - 496x^6 + 240x^4 + 232x^2 + \tfrac{8}{7}.$$

Hence, if we put $\quad \alpha^2 = 7,$

ψ_7 has the factor f_{12}, given by

$$f_{12} = x^{12} - (22+8\alpha)\,x^{10} - (229+88\alpha)\,x^8 + (1308+496\alpha)\,x^6$$
$$- (633+240\alpha)\,x^4 - (614+232\alpha)\,x^2 - \tfrac{1}{7}(21+8\alpha).$$

Next put
$$\beta^2 = 2\alpha,$$
it is found that f_{12} has the factor f_6, given by
$$f_6 = x^6-(11+4\alpha+7\beta+3\alpha\beta)\,x^4+(63+24\alpha+26\beta+10\alpha\beta)\,x^2$$
$$-\tfrac{1}{7}(371+140\alpha+161\beta+61\alpha\beta).$$

Finally, let
$$\gamma^2 = \beta(3-\alpha),$$
then f_6 has the factor f_3 given by
$$f_3 = x^3-\frac{\gamma}{4}(16+6\alpha+3\beta+\alpha\beta)\,x^2$$
$$+(5+2\alpha+3\beta+\alpha\beta)\,x-\frac{\alpha\beta\gamma}{28}(19+7\alpha+8\beta+3\alpha\beta).$$

By giving to α, β, γ all their different values, we obtain from f_3 the eight conjugate cubic factors of ψ_7. In particular, if we take α, β, γ all real and positive, the roots of $f_3 = 0$ are all real and positive, and are accordingly the values of x_1, x_2, x_3.

From an algebraical point of view this solution is as simple as can be desired; it may, however, be put into another shape, which is of much theoretical interest. In the transformation theory for $n = 7$, Klein's principal modulus τ is connected with the absolute invariant J by the relations
$$J : (J-1) : 1 = (\tau^2+13\tau+49)(\tau^2+5\tau+1)^3$$
$$: (\tau^4+14\tau^3+63\tau^2+70\tau-7)^2 : 1728\tau.$$

For the lemniscate functions $J = 1$, and consequently τ satisfies the equation
$$\phi(\tau) = \tau^4+14\tau^3+63\tau^2+70\tau-7 = 0.$$

This may be written
$$(\tau^2+7\tau+21)^2-28(\tau+4)^2 = 0,$$
whence
$$\tau^2+(7\pm 2\sqrt{7})\tau+(21\pm 8\sqrt{7}) = 0,$$
and hence
$$4\tau = -14+4\epsilon_1\sqrt{7}+\epsilon_2(\epsilon_1\sqrt{7}-1)\sqrt{(2\epsilon_1\sqrt{7})},$$
where ϵ_1, ϵ_2 are independent square roots of unity.

Let τ be any one of the four values of τ; then if we put
$$\alpha = -\tfrac{1}{18}(\tau^3+10\tau^2+14\tau-49),$$
$$\beta = \tfrac{1}{18}(\tau^3+16\tau^2+68\tau+35),$$

we have in virtue of $\phi(\tau) = 0$,

$$\alpha^2 = 7, \qquad \beta^2 = 2\alpha,$$

and moreover

$$\alpha\beta = \tfrac{1}{18}(5\tau^3+56\tau^2+196\tau+91).$$

By substituting these expressions, f_6 assumes the form

$$\begin{aligned} f_6 = x^6 &- \tfrac{1}{3}(3\tau^3+40\tau^2+168\tau+152)x^4 \\ &+ \tfrac{1}{9}(26\tau^3+368\tau^2+1696\tau+2065)x^2 \\ &- \tfrac{1}{63}(163\tau^3+2296\tau^2+10472\tau+12362). \end{aligned}$$

We have also, in terms of τ,

$$9\gamma^2 = -\tau^3-4\tau^2+4\tau+7,$$

so if we put

$$-\tau^3-4\tau^2+4\tau+7 = \sigma^2,$$

we have a cubic factor in the form

$$\begin{aligned} f_3 = x^3 &+ \tfrac{1}{108}(\tau^3+22\tau^2+158\tau+389)\sigma x^2 \\ &+ \tfrac{1}{3}(\tau^3+14\tau^2+62\tau+64)x \\ &+ \tfrac{1}{252}(17\tau^3+238\tau^2+1078\tau+1253)\sigma = 0. \end{aligned}$$

It may be noticed that we can put

$$8\tau = \beta^3+4\beta^2-2\beta-28,$$

with

$$\beta^4 = 28.$$

I am indebted to my colleague, Mr. W. E. H. Berwick, for checking all of my work, except the calculation of ψ_7, and especially for performing the actual resolution of ψ_7 into its factors; this is entirely his work, and I have only partly verified it. As to the value of ψ_7, Mr. T. G. Creak was kind enough to work it out according to my directions, and since his result agreed with one I had found myself by an entirely different process, it is practically certain that the value given is correct.

Reference should be made to a paper by Brioschi (*American Journal*, Vol. XIII, 1891, p. 381); use was made of this in the course of the investigation.

A DIRECT METHOD IN THE MULTIPLICATION THEORY OF THE LEMNISCATE FUNCTIONS AND OTHER ELLIPTIC FUNCTIONS

By G. B. MATHEWS.

[Read May 13th, 1915.]

IN what follows $\wp(u)$ means $\wp(u;\ 4,\ 0)$ and $\sigma(u)$ is defined by

$$\sigma(u) = \frac{2\omega}{\pi} \sin\frac{\pi u}{2\omega} \prod_1^\infty (1-q^{2n})^{-2} \prod_1^\infty \left(1-2q^{2n}\cos\frac{\pi u}{\omega}+q^{4n}\right),$$

with

$$q = e^{-\pi}, \qquad \omega = \int_0^1 \frac{dt}{\sqrt{(1-t^4)}}.$$

Putting

$$G(u) = e^{\pi u^2/8\omega^2}\sigma(u),$$

we have a function, discovered by Gauss, which has the property

$$G(iu) = iG(u),$$

and is expandible in the form

$$G(u) = u+c_1u^5+c_2u^9+\ldots = uP(u^4),$$

convergent for all values of u.

Let $\wp(u) = p$, and consider

$$f(p) = \wp(u)-\wp(nu)$$

as a function of p. Suppose, also, in the first place, that n is a real odd integer.

Then

$$f'(p) = 1-\frac{d\wp(nu)}{du}\,\frac{du}{dp} = 1-\frac{n\wp'(nu)}{\wp'(u)},$$

so that

$$\sqrt{(4p^3-4p)}\,[f'(p)-1] = -n\wp'(nu),$$

whence

$$\sqrt{(4p^3-4p)}\,f''(p)+\frac{(6p^2-2)f'(p)}{\sqrt{(4p^3-4p)}} = \frac{6p^2-2}{\sqrt{(4p^3-4p)}}-n^2\wp''(nu)\frac{du}{dp};$$

and therefore

$$(4p^3-4p)\,f''(p)+(6p^2-2)\,f'(p) = 6p^2-2-n^2[6\,(f-p)^2-2].$$

That is to say, $f(p)$ satisfies the equation

$$(2p^3-2p)\,\frac{d^2f}{dp^2}+(3p^2-1)\,\frac{df}{dp}+3n^2f^2-6n^2pf+(n^2-1)(3p^2-1) = 0. \quad (1)$$

The solution we want is a rational function of p, and for values of u sufficiently near to $\omega+i\omega$ can be expressed as a convergent power series in p containing odd powers only. Assuming

$$f = a_1 p+a_3 p^3+a_5 p^5+\ldots,$$

we find by substitution in (1), and equating successive coefficients to zero,

$$a_1 = -(n^2-1),$$
$$5a_3 = n^2(n^2-1)(n^2+1),$$
$$75a_5 = -n^2(n^2-1)(n^2+1)(2n^4-7),$$
$$13.75a_7 = n^2(n^2-1)(n^2+1)(3n^8-23n^4+55),$$

and so on.

Now let
$$\psi_n = \frac{\sigma(nu)}{\sigma(u)^{n^2}} = h(p),$$

considered as a function of p. We have

$$\wp'(u)\,\frac{h'(p)}{h(p)} = \frac{d}{du}\log\psi_n = n\mathrm{Z}(nu)-n^2\mathrm{Z}(u),$$

$$\wp'(u)\,\frac{d}{dp}\left\{\sqrt{(4p^3-4p)}\,\frac{h'(p)}{h(p)}\right\} = n^2\left\{\wp(u)-\wp(nu)\right\} = n^2f,$$

and accordingly

$$(4p^3-4p)\,\frac{d}{dp}\left(\frac{h'(p)}{h(p)}\right)+(6p^2-2)\,\frac{h'(p)}{h(p)} = n^2f(p). \quad (2)$$

We know beforehand that the ψ_n we want is a polynomial in p^2, so we assume

$$h(p) = 1+c_1 p^2+c_2 p^4+c_3 p^6+\ldots,$$

whence
$$\frac{h'(p)}{h(p)} = 2c_1 p+(4c_2-2c_1^2)\,p^3+(6c_3-6c_2 c_1+2c_1^3)\,p^5+\ldots,$$

$$\frac{d}{dp}\left(\frac{h'(p)}{h(p)}\right) = 2c_1+(12c_2-6c_1^2)\,p^2+(30c_3-30c_2 c_1+10c_1^3)\,p^4+\ldots,$$

so substituting in (2), equating coefficients, and using the calculated values of $a_1, a_3, a_5, \ldots,$ we find that

$$12c_1 = n^2(n^2-1),$$

$$10080c_2 = -n^2(n^2-1)(n^2-4)(n^2+75),$$

$$1995840c_3 = n^2(n^2-1)(n^2-4)(n^2-9)(17n^4+403n^2+9000),$$

$$2^{11}.3^5.5^3.7^2.11.13c_4 = n^2(n^2-1)(n^2-4)(n^2-9)$$
$$\times(283n^8-26978n^6-96173n^4+652068n^2+127764000),$$

and so on.

We know that there is an equivalence

$$\wp(nu) = \frac{p(n+b_1p^2+b_2p^4+\ldots)^2}{(1+c_1p^2+c_2p^4+\ldots)^2}, \tag{3}$$

where the denominator is ψ_n^2. Call the numerator $p\phi_n^2$. Still supposing that n is odd, change u into $u+\omega+i\omega$; then p becomes $-1/p$, and $\wp(nu)$ becomes $\wp(nu+\omega+i\omega)$, that is to say, $-1/\wp(nu)$. It follows that, except for a possible change of sign throughout, the coefficients of ϕ_n are those of ψ_n in the reverse order. Now, since

$$\wp(nu) = p-f(p) = (1-a_1)p-a_3p^3-a_5p^5-\ldots,$$

(3) gives the identity

$$\{(1-a_1)p-a_3p^3-\ldots\}\{1+2c_1p^2+(2c_2+c_1^2)p^4+\ldots\}$$
$$= p\{n^2+2nb_1p^2+(2nb_2+b_1^2)p^4+\ldots\},$$

so by equating coefficients we find

$$60b_1 = -n(n^2-1)(n^2+6),$$

$$10080b_2 = -n(n^2-1)(n^6-13n^4+36n^2+420),$$

$$2^7.3^4.5^2.7.11.13b_3 = n(n^2-1)(n^2-9)$$
$$\times(23n^8+2375n^6-14638n^4+116100n^2+693000),$$

and so on.

When we write $\qquad \psi_n = np^m+b_1p^{m-2}+\ldots,$

with $\qquad m = \tfrac{1}{2}(n^2-1),$

we find that the constant term is -1 for ψ_3, ψ_7, and $+1$ for ψ_5, ψ_9. The identity

$$\psi_{n+3}\psi_{n-3} = \psi_3^2\psi_{n-1}\psi_{n+1} - \psi_n^2\psi_2\psi_4,$$

enables us to prove by induction that the constant term is $(-1)^{\frac{1}{2}(n-1)}$ for all odd values of n. Thus with

$$2m = n^2-1, \qquad \epsilon = (-1)^{\frac{1}{2}(n-1)},$$

we have
$$\psi_n = np^m + b_1 p^{m-2} + b_2 p^{m-4} + b_3 p^{m-6} + \dots$$
$$\dots + \epsilon(c_4 p^8 + c_3 p^6 + c_2 p^4 + c_1 p^2 + 1),$$

with the b_i, c_i as above.

When n is even, ψ_n is of the form $\wp'(u)F$, where F is a polynomial in p. We might form a differential equation for F, but it is easier to proceed as follows. Changing n into $u+\omega+i\omega$, p becomes $-1/p$ as before, but $\wp(nu)$ is unaltered. We hence infer that F is a reciprocal polynomial. Moreover, $(n-1)$ being odd,

$$\frac{p\phi_{n-1}^2}{\psi_{n-1}^2} = p - \frac{\psi_{n-2}\psi_n}{\psi_{n-1}^2},$$

whence
$$\psi_n = \frac{p(\psi_{n-1}+\phi_{n-1})(\psi_{n-1}-\phi_{n-1})}{\psi_{n-2}}. \tag{4}$$

It thus appears that F is always reducible even after removing the factor $\psi_{\frac{1}{2}n}$. For instance,

$$\begin{aligned}\psi_6 &= -p'(3p^8+12p^6-46p^4+12p^2+3)(p^8-28p^6+6p^4-28p^2+1)\\ &= -p'(3p^4-6p^2-1)(p^4+6p^2-3)(p^8-28p^6+6p^4-28p^2+1),\end{aligned}$$

$$\begin{aligned}\psi_8 &= -4p'(p^{14}+15p^{12}-131p^{10}+51p^8+51p^6-131p^4+15p^2+1)\\ &\quad \times(p^{16}-88p^{14}+92p^{12}-872p^{10}+1990p^8-872p^6+92p^4-88p^2+1)\\ &= -4p'(p^2+1)(p^4-6p^2+1)(p^8+20p^6-26p^4+20p^2+1)\\ &\quad \times(p^{16}-88p^{14}+\dots-88p^2+1).\end{aligned}$$

It remains to consider complex multiplication. Let a, b be any two real integers,

$$\mu = a+bi, \qquad m = a^2+b^2 = N(\mu),$$

so that m is the norm of μ. If we put

$$\psi_\mu(u) = \exp\left\{\frac{(-b^2\pi+ab\pi i)u^2}{4\omega^2}\right\}\frac{\sigma(\mu u)}{\sigma(u)^m} = \frac{G(\mu u)}{G(u)^m},$$

we find that $\psi_\mu(u+2\omega) = \psi_\mu(u+2i\omega) = (-1)^{ab}\,\psi_\mu(u)$,

so that ψ_μ has the periods 2ω, $2i\omega$ unless a, b are both odd; that is, unless $\mu \equiv 1+i$ (mod 2), in which case we shall say that μ is half even. In every case ψ_μ^2 has the periods 2ω, $2i\omega$.

There are m incongruent residues of μ, including zero. The zeroes of ψ_μ are given by $2\lambda\omega/\mu$, where λ is any residue of μ not congruent to 0; the poles of ψ_μ are given by $2\lambda\omega$, where λ is any complex integer including 0, and we have near the origin

$$\psi_\mu = \frac{\mu}{u^{m-1}} + \ldots,$$

so each pole is of order $(m-1)$. Since $G(iu) = iG(u)$, we have

$$\psi_{i\mu} = i\psi_\mu, \qquad \psi_{-\mu} = -\psi_\mu, \qquad \psi_{-i\mu} = -i\psi_\mu. \tag{4}$$

Observing that

$$N(\mu+\nu)+N(\mu-\nu) = 2N(\mu)+2N(\nu),$$

we have, as in the real theory,

$$\begin{aligned}\frac{\psi_{\mu+\nu}\psi_{\mu-\nu}}{\psi_\mu^2\psi_\nu^2} &= \frac{G(\mu+\nu)u\;G(\mu-\nu)u}{G(\mu u)^2\;G(\nu u)^2} \\ &= \wp(\nu u)-\wp(\mu u) \\ &= \{\wp(u)-\wp(\mu u)\}-\{\wp(u)-\wp(\nu u)\} \\ &= \frac{\psi_{\mu+1}\psi_{\mu-1}}{\psi_\mu^2}-\frac{\psi_{\nu+1}\psi_{\nu-1}}{\psi_\nu^2}\end{aligned}$$

(since $\psi_1 = 1$), and therefore

$$\psi_{\mu+\nu}\psi_{\mu-\nu} = \psi_{\mu+1}\psi_{\mu-1}\psi_\nu^2 - \psi_{\nu+1}\psi_{\nu-1}\psi_\mu^2, \tag{5}$$

from which we derive other formulæ of reduction in the usual way.

For a complex μ the first few values of ψ_μ are

$$\begin{aligned}\psi_i &= i, \\ \psi_{1+i} &= (1+i)\sqrt{p}, \\ \psi_{1+2i} &= (1+2i)p^2-1, \\ \psi_{2+2i} &= -(1+i)p'(p^2-1), \\ \psi_{-1+3i} &= (1+i)\sqrt{p}\,\{(1+2i)p^2-1\}\{p^2-(1+2i)\}, \\ \psi_{3+2i} &= (3+2i)p^6+(7-4i)p^4+(-11+10i)p^2+1.\end{aligned}$$

In general, as may be proved by induction, ψ_μ is a polynomial in p^2 when μ is odd; the product of such a polynomial by p' when μ is even; and the product of such a polynomial by $\sqrt{p}$ when μ is half-even. Here $\sqrt{p}$ is that definite function of u which coincides with $1/\mathrm{sl}\,u$. In virtue of (4) above, and the fact that if μ_0 is the conjugate of μ, ψ_{μ_0} can be obtained from ψ_μ by changing the coefficients into their conjugates, only one-eighth of all possible arguments need be entered in a table of ψ_μ. When μ is prime the proper argument is that one of the eight associated ones which is congruent to 1 mod $(2+2i)$; for the other values of μ there does not seem to be any decisive rule.

When μ is odd, we can apply a direct method for calculating ψ_μ. We have

$$\frac{d^2}{du^2}\log\psi_\mu = m\wp(u)-\mu^2\wp(\mu u)$$

$$= (m-\mu^2)\,\wp(u)+\mu^2\left\{\wp(u)-\wp(\mu u)\right\}.$$

Now, just as before, we have

$$\wp(u)-\wp(\mu u) = -(\mu^2-1)\,p+\tfrac{1}{5}\mu^2(\mu^2-1)(\mu^2+1)\,p^3+\dots$$

$$= a_1 p+a_3 p^3+a_5 p^5+\dots,$$

where a_i is derived from a_i by writing μ for n. Hence, putting

$$\psi_\mu = h(p) = 1+\gamma_1 p^2+\gamma_2 p^4+\dots,$$

and proceeding as before, we have

$$(4p^3-4p)\,\frac{d}{dp}\left(\frac{h'(p)}{h(p)}\right)+(6p^2-2)\,\frac{h'(p)}{h(p)} = (m-\mu^4)\,p+\mu^2(a_3 p^3+a_5 p^5+\dots),$$

leading to $12\gamma_1 = \mu^4-m,$

$$10080\gamma_2 = -\mu^8-70m\mu^4+35m^2+336\mu^4-300m,$$

$$19958400\gamma_3 = 17\mu^{12}+165m\mu^8-1584\mu^8+5775m^2\mu^4$$
$$-104940m\mu^4-1925m^3+376992\mu^4$$
$$+49500m^2-324000m,$$

and so on.

If we assume, still with μ odd,

$$h(p) = \mu p^{\frac{1}{2}(m-1)}+\beta_1 p^{\frac{1}{2}(m-5)}+\dots,$$

we find in a similar way

$$60\beta_1 = -\mu(\mu^4+5m-6),$$

$$10080\beta_2 = -\mu(\mu^8-14m\mu^4+84\mu^4-35m^2+384m-420), \ \ldots .$$

If in β_i, γ_i we put $\mu = n$, $m = n^2$, they reduce to b_i, c_i as previously calculated; this is a good test of the accuracy of the work.

When μ is odd we have an equivalence

$$\wp(\mu u) = \pm \frac{p\phi_\mu^2}{\psi_\mu^2},$$

where ϕ_μ is related to ψ_μ in the same way as ϕ_n to ψ_n, and the upper or lower sign is to be taken according as ψ_μ ends with ± 1 or with $\pm i$.

When μ is even or half-even

$$\psi_\mu = \frac{p(\psi_{\mu-1}^2 \mp \phi_{\mu-1}^2)}{\psi_{\mu-2}}, \tag{6}$$

by the same argument as before, because in each case $\mu-1$ is odd.

Formula (6) is not one of reduction unless $N(\mu-2) < N(\mu)$; but by taking, if necessary, one of the three numbers $-\mu$, $i\mu$, $-i\mu$ instead of μ, we can always make it applicable. For example, suppose $\mu = -1+3i$; we have

$$\psi_{-1+3i} = \frac{G(-1+3i)u}{G(u)^{10}} = \frac{G\{i(3+i)u\}}{G(u)^{10}}$$

$$= \frac{iG(3+i)u}{G(u)^{10}} = i\psi_{3+i}.$$

We have $\quad \psi_{2+i} = (2+i)p^2-i, \qquad \phi_{2+i} = ip^2-(2+i),$

and hence, by (6),

$$\psi_{-1+3i} = i\psi_{3+i} = \frac{i\{\psi_{2+i}+i\phi_{2+i}\}\{\psi_{2+i}-i\phi_{2+i}\}}{\psi_{1+i}}$$

$$= \frac{ip\{(1+i)p^2+(1-3i)\}\{(3+i)p^2+(-1+i)\}}{(1+i)\sqrt{p}}$$

$$= (-1+i)\sqrt{p}\{p^2-(1+2i)\}\{(2-i)p^2+i\}$$

$$= (1+i)\sqrt{p}\{(1+2i)p^2-1\}\{p^2-(1+2i)\}$$

$$= \psi_{1+i}\psi_{1+2i}\{p^2-(1+2i)\},$$

where the factors ψ_{1+i}, ψ_{1+2i} arise from the fact that $1+i$ and $1+2i$ are divisors of $-1+3i$.

Of course, the direct method soon becomes far too laborious for practical application, but it does not lose its interest on that account. The differential equations (1) and (2) are quite different in character from the partial differential equations given by Jacobi (*Werke*, I, p. 266), and Halphen (*Fonctions Elliptiques*, I, p. 328) in the same connexion. At present I do not see how to deduce the ordinary equations from the partial equations, or conversely.

When n is a real odd number, and the invariants have the general values g_2, g_3, the direct method of this paper is still applicable; instead of (1) we have

$$(4p^3-g_2p-g_3)\frac{d^2f}{dp^2}+(6p^2-\tfrac{1}{2}g_2)\frac{df}{dp}+6n^2f^2-12n^2pf+(n^2-1)(6p^2-\tfrac{1}{2}g_2)=0, \tag{7}$$

and instead of (2),

$$(4p^3-g_2p-g_3)\frac{d}{dp}\left(\frac{h'(p)}{h(p)}\right)+(6p^2-\tfrac{1}{2}g_2)\frac{h'(p)}{h(p)}=n^2f. \tag{8}$$

So far as they go, the ordinary equations are more useful than the partial ones, when g_2, g_3 are actually given, especially when, as in our case, one of them is zero. Thus for the lemniscate functions, if we use Halphen's equation, we have to calculate a number of terms involving g_3, although g_3 is actually zero, on account of the term

$$\tfrac{2}{3}g_2^2\frac{\partial z}{\partial g_3}$$

occurring in the equation. Similarly in Jacobi's equation we have the term

$$2n(a^2-4)\frac{\partial z}{\partial a}$$

with $$a=(1+\kappa^2)/\kappa,$$

which in our case is zero.

If we eliminate f from (1) and (2) or from (7) and (8), we have an ordinary differential equation of the fourth order satisfied by ψ_n. The general solution is of course transcendental; the fact that it has a solution which is a finite polynomial is very noteworthy.

I am greatly indebted to Mr. T. G. Creak, of Llanberis, for helping me in the calculations. Almost all the coefficients were calculated by us independently; the values of b_3, γ_3 are due to Mr. Creak alone, but he has checked them by putting $\mu=n$, $m=n^2$ in γ_3, and by finding the values of b_3 for $n = 5$ and $n = 7$. These two values of b_3 came out right, so it is almost certain that the expression for b_3 is correct.

The coefficients c_1, c_2, c_3 were found by Gauss (see *Werke*, III, p. 411); he also gives γ_1, γ_2 (*ibid.*, p. 412). In the same place there are a number of real and complex multiplication formulæ corresponding to $n = 2, 3, 4, i$, $1+i$, $2+i$, $3+i$, $1+2i$, $1+3i$, $1+4i$. It is not clear how he obtained c_1, c_2, ..., but it was probably by a method similar to that of the present paper, since (*l.c.*, p. 408) he gives a differential equation of the fourth order satisfied by $G(u)$. Here, however, it is u (Gauss's ϕ) which is the independent variable. The functions ψ_{3+2i}, ψ_{1+4i}, ψ_5 are practically given by Eisenstein (*Mathematische Abhandlungen*, p. 133, or *Crelle*, XXIX, p. 177).

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